A DETERMINANTAL INEQUALITY
FOR POSITIVE DEFINITE MATRICES

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Let $H = (H_{i,j})$ $(1 \leq i, j \leq n)$ be an $nk \times nk$ matrix with complex coefficients, where each $H_{i,j}$ is itself a $k \times k$ matrix $(n, k \geq 2)$. Let $|H|$ denote the determinant of H and let $\|H\| = |(|H_{i,j}|)|$ $(1 \leq i, j \leq n)$. The purpose of this note is to prove the following theorem.

THEOREM. If H is positive definite Hermitian then $|H| \leq \|H\|$. Moreover, $|H| = \|H\|$ if and only if $H_{i,j} = 0$ whenever $i \neq j$.

The case $n = 2$ of this theorem is contained in [1].

Before proceeding to the proof, we introduce some notation. Suppose $2 \leq p \leq m$ and let $z_i = (z_{i,1}, z_{i,2}, \ldots, z_{i,m})$ have complex coefficients for $1 \leq i \leq p$. Then define $(z_1, z_2) = \Sigma_{r=1}^{m} z_{1,r} \bar{z}_{2,r}$ and define $z_1 \wedge z_2 \wedge \ldots \wedge z_p$ to be a vector with ${}_mC_p$ coordinates as follows: the coordinates of $z_1 \wedge z_2 \wedge \ldots \wedge z_p$ are the $p \times p$ minors of the matrix $Z = (z_{i,j})$ $(1 \leq i \leq p,\ 1 \leq j \leq m)$ where the ordering of the coordinates is lexicographic based upon the columns of Z. For example, if $p = 2$ and $m = 3$,

$$z_1 \wedge z_2 = \left(\begin{vmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{vmatrix}, \begin{vmatrix} z_{1,1} & z_{1,3} \\ z_{2,1} & z_{2,3} \end{vmatrix}, \begin{vmatrix} z_{1,2} & z_{1,3} \\ z_{2,2} & z_{2,3} \end{vmatrix} \right).$$

The proof of our theorem rests on the known fact [2] that if also $y_i = (y_{i,1}, y_{i,2}, \ldots, y_{i,m})$ for $1 \leq i \leq p$, then

$$(z_1 \wedge z_2 \wedge \ldots \wedge z_p, y_1 \wedge y_2 \wedge \ldots \wedge y_p) = |(z_i, y_j)|, \quad 1 \leq i, j \leq p.$$

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We now turn to the proof of our theorem. If $W = \operatorname{diag}(W_1, W_2, \ldots, W_n)$ is the direct sum of n non-singular $k \times k$ matrices $W_1, W_2, \ldots, W_n$ then $|WHW^*| = |WW^*|\ |H|$, and $\|WHW^*\| = |(|W_i H_{i,j} W_j^*|)| = \|W\|\ \|H\|\ \|W^*\| = |WW^*|\ \|H\|$. ($W^*$ is the conjugate transpose of W.) Thus, if $|WHW^*| \leq \|WHW^*\|$ then $|H| \leq \|H\|$, and if $|H| = \|H\|$ then $|WHW^*| = \|WHW^*\|$.

Since H is positive definite, $H = VV^*$ for some triangular V. We write $V = (V_{i,j})$ $(1 \leq i, j \leq n)$ where each $V_{i,j}$ is $k \times k$ and $V_{i,j} = 0$ if $i > j$. Let $W_i = (V_{i,i})^{-1}$ for $1 \leq i \leq n$. Then $WHW^* = (WV)(WV)^* = XX^*$ where

$$WV = X = \begin{pmatrix} I_k & X_{1,2} & X_{1,3} & \cdot & \cdot & \cdot & X_{1,n} \\ & I_k & X_{2,3} & \cdot & \cdot & \cdot & X_{2,n} \\ & & I_k & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & 0 & & & & \cdot & \\ & & & & & & I_k \end{pmatrix}.$$

Here each $X_{i,j}$ is $k \times k$ and I_k denotes the $k \times k$ identity matrix. Since $|XX^*| = 1$, to prove that $|H| \leq \|H\|$ it suffices to prove that $\|XX^*\| \geq 1$. Moreover, if $|H| = \|H\|$, then $\|XX^*\| = 1$. If we can show that this implies that $X = I_{nk}$ then $V = W^{-1}$ and hence $H = VV^*$ satisfies $H_{i,j} = 0$ for all $i \neq j$.

Let $x_1, x_2, \ldots, x_{nk}$ be the row vectors of the matrix X. Then

$$(x_{(i-1)k+1} \wedge x_{(i-1)k+2} \wedge \ldots \wedge x_{ik},\ x_{(j-1)k+1} \wedge x_{(j-1)k+2} \wedge \ldots \wedge x_{jk})$$

$$= |(x_{(i-1)k+s},\ x_{(j-1)k+t})|, \qquad 1 \leq s,\ t \leq k,$$

so that

$$\|XX^*\| = \|(x_i, x_j)\|, \quad 1 \leq i, j \leq nk,$$

$$= |(x_{(i-1)k+1} \wedge \ldots \wedge x_{ik}, x_{(j-1)k+1} \wedge \ldots \wedge x_{jk})|, \quad 1 \leq i, j \leq n,$$

$$= (x, x)$$

where

$$x = (x_1 \wedge x_2 \wedge \ldots \wedge x_k) \wedge (x_{k+1} \wedge x_{k+2} \wedge \ldots \wedge x_{2k}) \wedge \ldots \wedge (x_{(n-1)k+1} \wedge \ldots \wedge x_{nk}).$$

Then $\|XX^*\|$ is of the form $\Sigma|u_i|^2$ where the u_i are the co-ordinates of the vector x and are polynomials in the elements of the matrix X. We complete the proof by showing that among the u_i we find 1 and all of the non-zero off-diagonal coefficients of X. Let $X = (x_{i,j})$.

The first coordinate of $x_1 \wedge x_2 \wedge \ldots \wedge x_k$ is 1, and the first coordinate of $x_{(j-1)k+1} \wedge \ldots \wedge x_{jk}$ is zero for $2 \leq j \leq n$ since each such coordinate is the determinant of a matrix of zeros. Similarly, the coordinate of $x_{(i-1)k+1} \wedge \ldots \wedge x_{ik}$ constructed from columns $(i-1)k+1, (i-1)k+2, \ldots, ik$ of the matrix

$$A_i = \begin{pmatrix} x_{(i-1)k+1} \\ x_{(i-1)k+2} \\ \ldots \\ x_{ik} \end{pmatrix}$$

whose rows are the vectors $x_{(i-1)k+1}, x_{(i-1)k+2}, \ldots, x_{ik}$, is 1; and for all $j > i$ this coordinate in $x_{(j-1)k+1} \wedge \ldots \wedge x_{jk}$ is the determinant of the zero matrix. This means that if we form the matrix A whose rows are the vectors $x_{(i-1)k+1} \wedge \ldots \wedge x_{ik}$ for $1 \leq i \leq n$, then

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$$\begin{vmatrix} 1 & & & & \\ & 1 & & * & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & & \cdot \\ & & & & & 1 \end{vmatrix}$$

is one of the minors of A. (Here, the asterisk indicates elements whose precise values do not matter.) Thus one u_i is 1.

For fixed i $(1 \leqq i \leqq n - 1)$ let s, t be integers such that $1 \leqq s \leqq k$ and $ik \leqq t \leqq nk$. The minor of the matrix A_i constructed from columns $(i - 1)k + 1, \ldots, (i - 1)k + s - 1$, $(i - 1)k + s + 1, \ldots, ik, t$ has value $\pm x_{(i - 1)k + s, t}$. Hence one of the coordinates of $x_{(i - 1)k + 1} \wedge \cdots \wedge x_{ik}$ is $\pm x_{(i - 1)k + s, t}$. In $x_{(j - 1)k + 1} \wedge \cdots \wedge x_{jk}$ for $j > i$ this same coordinate is a determinant with at least $k - 1$ columns of zeros and hence is zero. Consequently, one of the minors of A is (after, possibly, a permutation of its columns)

$$\begin{matrix} i - 1 \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \\ \\ n - i \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \end{matrix} \begin{vmatrix} 1 & & & & & \\ & \cdot & & * & & \\ & & \cdot & & & \\ & & & 1 & & \\ & & & & \pm x_{(i - 1)k + s, t} & \\ & & & & 1 & \\ & & & & & \cdot \\ & 0 & & & & \cdot \\ & & & & & & 1 \end{vmatrix} .$$

Thus it follows that $\pm x_{(i - 1)k + s, t}$ is one of the coordinates of x.

It is now clear that

$$\|XX^*\| = 1 + \Sigma_{i, s, t} |x_{(i - 1)k + s, t}|^2 + \Sigma |u_i|^2$$

where the last sum is over the remaining u_i. Hence $\|XX^*\| \geqq 1$ and $\|XX^*\| = 1$ implies that all $x_{(i - 1)k + s, t}$ vanish so that $X = I_{nk}$.

The proof of the theorem is now complete.

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Everitt's proof of the case $n = 2$ depended on the fact that if A and B are positive definite $k \times k$ Hermitian matrices then $|A + B| > |A| + |B|$. We are now able to reverse the logic and deduce this inequality from our theorem. For let

$$C = \begin{pmatrix} A + B & A^{\frac{1}{2}} \\ A^{\frac{1}{2}} & I_k \end{pmatrix}$$

where $A^{\frac{1}{2}}$ is Hermitian and satisfies $(A^{\frac{1}{2}})^2 = A$. Let

$$T = \begin{pmatrix} I_k & -A^{\frac{1}{2}} \\ 0 & I_k \end{pmatrix}.$$

Then $TCT^* = \text{diag}(B, I_k)$ is positive definite so that C is also. Moreover, $|C| = |B|$. Applying our theorem to C, we find $|C| \leqq |A + B| - |A|$ or $|A + B| \geqq |A| + |B|$. We cannot have equality here since $A^{\frac{1}{2}} \neq 0$.

As another application of the case $n = 2$ we deduce an inequality due to Fischer [3]. Let

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an $(m + n) \times (m + n)$ positive definite Hermitian matrix where A is $m \times m$ and D is $n \times n$. Suppose $m \geqq n$ and let $H_1 = \text{diag}(H, I_{m-n})$. ($H_1 = H$ if $m = n$.) Write $H_1 = (H_{i,j})$ for $1 \leqq i, j \leqq 2$ where $H_{1,1} = A$ and $H_{2,2} = \text{diag}(D, I_{m-n})$. Applying our theorem we find that

$$|H| = |H_1| \leqq |H_{1,1}|\ |H_{2,2}| - |H_{1,2}|\ |H_{2,1}| \leqq |A|\ |D|$$

with equality if and only if $B = 0$.

Since Fischer's inequality implies Hadamard's inequality, it follows that the case $n = 2$ of our theorem also implies Hadamard's inequality.

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By a standard continuity argument, we may extend our result to non-negative Hermitian matrices.

COROLLARY. If H is non-negative Hermitian then $|H| \leqq \|H\|$.

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