REFERENCE FRAMES IN RELATIVISTIC SPACE-TIME

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ABSTRACT. Three fundamental concepts of reference frames in relativistic space-time are confronted: 1. the gravitational compass, 2. the stellar compass and 3. the inertial compass. It is argued that under certain conditions asymptotically fixed (stellar) reference frames can be introduced with the same rigour as local Fermi frames, thereby eliminating one possible psychological reason why the importance of Fermi frames frequently has been overestimated in the past. As applications of these three concepts we discuss: 1. a relativistic definition of the geoid, 2. a relativistic astrometric problem and 3. the post-Newtonian theory of a laser gyroscope fixed to the Earth's surface.

1. INTRODUCTION

The problem of reference frames in relativistic space-time† comprises at least two parts: the construction of a physical reference frame that is directly related to the observables and the choice of a suitable coordinate system. The physical reference frame of a single observer is mathematically described by the pair $(\gamma, e_{(\alpha)})$, where γ denotes the world line of the observer and $e_{(\alpha)}$ a bundle of orthonormal tetrad vectors (one timelike, three spacelike) along γ. The timelike part $e_{(0)}$ is related to proper time τ by $e_{(0)} = \partial/\partial\tau$, whereas the spacelike part $e_{(i)}$ represents the spatial reference directions. For practical purposes proper time τ as measured by an atomic clock on Earth is related with some barycentric coordinate time t by:

$$\tau \quad \leftrightarrow \quad \mathrm{TDT} \quad \leftrightarrow \quad \mathrm{TDB} \quad \leftrightarrow \quad t$$

where τ differs from terrestrial dynamical time TDT $= \tau_{geoid}$ and t from barycentric dynamical time TDB only by a constant factor. TDT differs from TDB only by periodic terms that have been obtained by *Moyer (1981)* and more accurately by *Hirayama et al. (1985)*.

We now want to concentrate upon the spatial reference directions $e_{(i)}$ and ask how they can be defined in a relativistic framework in an operative sense. Here, we would like to discuss three cases, where the spatial orientation is given 1. by gravity, 2. by

† for a detailed discussion of the subject the reader is referred to *(Soffel et al. 1986a)*

A. K. Babcock and G. A. Wilkins (eds.), The Earth's Rotation and Reference Frames for Geodesy and Geodynamics, 99–103.

light rays originating from distant fixed stars or extragalactic radio sources and 3. by gyroscopes.

2. ORIENTATION BY GRAVITY

Let us restrict our discussion to an accuracy of 10^{-11} g $\simeq 10^{-8}$ gal, a number that agrees roughly with the capability of modern superconducting gravimeters.

The PPN-metric *(Will (1981))* (parameters β and γ only) for the Earth can then in geocentric corotating coordinates be written as:

$$g_{00} = -1 + 2U/c^2 - 2\beta U^2/c^4 + \Omega^2(X^2+Y^2)/c^2$$
$$(g_{0i}) = \mathbf{L} - \mathbf{r} \times \boldsymbol{\Omega}/c$$
$$g_{ij} = (1 + 2\gamma U/c^2)\delta_{ij}$$

with

$$\mathbf{L} = -(\gamma+1)\frac{G\mathbf{J}_\oplus \times \mathbf{r}}{c^3 r^3},$$

where the β-term describes the gravitational redshift of second order and $\mathbf{L}$ the gravito - magnetic (Lense - Thirring) field of the Earth. From this we see that the frequency ratio of two atomic clocks fixed to the surface of the rotating Earth is given by:

$$\frac{f_1}{f_2} = \frac{\sqrt{-g_{00}}|_2}{\sqrt{-g_{00}}|_1} = \frac{1 - U_2^*/c^2}{1 - U_1^*/c^2}$$

with

$$U^* = U + \frac{1}{2}\Omega^2(X^2+Y^2) - \frac{(2\beta-1)}{2}\frac{U^2}{c^2}.$$

U^* can be considered as a relativistic version of the usual geopotential and we can define that U^* surface (a constant redshift surface) as u-geoid† that lies closest to mean sea level. In principle the realizability of such a geoid is given by the long - term stability s_{LT} of atomic clocks; the uncertainty in height is roughly given by $\Delta h \sim (s_{LT}/10^{-14}) \cdot 100\,\mathrm{m}$. Presently, $s_{LT} \sim 10^{-14}$; suggestions have already been made *(Knowles et al. 1982)* for a network of clocks for VLBI purposes with phase locking using a geostationary satellite link with $s_{LT} < 10^{-15}$.

An alternative way of defining a relativistic geoid is given by gravimetric measurements. Neglecting tidally induced deformations of the Earth that can be described in the usual Newtonian way by Love numbers, the gravimeter's test mass essentially maintains a constant "proper distance" from the Earth by a four - acceleration $\boldsymbol{a}$ ($a^\mu = Du^\mu/D\tau$). This four - acceleration can then be used to define the direction of the plumbline $e_{(a)} = -\boldsymbol{a}/|\boldsymbol{a}|$ as spatial reference direction and corresponding a-level surfaces $\tilde{U}$. If $\boldsymbol{b}$ denotes a vector in $\tilde{U}$, we require that $\boldsymbol{b}$ lies in the restspace of a corotating observer and is orthogonal to the acceleration vector, i.e.

$$(\boldsymbol{b}, \boldsymbol{u}) = (\boldsymbol{b}, \boldsymbol{a}) = 0.$$

Using these properties one finds that the a-level surfaces $\tilde{U}$ agree with the U^* surfaces of constant redshift for any stationary metric. Hence, it is not necessary to distinguish between the two classes of level surfaces.

† u referring to the four - velocity $\boldsymbol{u} = \partial/\partial\tau$ of the atomic clocks.

At our level of accuracy there are no relativistic contributions to the Newtonian tidal forces. One finds that the Newtonian expression for the Earth's gravity is modified by i) a U^2 term according to the structure of the post-Newtonian geopotential and ii) a term proportional to $(3 - 4\beta + \gamma) \sum_a m_a / c^2 r_{\oplus a}$ where the sum extends over all bodies of the solar system except for the Earth (see also *Nordtvedt 1971; Will 1971,1981)*. The latter term is related to a violation of the strong equivalence principle (Nordtvedt effect) and due to the eccentricity of the Earth's orbit has a periodic contribution with period of a year and amplitude of $\sim 2 \cdot 10^{-10}$ g.

3. ORIENTATION BY FIXED STARS

Let us consider the solar system as being isolated -i.e. space-time approaches flat Minkowski space 'far away' from the solar system but still in a regime where effects from Hubble expansion are not appreciable *(Soffel et al. 1985)*. In that case the notion of fixed stars and light rays originating from one and the same fixed - star can be defined rigourously in a coordinate independent manner. This can elegantly be done by means of a conformal transformation that maps the various asymptotic ('far away') parts of space-time into a finite location . Past null infinity *(e.g. Misner et al. 1973)* where all light rays must originate in the distant past can then be identified with the celestial sphere (folded with time) and light rays are considered to originate from a definite point on the celestial sphere if their tangent vectors are parallel close to past null infinity $\mathcal{J}^-$. This parallelity of two null vectors close to $\mathcal{J}^-$ can be ensured if they have identical components w.r.t. some asymptotically cartesian and centered (e.g. at rest w.r.t. the barycenter) coordinate system (for more details on this stellar compass see *Soffel et al. 1986b)*.

As simple application we consider the light deflection in the gravitational field of the Sun. Let k denote the 4-momentum of a light ray originating from the asymptotic regime (no parallax and proper motion) and $e_{(i)}$ the spatial axes defining the astronomical coordinates (α, δ) of the stellar image. Neglecting effects from aberration we can then obtain the directional cosines of the stellar image as:

$$\bar{m}_i \equiv -(e_{(i)}, \frac{\bar{k}}{|k|}) \quad ; \quad \bar{m}_x = \cos\bar{\delta}\cos\bar{\alpha}, \quad \text{etc.}$$

where $\bar{k}$ is obtained from k by projection into the spacelike hypersurface (3-space) of the observer:

$$\bar{k}^\mu = P^\mu_\lambda \quad ; \quad P^\mu_\lambda = \delta^\mu_\lambda + u_\lambda u^\mu$$

From this we recover the formulas (e.g. *Kaplan 1981)*:

$$\bar{\alpha} - \alpha \simeq \frac{\delta\theta}{\sin\chi} \sec\delta \cos\delta_\odot \sin(\alpha - \alpha_\odot)$$
$$\bar{\delta} - \delta \simeq \frac{\delta\theta}{\sin\chi} \{\sin\delta\cos\delta_\odot\cos(\alpha - \alpha_\odot) - \sin\delta_\odot\cos\delta\}$$

with

$$\frac{\delta\theta}{\sin\chi} = \frac{2Gm_\odot}{c^2 r_{\odot\oplus}\sin^2\chi}(1 + \cos\chi) \quad ; \quad \frac{2Gm_\odot}{c^2 r_{\odot\oplus}} \simeq 4 \cdot 10^{-3}{}''$$

$$\cos\chi = \sin\delta\sin\delta_\odot + \cos\delta\cos\delta_\odot\cos(\alpha - \alpha_\odot)$$

Here, (α, δ) might be thought of representing the astronomical coordinates of the stellar image in the asymptotic regime that are independent of the solar position in the annual cycle. It should be stressed that our formulation of the stellar compass is generally applicable for astrometric problems and not just for this simple example.

4. ORIENTATION BY GYROSCOPES

The spin axis of a gyroscope can be used as spatial reference direction as long as the torques acting on the gyro are known to sufficient accuracy. For vanishing external torques a spin 4-vector S can be assumed to obey the usual Fermi - Walker transport law and to define a local inertial axis. For stationary fields fixed star oriented axes can be introduced w.r.t. which the inertial axes precess with angular velocity $\boldsymbol{\Omega}$ (e.g. *Misner et al. 1973)*:

$$\boldsymbol{\Omega} = -\frac{1}{2c^2}(\mathbf{v} \times \mathbf{a}) - \frac{c}{2}\nabla \times \mathbf{L} + (\gamma + \frac{1}{2})\mathbf{v} \times \nabla U/c^2$$

The last term leads to the well known geodetic precession of the Earth's rotation axis about the normal to the ecliptic by 2" per century. Let us now consider a special type of gyroscope namely a laser gyroscope of the Sagnac type. In such an instrument we find two laser beams travelling around a closed circuit in opposite sense and some means to extract an interference pattern from the system. This interference pattern will change if the system's state of rotation changes w.r.t. local inertial axes and in this sense may serve for the realization of such axes. One can then employ Maxwell's equations in the eikonal approximation to show that the Sagnac phase shift Φ is given by *(Scully et al. 1981)*:

$$\Delta\Phi \simeq \frac{4\pi}{\lambda}\mathbf{A} \cdot (\nabla \times (g_{0i}))$$

where λ is the mean wavelength of the laser beams, $\mathbf{A}$ denotes the 3-vector of oriented area enclosed by the laser beams and g_{0i} refers to axes at rest in the Sagnac platform. For a laser gyro on the Earth's surface the curl of (g_{0i}) to post-Newtonian order is given by:

$$\begin{aligned}(\nabla \times (g_{0i})) &= 2\,\frac{\boldsymbol{\Omega}_T}{c} = \frac{2}{c}\,[\boldsymbol{\Omega}_\oplus - \boldsymbol{\Omega}] \\ &= \frac{2}{c}\,[\boldsymbol{\Omega}_\oplus + \frac{c}{2}\nabla \times \mathbf{L} - \gamma\mathbf{v} \times \nabla U/c^2]\end{aligned}$$

Here, the first term just gives the classical Sagnac result. The next two terms indicate the influence of the Lense - Thirring effect and the geodetic precession (corrected for the Thomas term) upon the Sagnac phase shift; they are roughly 10^{-10} times smaller than the classical term. Presently, various laboratories plan or already develop such Sagnac devices that seem to be ideally suited for geodetic purposes. E.g. the passive resonant ring laser gyroscope developed at Seiler Research Laboratory (USAF, Colorado Springs) *(Rotge et al. 1985)* has an area of $\sim 60\,\mathrm{m}^2$ and is expected to reach a sensitivity of $\Delta\Omega/\Omega \sim 10^{-10} \sim 10\,\mu\mathrm{sec/day}$ with averaging times of a few minutes (VLBI: $\Delta\Omega/\Omega \sim 5 \cdot 10^{-10}$; averaging time $\sim$ 1 day). So perhaps in the future monitoring UT1 variations and polar motion will be based upon local inertial measurements instead of observing electromagnetic signals from objects at cosmic distance.

* Research supported in part by the Deutsche Forschungsgemeinschaft (DFG), SFB78

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DISCUSSION

J. Hughes: Is the 60 m^2 based on fiber optics?

Reply by Soffel: No, they use an isolation-pad of 25 × 25 feet as the base for their system (The material is crushed granite aggregate and weights 200,000 kg.)