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TENSOR PRODUCTS OF HOLOMORPHIC DISCRETE SERIES REPRESENTATIONS

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1. Introduction. We discuss the decomposition of tensor products of holomorphic discrete series representations, generalizing a technique used in [**9**] for representations of $SL_2(\mathbf{R})$, based on a suggestion of Roger Howe. In the case of two representations with highest weights, the discussion is entirely algebraic, and is best formulated in the context of generalized Verma modules (see § 3). In the case when one representation has a highest weight and the other a lowest weight, the approach is more analytic, relying on the realization of these representations on certain spaces of holomorphic functions.

For a simple group, these two cases exhaust the possibilities; for a non-simple group, one has to piece together representations on the various factors.

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2. Notation and preliminaries. We use the notation and conventions of [**3**], [**4**], [**5**], and [**6**]. Let $\mathfrak{g}_0$ be a non-compact simple real Lie algebra. Define $\mathfrak{k}_0$ and $\mathfrak{p}_0$ as in [**3**], § 2, and let $\mathfrak{h}_0$ be a maximal abelian subalgebra of $\mathfrak{k}_0$, which we assume is also maximal abelian in $\mathfrak{g}_0$. Complexify $\mathfrak{g}_0$, $\mathfrak{h}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ to $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{p}$, respectively. Choose a system Σ of positive roots and let Σ_+ be the set of non-compact positive roots. We shall assume that the non-compact positive roots are *totally positive*, in the terminology of Harish-Chandra (see [**5**]; § 3, § 4, § 5); this implies that $\mathfrak{k}_0$ has non-trivial centre (see [**5**]; corollaries to Lemma 13). Let 2ρ denote the sum of all the positive roots, and for each root α, choose X_α in the corresponding root space, such that $\alpha(H_\alpha) = 2$, where $H_\alpha = [X_\alpha, X_{-\alpha}]$.

Let $\mathfrak{p}_+ = \sum_{\gamma\in\Sigma_+}\mathbf{C}X_\gamma$, $\mathfrak{p}_- = \sum_{\gamma\in\Sigma_+}\mathbf{C}X_{-\gamma}$, and let $\mathfrak{P}_+$, $\mathfrak{P}_-$ denote their respective universal enveloping algebras. Let $\mathfrak{b} = \mathfrak{k} \oplus \mathfrak{p}_+$, a subalgebra of $\mathfrak{g}$.

Let G be the simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, let G_0 be the (real) semisimple Lie subgroup with Lie algebra $\mathfrak{g}_0$, and let K_0 be the (compact) subgroup with Lie algebra $\mathfrak{k}_0$.

Suppose $\Lambda \in \mathfrak{h}^*$ satisfies:

(1) $\quad \Lambda(H_\alpha)$ is a nonnegative integer, for all $\alpha \in \Sigma, \alpha \notin \Sigma_+$

(2) $\quad (\Lambda + \rho)(H_\gamma) < 0, \quad$ for all $\gamma \in \Sigma_+$.

Suppose too that L is an irreducible representation of K_0 on a Hilbert space V,

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which has highest weight Λ. Then (see [**6**]; [**5**], Theorems 2 and 3), Harish-Chandra constructs a (unique) unitary irreducible representation of G_0 so that the corresponding representation of $\mathfrak{g}$ has highest weight Λ. This representation (and the corresponding Lie algebra representation) will be denoted π_Λ, and called a *holomorphic discrete series representation.*

We extend L from a representation of $\mathfrak{k}$ to one of $\mathfrak{b}$ by letting $\mathfrak{p}_+$ act trivially. Then (see [**10**]), as a representation of the universal enveloping algebra of $\mathfrak{g}$, π_Λ is a *generalized Verma module*, the representation algebraically induced from the representation L of the universal enveloping algebra of $\mathfrak{b}$. In particular (see [**2**]), the space on which π_Λ acts is isomorphic to $\mathfrak{P}_- \otimes V$.

Let $\gamma_1, \ldots, \gamma_q$ be the distinct non-compact roots, let $\Lambda = \Lambda_0, \ldots, \Lambda_r$ be the distinct weights of V, and μ_i the multiplicity of Λ_i in V. Then the weights of π_Λ are all of the form $\lambda = \Lambda_i - (m_1\gamma_1 + \ldots + m_q\gamma_q)$, where the m_i are nonnegative integers. And by the Corollary to Lemma 21, [**5**], the multiplicity of a weight λ in π_Λ is given by

$$n_\Lambda(\lambda) - \textstyle\sum_0^r \mu_i n_i(\lambda),$$

where $n_i(\lambda)$ is the number of distinct sequences $(m_1, \ldots, m_q)$ of nonnegative integers such that $\lambda = \Lambda_i - (m_1\gamma_1 + \ldots + m_q\gamma_q)$. In particular, $n_\Lambda(\Lambda) = 1$.

We shall also need a simple fact about holomorphic functions:

LEMMA 2.1. *Let Ω be a connected open subset of $\mathbf{C}^n$, V a finite dimensional complex vector space. Suppose $F : \Omega \times \Omega \to V$ is holomorphic in the first argument, antiholomorphic in the second. If the restriction of F to the diagonal in $\Omega \times \Omega$ is identically zero, then $F \equiv 0$.*

Proof. The proof is by an induction argument on n. We may assume there is a polydisc P, centred at the origin, with $P \subseteq \Omega$. Fix $z_1, z_2, \ldots, z_{n-1}$ such that for small z_n, $(z_1, \ldots, z_{n-1}, z_n) \in P$. For any real θ, consider the function $g(z) = F(z_1, \ldots, z_{n-1}, z; z_1, \ldots, z_{n-1}, e^{i\theta}\bar{z})$. It is holomorphic and vanishes on the line $z = e^{i\theta}\bar{z}$; hence $g \equiv 0$. Thus for small z_n, $F(z_1, \ldots, z_n; z_1, \ldots, z_{n-1}, z)$ vanishes whenever $|z| = |z_n|$, hence vanishes for all z.

We have shown that F vanishes, not just on the diagonal in $\Omega \times \Omega$, but whenever the first $n - 1$ variables agree. This proof of the inductive step also works for $n = 1$, and the lemma is proved.

3. Two representations with highest weights. Suppose Λ, Λ' satisfy (1) and (2) of § 2; let L, L' be irreducible representations of K_0 on V, V', with highest weights Λ, Λ', respectively, and let π_Λ, $\pi_{\Lambda'}$ be the corresponding representations of G_0. We consider $\pi_\Lambda \otimes \pi_{\Lambda'}$ and will show that it is a direct sum of representations $\pi_{\Lambda''}$. Indeed, let Λ_i, μ, $0 \leqq i \leqq r$, be the weights of L and their multiplicities; Λ_i', μ_i', $0 \leqq i \leqq r'$, the weights and multiplicities for L'; and let Λ_i'', μ_i'', $0 \leqq i \leqq s$, be the weights of $L \otimes L'$ and their multiplicities. Then the weights of $\pi_\Lambda \otimes \pi_{\Lambda'}$ are sums of weights of π_Λ and $\pi_{\Lambda'}$, and are all of the form $\nu = \Lambda_i'' - (m_1\gamma_1 + \ldots + m_q\gamma_q)$, for some nonnegative integers m_i.

It is easily seen that we may calculate $n''(\nu)$, the multiplicity with which ν occurs in $\pi_\Lambda \otimes \pi_{\Lambda'}$, in either of two different ways. Indeed,

$$n''(\nu) = \sum_{\lambda+\lambda'=\nu} n_\Lambda(\lambda)\cdot n_{\Lambda'}(\lambda').$$

Also $n''(\nu) = \sum_0^s \mu_i''\cdot n_i''(\nu)$, where $n_i''(\nu)$ is the number of distinct sequences $(m_1, \ldots, m_q; m_1', \ldots, m_q')$ such that $\nu = \Lambda_i'' - ((m_1 + m_1')\gamma_1 + \ldots + (m_q + m_q')\gamma_q)$. In particular, $n''(\Lambda + \Lambda') = 1$.

THEOREM 1. *$\pi_\Lambda \otimes \pi_{\Lambda'}$ is a direct sum of representations of the form $\pi_{\Lambda''}$, with finite multiplicities. The Λ'' which occur are all of the form $\Lambda'' = \Lambda_i'' - (m_1\gamma_1 + \ldots + m_q\gamma_q)$, where Λ_i'' $(0 \leqq i \leqq s)$ is a weight of $L \otimes L'$, and m_i are nonnegative integers. The multiplicity $M_{\Lambda''}$, of $\pi_{\Lambda''}$ is given by the inductive formula*

$$M_{\Lambda''} = n''(\Lambda'') - \sum_{\lambda\neq\Lambda''} M_\lambda n_\lambda(\Lambda'').$$

Note that only finitely many terms in the sum are non-zero. Indeed if $\{\Lambda_i^0\}$ are the weights of the representation of K_0 with highest weight λ, then $n_\lambda(\Lambda'') = 0$ unless $\Lambda'' = \Lambda_i^0 - \gamma$, for some $\gamma = m_1\gamma_1 + \ldots + m_q\gamma_q$, with m_i nonnegative integers, and $M_\lambda = 0$ unless $\lambda = \Lambda_{j'}'' - \gamma'$ for some γ' of the same form.

In particular, $\pi_{\Lambda+\Lambda'}$ occurs with multiplicity one.

Proof. We have already remarked that the weights of $\pi_\Lambda \otimes \pi_{\Lambda'}$ are all of the stated form. Consequently, we may apply Lemma 4.4 of [**2**] to conclude that (as a representation of the universal enveloping algebra) $\pi_\Lambda \otimes \pi_{\Lambda'}$ has an infinite composition series $0 = X_0 \subset X_1 \subset X_2 \subset \ldots$ whose union is the whole space and such that X_{i+1}/X_i is a representation of the form $\pi_{\Lambda''}$, for some Λ'' (which must, of course, be a weight of $\pi_\Lambda \otimes \pi_{\Lambda'}$). But since we are concerned with a *unitary* representation of G_0, X_{i+1}/X_i is (isomorphic to) a $\mathfrak{g}$-invariant subspace (the orthocomplement of X_i in X_{i+1}), and the composition series is actually a direct sum.

All that remains is the formula for $M_{\Lambda''}$. But the multiplicity of $\pi_{\Lambda''}$ is the number of times Λ'' occurs as a highest weight, which is the number of times it occurs in $\pi_\Lambda \otimes \pi_{\Lambda'}$ less the number of times it occurs as a weight of some irreducible subrepresentation other than $\pi_{\Lambda''}$. That is exactly what the formula says.

Remark. This result can be proved directly, without appeal to generalized Verma module theory. We notice that the tensor product has $\Lambda + \Lambda'$ as a highest weight, so must contain a copy of $\pi_{\Lambda+\Lambda'}$; then find a highest weight Λ'' in the complementary subspace and conclude that $\pi_{\Lambda''}$ occurs. Then just continue inductively, removing representations and finding highest weights in

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what is left. Of course, this is also essentially the same as the argument which is used to prove the above-cited lemma.

4. Holomorphic and anti-holomorphic discrete series. Now, as before, let G be the simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, let G_0 and K_0 be the (real) Lie subgroups with Lie algebras $\mathfrak{g}_0$ and $\mathfrak{k}_0$; also let P_+, P_-, and K be the subgroups corresponding to $\mathfrak{p}_+$, $\mathfrak{p}_-$, and $\mathfrak{k}$, respectively.

In [**6**], § 2, Harish-Chandra shows that the product map $P_- \times K \times P_+ \to P_-\cdot K\cdot P_+$ is an isomorphism onto an open subset of G which contains G_0, and that $P_-\cdot K\cdot G_0$ is open in G, with

$$(4.1) \quad P_-\cdot K \cap G_0 = K_0.$$

Using this, he then shows that there is a real analytic isomorphism φ from $K_0\backslash G_0$ onto an open subset of $\mathfrak{p}_+$, and this can be used to describe a complex structure on $K_0\backslash G_0$, with respect to which $K_0\backslash G_0$ is a bounded domain (see [**6**], § 2, § 3; [**7**], Lemma 21; or [**1**], Theorem 1). Following [**1**], we let $\Omega = \varphi(K_0\backslash G_0)$ and $W = P_-\cdot K\cdot G_0$.

Now suppose $\Lambda \in \mathfrak{h}^*$ satisfies conditions (1) and (2) of § 2. Suppose too that there is an irreducible representation L of K_0 on a Hilbert space V, such that the corresponding representation of $\mathfrak{k}$ or K (also denoted L) has highest weight Λ.

We shall now present Harish-Chandra's construction of the irreducible representation π_Λ of G_0, alluded to in § 2. We shall follow the treatment of [**1**]. Indeed, let $\mathscr{H}_L$ (respectively $\mathscr{H}_{L,\mathrm{hol}}$) be the space of measurable (resp. holomorphic) functions $f : W \to V$ such that

(i) $f(pkw) = L(k)f(w), \quad \text{for } p \in P_-, \quad k \in K, \quad w \in W$

(ii) $||f||^2 = \int_{G_0} ||f(g)||^2 dg < \infty.$

With respect to the norm defined in (ii), $\mathscr{H}_L$ (resp. $\mathscr{H}_{L,\mathrm{hol}}$) is a Hilbert space on which G_0 acts unitarily by right translation. The action of G_0 on $\mathscr{H}_{L,\mathrm{hol}}$ is π_Λ, and by (4.1), the action of G_0 on $\mathscr{H}_L$ is isomorphic to $\mathrm{Ind}_{K_0}{}^{G_0}L$.

However, for our purposes, another realization is also important (see [**1**]). Noting that $\exp \Omega$ is a complex analytic section of W over $P_-\cdot K$, for $f \in \mathscr{H}_L$ or $\mathscr{H}_{L,\mathrm{hol}}$ and $X \in \Omega$, we let $f^\sim(X) = f(\exp X)$. We shall see that the map $f \mapsto f^\sim$ is an isomorphism from $\mathscr{H}_L$ (resp. $\mathscr{H}_{L,\mathrm{hol}}$) onto a space $\mathscr{H}_L{}^\sim$ (resp. $\mathscr{H}_{L,\mathrm{hol}}{}^\sim$) of measurable (resp. holomorphic) functions $f^\sim : \Omega \to V$.

If $X \in \Omega$, then $\exp X \in W = P_-\cdot K\cdot G_0$. We let $\exp X = p(X)k(X)g(X)$, with $p(X) \in P_-, \quad k(X) \in K, \quad g(X) \in G_0$. We also use the analytic isomorphism $X \mapsto g(X)$ to transport Haar measure $d\dot{g}$ on $K_0\backslash G_0$ to a measure $d\mu(X)$ on Ω. Choosing the norm on V so that K_0 acts unitarily, we find that for $f \in \mathscr{H}_L$ or $\mathscr{H}_{L,\mathrm{hol}}$,

$$\begin{aligned}||f||^2 &= \int_{G_0} ||f(g)||^2 dg = \int_{K_0\backslash G_0} ||f(g)||^2 d\dot{g} \\ &= \int_\Omega ||f(g(X))||^2 d\mu(X) \\ &= \int_\Omega ||L(k(X)^{-1})f(p(X)k(X)g(X))||^2 d\mu(X) \\ &= \int_\Omega ||f^\sim(X)||^2_{L,X} d\mu(X),\end{aligned}$$

where, for $X \in \Omega$, we define a new norm on V by $||v||_{L,X} = ||L(k(X)^{-1})v||$.

By (4.1) and (i) above, we see that elements of $\mathscr{H}_L$ or $\mathscr{H}_{L,\text{hol}}$ can be regarded as functions from $K_0\backslash G_0$ to V. Using the exponential map, we regard them as functions from Ω to V. The above calculation shows how to calculate the norm of such a function; the norms $||\cdot||_{L,X}$ on V must be introduced because K does not act unitarily on V, even though K_0 does. We let $\mathscr{H}_L^\sim$ (resp. $\mathscr{H}_{L,\text{hol}}^\sim$) be the Hilbert space of all measurable (resp. holomorphic) functions $f^\sim : \Omega \to V$, with the norm defined by the final integral above.

Now Ω is a bounded subset of a finite dimensional complex vector space. The coordinate functions correspond to the non-compact positive roots. We note (see [**1**], remarque 1 and proof of théorème 2) that the (holomorphic) polynomial functions from Ω to V are contained in $\mathscr{H}_L^\sim (\mathscr{H}_{L,\text{hol}}^\sim)$, and of course are dense. In fact, because of the above observation about the coordinate functions, the holomorphic polynomials are just the K_0-finite vectors in $\mathscr{H}_{L,\text{hol}}^\sim$; the set of all holomorphic polynomial functions is in an obvious way isomorphic to $\mathfrak{P}_- \otimes V$, which as we have already remarked is the space of the corresponding generalized Verma module.

If $\Lambda' \in \mathfrak{h}^*$ is such that $-\Lambda'$ satisfies conditions (1) and (2) of § 2, and if L' is a representation of K_0 on V' with lowest weight Λ', then in the same way we construct a representation $\pi_{\Lambda'}$ with the lowest weight Λ'. Indeed, let $\mathscr{H}_{L',\text{antihol}}$ (resp. $\mathscr{H}_{L',\text{antihol}}^\sim$) be the Hilbert space of anti-holomorphic functions $f : W \to V'$ (resp. $f^\sim : \Omega \to V'$) with norm defined analogously to the norm for $\mathscr{H}_{L,\text{hol}}$ (resp. $\mathscr{H}_{L,\text{hol}}^\sim$). The action of G_0 by right translation on $\mathscr{H}_{L',\text{antihol}}$ is unitary and irreducible, with lowest weight Λ'.

The representation $\pi_\Lambda \otimes \pi_{\Lambda'}$ is realized on $\mathscr{H}_{L,\text{hol}} \otimes \mathscr{H}_{L',\text{antihol}}$ or $\mathscr{H}_{L,\text{hol}}^\sim \otimes \mathscr{H}_{L',\text{antihol}}^\sim$. The latter space consists of all functions $F : \Omega \times \Omega \to V \otimes V'$ which are holomorphic in the first variable, anti-holomorphic in the second, and such that

$$\iint_{\Omega\times\Omega} ||F(X, X')||^2_{L\otimes L',(X,X')} d\mu(X) d\mu(X') < \infty$$

where the norms on $V \otimes V'$ are defined by letting

$$||v \otimes v'||_{L\otimes L',(X,X')} = ||v||_{L,X} \cdot ||v'||_{L',X'}.$$

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This space contains all polynomial maps which are holomorphic and anti-holomorphic in the first and second variables, respectively.

Every $F \in \mathscr{H}_{L,\text{hol}}{}^{\sim} \otimes \mathscr{H}_{L',\text{antihol}}{}^{\sim}$ is a continuous function, so we may let $\mathbf{R}F : \Omega \to V \otimes V'$ denote the restriction of F to the diagonal of $\Omega \times \Omega$ (which is isomorphic to Ω); i.e. $\mathbf{R}F(X) = F(X, X)$. The image under $\mathbf{R}$ of a polynomial function is contained in $\mathscr{H}_{L\otimes L'}{}^{\sim}$, and we see that we have a densely defined map

$$\mathbf{R} : \mathscr{H}_{L,\text{hol}}{}^{\sim} \otimes \mathscr{H}_{L',\text{antihol}}{}^{\sim} \to \mathscr{H}_{L\otimes L'}{}^{\sim}.$$

PROPOSITION 4.1. *The densely defined map* $\mathbf{R}$ *is closed, with dense image and trivial kernel.*

Proof. L^2-convergence of holomorphic functions implies pointwise convergence, so a convergent sequence must converge (pointwise) on the diagonal. Consequently, if the restrictions to the diagonal converge in $\mathscr{H}_{L\otimes L'}{}^{\sim}$, the limit must in fact be the restriction of the limit function. This shows that $\mathbf{R}$ is closed.

That the kernel is trivial is an obvious consequence of Lemma 2.1.

The image of $\mathbf{R}$ contains all polynomial functions; since Ω is a bounded domain, the Stone–Weierstrass Theorem says these functions are sup-norm dense in $C_c(\Omega, V \otimes V')$. But $C_c(\Omega, V \otimes V')$ is dense in $\mathscr{H}_{L\otimes L'}{}^{\sim}$, and sup-norm convergence of bounded functions implies convergence in $\mathscr{H}_{L\otimes L'}{}^{\sim}$, using dominated convergence and the fact that the constant functions are in $\mathscr{H}_{L\otimes L'}{}^{\sim}$. This concludes the proof.

THEOREM 2. *If* Λ, $-\Lambda'$ *satisfy* (1) *and* (2) *of* § 2, *and* L (*resp.* L') *is an irreducible representation of* K_0 *with highest weight* Λ (*resp. lowest weight* Λ'), *then, as representations of* G_0,

$$\pi_\Lambda \otimes \pi_{\Lambda'} \approx \operatorname{Ind}_{K_0}{}^{G_0} L \otimes L'.$$

Proof. By Schur's Lemma and Proposition 4.1, the tensor product is unitarily equivalent to the action of G_0 on $\mathscr{H}_{L\otimes L'}{}^{\sim}$, and this representation is isomorphic to $\mathscr{H}_{L\otimes L'}$, for which the action of G_0 by right translation is indeed $\operatorname{Ind}_{K_0}{}^{G_0} L \otimes L'$.

5. Examples. In the case of $SL_2(\mathbf{R})$, the above results are especially easy to describe. In fact, they have already been dealt with, in a more ad hoc manner, in [**9**]. Here $K_0 = SO(2)$ is abelian; it is the circle group. We denote by χ_n the nth character of K_0:

$$\chi_n : \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \mapsto e^{in\theta}.$$

Following Lang ([**8**]), for $n \geqq 2$ (resp. $n \leqq -2$) we let T_n denote the discrete series representation with lowest (resp. highest) weight n (corresponding to the character χ_n). The weights of T_n are n, $n+2$, $n+4, \ldots$ (resp. n, $n-2$,

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$n-4, \ldots$), each occurring with multiplicity one. So if $m, n \geqq 2$, the weights of $T_\Lambda \otimes T_n$ are $m+n, m+n+2, \ldots, m+n+2k, \ldots$, where $m+n+2k$ occurs with multiplicity $k+1$. Consequently, by Theorem 1,

$$T_m \otimes T_n \approx T_{m+n} \oplus T_{m+n+2} \oplus T_{m+n+4} \oplus \ldots = \bigoplus_{k=0}^{\infty} T_{m+n+2k} \quad \text{and}$$
$$T_{-m} \otimes T_{-n} \approx \bigoplus_{k=0}^{\infty} T_{-m-n-2k}.$$

On the other hand, χ_n is itself the irreducible representation of K_0 with highest (and lowest) weight n, so for $m, n \geqq 2$, by Theorem 2,

$$T_m \otimes T_{-n} \approx \operatorname{Ind}_{K_0}{}^{G_0}\chi_m \otimes \chi_{-n} = \operatorname{Ind}_{K_0}{}^{G_0}\chi_{m-n}.$$

This induced representation is easily decomposed using Frobenius Reciprocity; it contains one copy of the direct integral of the principal series representations of the appropriate parity (the same as the parity of $m-n$), and (at most) finitely many discrete series representations, namely those which contain the weight $m-n$. Each such component occurs with multiplicity one. See [**9**] for details, and for the connection with holomorphic functions on the upper half-plane. It is also proved there that the same results (i.e. same formulae) carry over to the "limits of discrete series" or "mock discrete series," i.e. when n and/or m is allowed to equal 1.

The other example we pursue is that of $Sp(2, \mathbf{R})$. Here $K_0 \approx U(2)$, and the (compact) Cartan subalgebra $\mathfrak{h}$ consists of all matrices of the form

$$\begin{bmatrix} 0 & \operatorname{diag}(t_1, t_2) \\ \operatorname{diag}(-t_1, -t_2) & 0 \end{bmatrix}$$

As simple positive roots we choose $\alpha = t_1 - t_2$ and $\gamma = 2 \cdot t_2$. The positive roots are $\alpha, \gamma, \gamma+\alpha, \gamma+2\alpha$, of which α is compact and the others are non-compact. We find that $\rho = 2\alpha + 3/2\gamma$.

If we write $\Lambda = m\alpha + n\gamma$, with $m, 2n \in \mathbf{Z}$, then condition (1) of § 2 amounts to $m \geqq n$, and condition (2) amounts to $m < -2$, since, by (1), $m \geqq n$.

The group $U(2)$ has a natural action on $\mathbf{C}^2$; we denote by σ_p its action on the symmetric tensors of degree p, and by $\sigma_{p,q}$ the representation $\det^q \otimes \sigma_p$. The weights of $\sigma_{p,q}$ are $p(\alpha + \frac{1}{2}\gamma) + q(\alpha+\gamma) - k\alpha$, $\quad 0 \leqq k \leqq p$. If $\Lambda = m\alpha + n\gamma$ as above, then the representation of $U(2)$ with highest weight Λ is $\sigma_{2(m-n),2n-m}$. If $\Lambda' = m'\alpha + n'\gamma$ is another such weight, then we may assume $m - n \geqq m' - n'$.

Consequently, in $\pi_\Lambda \otimes \pi_{\Lambda'}$, the weight $\Lambda + \Lambda' - k\alpha$ has multiplicity $k+1$, for $0 \leqq k \leqq 2(m'-n')$, and we see that $\pi_{\Lambda+\Lambda'-k\alpha}$ occurs in the tensor product with multiplicity one when $0 \leqq k \leqq 2(m'-n')$, and not otherwise. Similarly, each representation $\pi_{\Lambda+\Lambda'-k\gamma}$ occurs in $\pi_\Lambda \otimes \pi_{\Lambda'}$ with multiplicity one. Other representations $\pi_{\Lambda+\Lambda'-k\alpha-j\gamma}$ also occur (e.g. $\pi_{\Lambda+\Lambda'-\alpha-\gamma}$ occurs with multiplicity 2) but it gets increasingly messy to calculate the multiplicities.

If $\Lambda = m\alpha + n\gamma$ and $\Lambda' = m'\alpha + n'\gamma$ are such that Λ and $-\Lambda'$ both satisfy (1) and (2), and $m, m', 2n, 2n' \in \mathbf{Z}$, then $L = \sigma_{2(m-n),2n-m}$ is a represen-

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tation of K_0 with highest weight Λ, and $L' = \sigma_{2(n'-m'),m'}$ has lowest weight Λ' (note that since $-\Lambda'$ satisfies (1), we have $2(n' - m') \geqq 0$). So, by Theorem 2,

$$\pi_\Lambda \otimes \pi_{\Lambda'} \approx \mathrm{Ind}_{K_0}{}^{G_0} L \otimes L'.$$

Let us pause to decompose $L \otimes L'$. Consider first the tensor product $\sigma_p \otimes \sigma_{p'}$, where $p \geqq p'$. By a calculation analogous to our analysis of the weights in the tensor product of two representations with highest weights, one finds that

$$\sigma_p \otimes \sigma_{p'} = \sigma_{p+p',0} \oplus \sigma_{p+p'-2,1} \oplus \sigma_{p+p'-4,2} \oplus \ldots \oplus \sigma_{p-p',p'}.$$

Putting in the determinants, with L, L' as above,

$$\begin{aligned} L \otimes L' &= \sigma_{2(m-n),2n-m} \otimes \sigma_{2(n'-m'),m'} \\ &= \det{}^{2n-m+m'} \otimes \sigma_{2(m-n)} \otimes \sigma_{2(n'-m')}. \end{aligned}$$

If, for example, $m - n \geqq n' - m'$, then

$$\begin{aligned} L \otimes L' &= \det{}^{2n-m+m'} \otimes \left(\bigoplus_{k=0}^{2(n'-m')} \sigma_{2(m-n+n'-m'-k),k}\right) \\ &= \bigoplus_{k=0}^{2(n'-m')} \sigma_{2(m-n+n'-m'-k),k+2n-m+m'}. \end{aligned}$$

So $\pi_\Lambda \otimes \pi_{\Lambda'}$ is the direct sum of the representations of G_0 induced from these irreducible representations of K_0; their structure can be analysed by Frobenius Reciprocity.

It should be clear that in principle one can always do the same thing. The decomposition of tensor products of holomorphic discrete series representations will follow easily from an analysis of irreducible representations of K_0—their weights and the decomposition of their tensor products. On the other hand, it is also clear that these calculations quickly become fairly involved.

6. The general case. Our final remark is that the same methods can be applied even if the group is not simple. It is of course possible in this case to find two representations which are both holomorphic on some of the simple factors but one of which is holomorphic and the other anti-holomorphic on some other factors. In such a situation, one would apply the above results to each of the simple factors separately.

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