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Fourier–Jacobi Type Spherical Functions for Discrete Series Representations of $Sp(2, \mathbb{R})$

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Abstract. In this paper we define a kind of generalized spherical functions on $Sp(2, \mathbb{R})$. We call it 'Fourier–Jacobi type', since it can be considered as a generalized Whittaker model associated with the Jacobi maximal parabolic subgroup. Also we give the multiplicity theorem and an explicit formula of these functions for discrete series representations of $Sp(2, \mathbb{R})$.

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1. Introduction

Let $\mathcal{H}_2$ be the Siegel upper half plane of degree 2, and

$$F(Z) = F(t, z, t'), \quad Z = \begin{pmatrix} t & z \\ z & t' \end{pmatrix} \in \mathcal{H}_2,$$

be a Siegel modular form of degree 2 and weight k with respect to $\Gamma_2 = Sp(2, \mathbb{Z})$. As is well known, $F(Z)$ admits an expansion of the form

$$F(Z) = \sum_{m=0}^{\infty} \phi_m(t, z) e^{2\pi\sqrt{-1}mt'},$$

called the Fourier–Jacobi expansion, and each coefficient function $\phi_m(t, z)$ becomes a Jacobi form of weight k and index m with respect to $\Gamma_1 = SL(2, \mathbb{Z})$. This expansion is fundamental in the excellent proof of the Saito–Kurokawa lifting by Maaß, Andrianov, and Zagier, which is a good lifting from ordinary modular forms of weight $2k - 2$ with respect to Γ_1 to Siegel modular forms of degree 2 and weight k with respect to Γ_2 (cf. [3]).

We wish to consider the Fourier–Jacobi expansion for non-holomorphic Siegel modular forms and we are interested in some correspondence between three distinct non-holomorphic automorphic forms, that is, Siegel modular forms, ordinary modular forms, and Jacobi forms. However, we have no suitable formulation of

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this expansion for non-holomorphic forms, which we regard as the one of the most fundamental tools for the study of automorphic forms. Under this motivation, we study a kind of generalized Whittaker model of irreducible admissible representations of $G = Sp(2, \mathbb{R})$ explained below.

Let $M_J A_J N_J$ be a Langlands decomposition of the Jacobi maximal parabolic subgroup P_J of G corresponding to the long root. Then the unipotent radical N_J of P_J is isomorphic to the three-dimensional Heisenberg group $H(\mathbb{R})$ and the identity component M_J° of M_J is isomorphic to $SL(2, \mathbb{R})$. Remark that M_J is the centralizer of the center of N_J in the Levi part $M_J A_J$ of P_J with respect to the conjugate action. The semidirect product group $R_J = M_J^\circ \ltimes N_J$ is not reductive and is isomorphic to the Jacobi group $SL(2, \mathbb{R}) \ltimes H(\mathbb{R})$. Now we consider the intertwining space

$$\mathcal{I}_{\rho,\pi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C}, K)}(\pi, C^\infty \mathrm{Ind}_{R_J}^G(\rho))$$

from an irreducible admissible representation π of G into the reduced generalized Gelfand–Graev representation $C^\infty \mathrm{Ind}_{R_J}^G(\rho)$ induced from an irreducible unitary representation ρ of R_J. Here $\mathfrak{g}_\mathbb{C}$ is the complexification of the Lie algebra of G and K is a maximal compact subgroup of G. These embeddings of π into $C^\infty \mathrm{Ind}_{R_J}^G(\rho)$ are a variety of generalized Whittaker models for π (cf. [18]).

We treat this intertwining space $\mathcal{I}_{\rho,\pi}$ restricting to a multiplicity one K-type (τ^*, V_{τ^*}) of π, where the asterisk means the contragradient representation. Then the restriction to τ^* induces the isomorphism from $\mathcal{I}_{\rho,\pi}$ to a subspace of the space $C^\infty_{\rho,\tau}(R_J \backslash G/K)$ of $\mathcal{F}_\rho \otimes V_\tau$-valued smooth functions on G such that

$$f(rgk) = (\rho(r) \otimes \tau(k)^{-1}) f(g), \qquad \text{for } (r, g, k) \in R_J \times G \times K.$$

Now we define *the space of Fourier–Jacobi type spherical functions* $\mathcal{J}_{\rho,\pi}(\tau)$ *of type* $(\rho, \pi; \tau)$ to be the above subspace of $C^\infty_{\rho,\tau}(R_J \backslash G/K)$. When π belongs to the holomorphic discrete series, these functions which are moderate growth on A_J appear in the coefficients of the Fourier–Jacobi expansion of (holomorphic) Siegel modular forms. Also we can consider these generalized spherical functions as the real local Whittaker–Shintani functions on $Sp(2, \mathbb{R})$ of Fourier–Jacobi type in the paper of Murase and Sugano [12; §§4,5].

Our main interests for the Fourier–Jacobi type spherical functions are twofold: one is to decide the multiplicity for these functions. The other is to obtain explicit formulas of the radial parts of these functions. In this paper we restrict these problems to the case that the representation π with the minimal K-type τ^* belongs to the discrete series and attack them using the theorem of Yamashita [19; Theorem 2.4] explained in Section 5. The theorem of Yamashita asserts that the space $\mathcal{J}_{\rho,\pi}(\tau)$ is a subspace of $C^\infty_{\rho,\tau}(R_J \backslash G/K)$ characterized by the Schmid operators. To decide this subspace explicitly, we write down the action of the Schmid operators and solve the resulting (generally infinite-dimensional) system of differential equations. The main result of this paper is the following theorem:

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MAIN THEOREM (see the theorems in Sections 6 and 7). *Let π be a discrete series representation of $Sp(2, \mathbb{R})$ and τ^* be the minimal K-type of π. For each irreducible unitary representation ρ of R_J of type m, let $\mathcal{I}_{\rho,\pi}$ and $\mathcal{J}_{\rho,\pi}(\tau)$ be as above and let $\mathcal{J}^{\circ}_{\rho,\pi}(\tau)$ be the subspace of $\mathcal{J}_{\rho,\pi}(\tau)$ consisting of all $f \in \mathcal{J}_{\rho,\pi}(\tau)$ which are moderate growth on A_J. Then we have*

$$\dim \mathcal{I}_{\rho,\pi} = \dim \mathcal{J}_{\rho,\pi}(\tau) \leqslant 3, \quad \dim \mathcal{J}^{\circ}_{\rho,\pi}(\tau) \leqslant 1.$$

Moreover, the radial parts of the functions in $\mathcal{J}^{\circ}_{\rho,\pi}(\tau)$ are expressed by Meijer's G-function $G^{3,0}_{2,3}\left(x\middle|\begin{matrix} a_1, a_2 \\ b_1, b_2, b_3\end{matrix}\right)$ or more degenerate similar functions.

Here the term 'type m' means that the stabilizer of the equivalent class of $\rho|_{N_J}$ in R_J is R_J itself. We treat the case of holomorphic and anti-holomorphic discrete series representations in Section 6, and the case of large (in the sense of Vogan [16]) ones in Section 7. For irreducible unitary representations of R_J of not type m, the Fourier–Jacobi type spherical functions are reduced essentially to the ordinary Whittaker functions on $Sp(2, \mathbb{R})$. We treat this case in Section 8.

We remark that Meijer's G-function $G^{3,0}_{2,3}\left(x\middle|\begin{matrix} a_1, a_2 \\ b_1, b_2, b_3\end{matrix}\right)$ ([11]) with special parameters a_i and b_j degenerates into the (classical) Whittaker function $W_{\kappa,\mu}(x)$, the function of the form $x^{\alpha}e^{\beta x}$, and so on (see Appendix). This fact also interests us in the study of the process of the degeneration of generalized spherical functions as special functions.

A similar problem for generalized principal series representations of G induced from the Jacobi maximal parabolic subgroup P_J (P_J-principal series representations) is treated in [7]. We will discuss other generalized principal series representations in a forthcoming paper.

This paper is half of the author's thesis [6], except for some additions and modifications. He would like to express his gratitude to Professor T. Oda for many valuable guidance.

2. Preliminaries

2.1. GROUPS AND ALGEBRAS

As usual, we denote by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of rational integers, the real number field and the complex number field, respectively, and by $\mathbb{Z}_{\geqslant m}$ the set of integers n such that $n \geqslant m$. Moreover, we use the convention throughout this paper that unwritten components of a matrix are zero. Let $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ be the space of real and complex matrices of size n, respectively. Put

$$J_2 = \begin{pmatrix} O_2 & 1_2 \\ -1_2 & O_2 \end{pmatrix} \in M_4(\mathbb{R}),$$

where $1_2 \in M_2(\mathbb{R})$ is a unit matrix. Let G be the real symplectic group $Sp(2, \mathbb{R})$ of

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degree two given by

$$Sp(2,\mathbb{R}) = \{g \in M_4(\mathbb{R}) \mid {}^t g J_2 = J_2 g^{-1}, \det g = 1\}.$$

Here ${}^t g$ means the transpose of g, $\det g$ the determinant of g, and g^{-1} the inverse of g. Let $\theta(g) = {}^t g^{-1}$ $(g \in G)$ be a Cartan involution of G. Then

$$K = \{g \in G \mid \theta(g) = g\} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \;\middle|\; A, B \in M_2(\mathbb{R}) \right\}$$

is a maximal compact subgroup of G and is isomorphic to the unitary group $U(2)$ via the homomorphism

$$K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in U(2) = \{g \in M_2(\mathbb{C}) \mid {}^t\bar{g} \cdot g = 1_2, \det g \neq 0\}.$$

Let $\mathfrak{g} = \{X \in M_4(\mathbb{R}) \mid J_2 X + {}^t X J_2 = 0\}$ be the Lie algebra of G. If we denote the differential of θ again by θ, then we have $\theta(X) = -{}^t X$ $(X \in \mathfrak{g})$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces of θ in $\mathfrak{g}$, respectively. Then

$$\mathfrak{k} = \left\{ X \in \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \;\middle|\; A, B \in M_2(\mathbb{R}), {}^t A = -A, {}^t B = B \right\},$$
$$\mathfrak{p} = \left\{ X \in \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \;\middle|\; A, B \in M_2(\mathbb{R}), {}^t A = A, {}^t B = B \right\},$$

and we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Remark that $\mathfrak{k}$ is the Lie algebra of K and is isomorphic to the unitary algebra $\mathfrak{u}(2)$ via the linear map

$$\mathfrak{k} \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in \mathfrak{u}(2) = \{X \in M_2(\mathbb{C}) \mid X + {}^t\bar{X} = 0\}.$$

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbb{C}} = \mathfrak{l} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{l}$ and by $U(\mathfrak{l}_{\mathbb{C}})$ the universal enveloping algebra of $\mathfrak{l}_{\mathbb{C}}$. Let $\{Z, H', Y, Y'\}$ be a basis of $\mathfrak{k}_{\mathbb{C}}$, where Z (resp. H', Y, Y') is an element $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}}$ with $(A, B) = (O_2, -\sqrt{-1} \cdot 1_2)$ (resp. $\left(O_2, -\sqrt{-1}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right)$, $\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, O_2\right)$, $\left(O_2, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right)$). Remark that $\{H', X, \bar{X}\}$ with $X = \frac{1}{2}(Y - \sqrt{-1}Y')$ and $\bar{X} = \frac{1}{2}(-Y - \sqrt{-1}Y')$ is an $\mathfrak{sl}_2$-triple, i.e. $[H', X] = 2X$, $[H', \bar{X}] = -2\bar{X}$, and $[X, \bar{X}] = H'$.

Put

$$T_1 = \tfrac{1}{2}\sqrt{-1}(Z + H'), \qquad T_2 = \tfrac{1}{2}\sqrt{-1}(Z - H') \quad \text{and} \quad \mathfrak{h} = \mathbb{R}T_1 \oplus \mathbb{R}T_2.$$

Then $\mathfrak{h}$ is a compact Cartan subalgebra of $\mathfrak{g}$. For a linear form $\beta\colon \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$, we write $\beta(T_i) = \sqrt{-1}\beta_i \in \mathbb{C}$ and identify β with $(\beta_1, \beta_2) \in \mathbb{C}^2$. Then the set of roots $\Delta = \Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ is given by $\Delta = \{\pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1)\}$. We fix a positive root system $\Delta^+ = \{(2,0), (0,2), (1,1), (1,-1)\}$. For each root $\beta \in \Delta$, denote the root space for β by $\mathfrak{g}_\beta$ and define a root vector X_β in $\mathfrak{g}_\beta$ by

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$$X_{(2,0)} = \left(\begin{array}{cc|cc} 1 & & \sqrt{-1} & \\ & 0 & & 0 \\ \hline \sqrt{-1} & & -1 & \\ & 0 & & 0 \end{array}\right), \quad X_{(1,1)} = \left(\begin{array}{cc|cc} & 1 & & \sqrt{-1} \\ 1 & & \sqrt{-1} & \\ \hline & \sqrt{-1} & & -1 \\ \sqrt{-1} & & -1 & \end{array}\right),$$

$$X_{(0,2)} = \left(\begin{array}{cc|cc} 0 & & 0 & \\ & 1 & & \sqrt{-1} \\ \hline 0 & & 0 & \\ & \sqrt{-1} & & -1 \end{array}\right), \quad X_{(1,-1)} = \left(\begin{array}{cc|cc} & 1 & & -\sqrt{-1} \\ -1 & & -\sqrt{-1} & \\ \hline & \sqrt{-1} & & 1 \\ \sqrt{-1} & & -1 & \end{array}\right),$$

and $X_{-\beta} = {}^t\overline{X_\beta}$ $(\beta \in \Delta^+)$. Here $\overline{X}$ means the complex conjugate of X. Put Δ_c^+ and Δ_n^+ the set of compact and non-compact positive roots, respectively. Then $\Delta_c^+ = \{(1,-1)\}$ and $\Delta_n^+ = \{(2,0),\ (0,2),\ (1,1)\}$, and we have a decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with $\mathfrak{p}_+ = \sum_{\beta\in\Delta_n^+} \mathfrak{g}_\beta$ and $\mathfrak{p}_- = \sum_{\beta\in\Delta_n^+} \mathfrak{g}_{-\beta}$. Moreover, put $\|\beta\| = \sqrt{\beta_1^2+\beta_2^2}$ for each $\beta \in \Delta$. Then the set

$$\left\{c\|\beta\|(X_\beta + X_{-\beta}),\ \frac{c}{\sqrt{-1}}\|\beta\|(X_\beta - X_{-\beta}) \quad (\beta \in \Delta_n^+)\right\} \tag{2.1}$$

forms an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form for some constant c. Here we remark that $\|\beta\|^2 = 4$ or 2.

Put $H_1 = \mathrm{diag}(1,0,-1,0)$, $H_2 = \mathrm{diag}(0,1,0,-1)$ and $\mathfrak{a}_\mathfrak{p} = \mathbb{R}H_1 \oplus \mathbb{R}H_2$. Then $\mathfrak{a}_\mathfrak{p}$ is a maximal Abelian subalgebra of $\mathfrak{p}$. Let us define $e_i \in \mathfrak{a}_\mathfrak{p}{}^*$ $(i=1,2)$ by $e_i(a_1H_1 + a_2H_2) = a_i$. Then the restricted root system $\Sigma = \Sigma(\mathfrak{a}_\mathfrak{p}, \mathfrak{g})$ of $(\mathfrak{a}_\mathfrak{p}, \mathfrak{g})$ is given by $\Sigma = \{\pm 2e_1,\ \pm 2e_2,\ e_1 \pm e_2,\ -e_1 \pm e_2\}$, and $\Sigma^+ = \{2e_1,\ 2e_2,\ e_1 \pm e_2\}$ forms a positive root system. Denote the restricted root space for each $\alpha \in \Sigma$ by $\mathfrak{g}_\alpha$ and choose a restricted root vector E_α in $\mathfrak{g}_\alpha$ as

$$E_{2e_1} = E_{1,3}, \quad E_{2e_2} = E_{2,4}, \quad E_{e_1+e_2} = E_{1,4} + E_{2,3}, \quad E_{e_1-e_2} = E_{1,2} - E_{4,3},$$

and $E_{-\alpha} = \theta E_\alpha = -{}^tE_\alpha$ $(\alpha \in \Sigma^+)$. Here $E_{i,j}$ is a matrix whose (i,j)th entry is 1 and the others are 0. If we put $\mathfrak{n}_\mathfrak{p} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$, then we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{k}$.

Set

$$\mathfrak{a}_J = \mathbb{R}H_1,\ \mathfrak{n}_J = \mathfrak{g}_{2e_1} \oplus \mathfrak{g}_{e_1+e_2} \oplus \mathfrak{g}_{e_1-e_2} \simeq \mathfrak{h}(\mathbb{R}),$$
$$\mathfrak{m}_J = \mathbb{R}H_2 \oplus \mathfrak{g}_{2e_2} \oplus \mathfrak{g}_{-2e_2} \simeq \mathfrak{sl}(2,\mathbb{R}),$$

and

$$A_J = \exp \mathfrak{a}_J, \quad N_J = \exp \mathfrak{n}_J \simeq H(\mathbb{R}),$$
$$M_J = Z_K(\mathfrak{a}_J) \exp \mathfrak{m}_J \simeq \{\pm 1\} \times SL(2,\mathbb{R}).$$

Here $Z_K(\mathfrak{a}_J)$ is the centralizer of $\mathfrak{a}_J$ in K, and $\mathfrak{h}(\mathbb{R})$ and $H(\mathbb{R})$ are the three-dimensional Heisenberg algebra and group, respectively. Remark that the center of $\mathfrak{n}_J$ (resp. N_J) equals to $\mathfrak{g}_{2e_1}$ (resp. $\exp \mathfrak{g}_{2e_1}$). If we put $P_J = M_J A_J N_J$, then P_J

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is a maximal parabolic subgroup of G corresponding to the long root $2e_2$ and the right-hand side gives its Langlands decomposition. Of course, $\mathfrak{p}_J = \mathfrak{m}_J \oplus \mathfrak{a}_J \oplus \mathfrak{n}_J$ is the Lie algebra of P_J. We call P_J (resp. $\mathfrak{p}_J$) *the Jacobi maximal parabolic subgroup* (resp. *subalgebra*) of G (resp. $\mathfrak{g}$). For every integer n, put $\mathfrak{g}_n = \{X \in \mathfrak{g} \mid [H_1, X] = nX\}$. Then

$$\mathfrak{g}_0 = \mathfrak{m}_J \oplus \mathfrak{a}_J, \quad \mathfrak{g}_1 = \mathfrak{g}_{e_1+e_2} \oplus \mathfrak{g}_{e_1-e_2}, \quad \mathfrak{g}_2 = \mathfrak{g}_{2e_1}, \quad \mathfrak{g}_{-i} = \theta\mathfrak{g}_i \ (i = 1, 2),$$

and $\mathfrak{g}_n = \{0\}$ for $n \neq 0, \pm 1, \pm 2$.

The Levi part $M_J A_J$ of P_J acts on N_J via the conjugate action. Let R_J be the semidirect product group $M_J^\circ \ltimes N_J$ with respect to this action. Here M_J° is the identity component of the centralizer M_J of the center of N_J in $M_J A_J$. Then R_J is not reductive and is isomorphic to *the Jacobi group* $SL(2, \mathbb{R}) \ltimes H(\mathbb{R})$. The Lie algebra $\mathfrak{r}_J$ of R_J is given by $\mathfrak{r}_J = \mathfrak{m}_J \oplus \mathfrak{n}_J \simeq \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{h}(\mathbb{R})$. Because of Iwasawa decomposition, we have $G = R_J A_J K$ and $\mathfrak{g} = \mathfrak{r}_J \oplus \mathfrak{a}_J \oplus \mathfrak{k}$.

The following decompositions of the root vectors X_β for each $\beta \in \Delta_n = \Delta_n^+ \cup (-\Delta_n^+)$ are required in the later sections.

LEMMA 2.1.

$$\begin{aligned}
X_{\pm(2,0)} &= \pm\tfrac{1}{2}(Z + H') + H_1 \pm 2\sqrt{-1}E_{2e_1},\\
X_{\pm(1,1)} &= (\bar{X} - X) \pm (\bar{X} + X) + 2E_{e_1-e_2} \pm 2\sqrt{-1}E_{e_1+e_2},\\
X_{\pm(0,2)} &= \pm\tfrac{1}{2}(Z - H') + H_2 \pm 2\sqrt{-1}E_{2e_2},
\end{aligned}$$

Proof. These are obtained by direct computation. □

2.2. PARAMETRIZATION OF REPRESENTATIONS OF K

Since the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is compact, $\Delta_c^+ = \{(1, -1)\}$ is a positive system of $\Delta(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and the set $\Lambda = \{\lambda = (\lambda_1, \lambda_2) \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geqslant \lambda_2\}$ parametrizes the Δ_c^+-dominant weights, and thus the equivalence classes of irreducible representations of K, as can be seen from the highest weight theory (cf. Knapp [8; Theorem 4.28]). For each $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, put $d_\lambda = \lambda_1 - \lambda_2$ and let $V_\lambda = \bigoplus_{k=0}^{k=d_\lambda} \mathbb{C}v_k^\lambda$ be a $(d_\lambda + 1)$-dimensional vector space with a basis $\{v_k^\lambda\}_{0 \leqslant k \leqslant d_\lambda}$. Now let us define the action τ_λ of $\mathfrak{k}_\mathbb{C}$ on V_λ by

$$\begin{aligned}
&\tau_\lambda(Z)v_k^\lambda = (\lambda_1 + \lambda_2)v_k^\lambda, \quad \tau_\lambda(X)v_k^\lambda = (k+1)v_{k+1}^\lambda,\\
&\tau_\lambda(H')v_k^\lambda = (2k - d_\lambda)v_k^\lambda, \quad \tau_\lambda(\bar{X})v_k^\lambda = (d_\lambda + 1 - k)v_{k-1}^\lambda,
\end{aligned}$$

for $0 \leqslant k \leqslant d_\lambda$. Here we understand $v_{-1}^\lambda = v_{d_\lambda+1}^\lambda = 0$. Then $(\tau_\lambda, V_\lambda)$ can be globalized to K. The basis $\{v_k^\lambda\}_{0 \leqslant k \leqslant d_\lambda}$ is called *the standard basis of* V_λ.

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Both of $\mathfrak{p}_\pm$ become K-modules via the adjoint representation of K, and we have isomorphisms $\mathfrak{p}_+ \simeq V_{(2,0)}$ and $\mathfrak{p}_- \simeq V_{(0,-2)}$ by the correspondence of the basis

$$(X_{(0,2)}, X_{(1,1)}, X_{(2,0)}) \longmapsto (v_0^{(2,0)}, v_1^{(2,0)}, v_2^{(2,0)}),$$
$$(X_{(-2,0)}, X_{(-1,-1)}, X_{(0,-2)}) \longmapsto (v_0^{(0,-2)}, -v_1^{(0,-2)}, v_2^{(0,-2)}),$$

respectively. For a given irreducible K-module V_λ with the parameter $\lambda \in \Lambda$, the tensor products $\mathfrak{p}_\pm \otimes V_\lambda$ have the irreducible decompositions $V_\lambda \otimes \mathfrak{p}_\pm \simeq \oplus_{\beta \in \Delta_n^+} V_{\lambda \pm \beta}$. For each $\beta \in \Delta_n^+$, let $P^{\pm\beta}\colon V_\lambda \otimes \mathfrak{p}_\pm \to V_{\lambda\pm\beta}$ be the projectors into the irreducible factors of $V_\lambda \otimes \mathfrak{p}_\pm$. In the later sections, we need the following Clebsch–Gordan coefficients.

LEMMA 2.2.

(1) *For* $(\beta, \gamma) = ((2,0),(2,0))$ *or* $((0,-2),(0,-2))$, *we have*

$$P^\beta(v_k^\lambda \otimes v_2^\gamma) = \tfrac{1}{2}(k+1)(k+2)v_{k+2}^{\lambda+\beta}, \quad 0 \leqslant k \leqslant d_\lambda$$
$$P^\beta(v_k^\lambda \otimes v_1^\gamma) = (k+1)(d_\lambda + 1 - k)v_{k+1}^{\lambda+\beta}, \quad 0 \leqslant k \leqslant d_\lambda$$
$$P^\beta(v_k^\lambda \otimes v_0^\gamma) = \tfrac{1}{2}(d_\lambda + 1 - k)(d_\lambda + 2 - k)v_k^{\lambda+\beta}, \quad 0 \leqslant k \leqslant d_\lambda.$$

(2) *For* $(\beta, \gamma) = ((1,1),(2,0))$ *or* $((-1,-1),(0,-2))$, *we have*

$$P^\beta(v_k^\lambda \otimes v_2^\gamma) = (k+1)v_{k+1}^{\lambda+\beta}, \quad 0 \leqslant k \leqslant d_\lambda - 1,$$
$$P^\beta(v_k^\lambda \otimes v_1^\gamma) = (d_\lambda - 2k)v_k^{\lambda+\beta}, \quad 0 \leqslant k \leqslant d_\lambda,$$
$$P^\beta(v_k^\lambda \otimes v_0^\gamma) = -(d_\lambda + 1 - k)v_{k-1}^{\lambda+\beta}, \quad 1 \leqslant k \leqslant d_\lambda,$$

and the others are 0.

(3) *For* $(\beta, \gamma) = ((0,2),(2,0))$ *or* $((-2,0),(0,-2))$, *we have*

$$P^\beta(v_k^\lambda \otimes v_2^\gamma) = v_k^{\lambda+\beta}, \quad 0 \leqslant k \leqslant d_\lambda - 2,$$
$$P^\beta(v_k^\lambda \otimes v_1^\gamma) = -2v_{k-1}^{\lambda+\beta}, \quad 1 \leqslant k \leqslant d_\lambda - 1,$$
$$P^\beta(v_k^\lambda \otimes v_0^\gamma) = v_{k-2}^{\lambda+\beta}, \quad 2 \leqslant k \leqslant d_\lambda,$$

and the others are 0.

Proof. Omit (cf. Oda [13; Lemma 3.1, 3.2, 3.3]). □

2.3. REPRESENTATION THEORY OF R_J

To describe the unitary dual of the Jacobi group, we recall a parametrization of that of the universal cover $\widetilde{G}_1 = \widetilde{SL}(2,\mathbb{R})$ of $G_1 = SL(2,\mathbb{R})$.

PROPOSITION 2.3. *We have the following representatives for the unitary equivalence classes of irreducible unitary representations of* $\widetilde{G}_1$.

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(1) *(unitary principal series)* $\mathcal{P}_s^\tau$, $s \in \sqrt{-1}\mathbb{R}$, $\tau = 0, 1, \pm\frac{1}{2}$ *except for the case* $(s, \tau) = (0, 1)$.
(2) *(complementary series)* $\mathcal{C}_s^\tau$, $0 < s < 1$ *for* $\tau = 0, 1$ *and* $0 < s < \frac{1}{2}$ *for* $\tau = \pm\frac{1}{2}$.
(3) *((limit of) discrete series)* $\mathcal{D}_k^\pm$, $k \in \frac{1}{2}\mathbb{Z}_{\geqslant 2}$.
(4) *(quotient representation)* $\mathcal{D}_{\frac{1}{2}}^-$, $\mathcal{D}_{\frac{1}{2}}^+$.
(5) *The trivial representation* 1_{G_1} *of* G_1.

Here only the representations $\mathcal{P}_s^\tau$ *and* $\mathcal{C}_s^\tau$ *with* $\tau = 0, 1$, $\mathcal{D}_k^\pm$ *with* $k \in \mathbb{Z}_{\geqslant 1}$, *and* 1_{G_1} *give those of* G_1.

Proof. See Gelbart [5; Lemma 4.1, 4.2]. □

For each irreducible unitary representation π, let $L = L_\pi$ be the set of indices defined by

$$L = \begin{cases} \{\tau + 2\mathbb{Z}\}, & \text{for } \pi = \mathcal{P}_s^\tau \text{ or } \mathcal{C}_s^\tau, \\ \{\pm k \pm 2\mathbb{Z}_{\geqslant 0}\}, & \text{for } \pi = \mathcal{D}_k^\pm, \\ \{0\}, & \text{for } \pi = 1_{G_1}. \end{cases}$$

Then there exists a basis $\{w_l\}_{l \in L}$ of the underlying Harish–Chandra module of π such that the action of the complexification $(\mathfrak{g}_1)_\mathbb{C}$ of the Lie algebra $\mathfrak{g}_1$ of $\widetilde{G}_1$ is given by $\pi(U)w_l = lw_l$ and $\pi(V_\pm)w_l = \frac{1}{2}(z_1 + 1 \pm l)w_{l\pm 2}$. Here $\{U, V_\pm\}$ with $U = -\sqrt{-1}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ and $V_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm\sqrt{-1} \\ \pm\sqrt{-1} & -1 \end{pmatrix}$ is a basis of $(\mathfrak{g}_1)_\mathbb{C}$, and we understand that $w_l = 0$ if $l \notin L$ and that $z_1 = s$ (resp. $k - 1$, -1) for $\pi = \mathcal{P}_s^\tau$ and $\mathcal{C}_s^\tau$ (resp. $\mathcal{D}_k^\pm$, 1_{G_1}).

Next we investigate the irreducible unitary representations of the Jacobi group (*cf.* Berndt and Schmidt [1; Chapter 2]). Let $N_J \simeq H(\mathbb{R})$ be the unipotent radical of the Jacobi maximal parabolic subgroup P_J defined in Section 2.1. Remark that the center Z of N_J is equal to $\exp \mathfrak{g}_{2e_1}$ and is also the center of R_J. For an irreducible unitary representation ν of N_J, there exist $m \in \mathbb{R}$ uniquely such that $\nu(\exp E_{2e_1}) = e^{2\pi\sqrt{-1}m}$ by Schur's lemma. Then we call such ν *of type* m. The following lemma is an immediate consequence of the well known theorem of Stone-von Neumann.

LEMMA 2.4.

(1) *Let* ν *be an irreducible unitary representation of* N_J *of type* m. *If* $m = 0$, *then* ν *is a character. On the other hand,* ν *is infinite dimensional if* $m \neq 0$.
(2) *Let* ν *and* ν' *be any two irreducible unitary representations of* N_J *of the same type* $m \neq 0$. *Then* ν *is unitary equivalent with* ν'.

Proof. See Corwin-Greenleaf [2; pp. 46–47, 51–52]. □

Fix an irreducible unitary representation $(\nu_m, \mathcal{U}_m)$ of N_J of type $m \neq 0$. Since the equivalence class of ν_m is determined by m from Lemma 2.4, the stabilizer of the equivalence class of ν_m in the Levi part $M_J A_J$ of P_J with respect to the conjugate action coincides with the centralizer of the center of N_J, i.e. M_J. From the theory of the Weil representation, ν_m can be extended to a continuous projective unitary rep-

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resentation $(\tilde{\nu}_m, \mathcal{U}_m)$ of R_J by $\tilde{\nu}_m(\tilde{n}) = W_m(g)\nu_m(n)$ for $\tilde{n} = g \cdot n \in M_J^\circ \ltimes N_J$ with the Weil representation W_m of M_J°. Here $\tilde{\nu}_m$ has a factor set α which is a proper 2-cocycle, and W_m considered as the representation of $\widetilde{G}_1$ has the irreducible decomposition $W_m = \mathcal{D}_{\frac{1}{2}}^{\mathrm{sign}(m)} \oplus \mathcal{D}_{\frac{3}{2}}^{\mathrm{sign}(m)}$.

PROPOSITION 2.5 (Satake [14; Appendix I, Proposition 2]). *Let $(\tilde{\nu}_m, \mathcal{U}_m)$ with $m \neq 0$ be as above. For every irreducible unitary representation $(\pi_1, \mathcal{W}_{\pi_1})$ of $\widetilde{M}_J^\circ$ which does not factor through M_J°, put $\rho_{\pi_1,m}(\tilde{n}) = \pi_1(g) \otimes \tilde{\nu}_m(\tilde{n})$ for $\tilde{n} = g \cdot n \in M_J^\circ \ltimes N_J$. Then $\rho_{\pi_1,m}$ is an irreducible unitary representation of R_J. Conversely, all irreducible unitary representations of R_J with non-trivial central character are obtained in this manner. Moreover $\rho_{\pi_1,m}$ is square-integrable iff π_1 is so.*

Next let us consider the irreducible unitary representation $(\rho, \mathcal{F}_\rho)$ of R_J which is trivial on the center Z. Then such ρ can be considered as a representation of $R_J/Z \simeq G_1 \ltimes \mathbb{R}^2$. Now we investigate the unitary dual of $G_1 \ltimes \mathbb{R}^2$ by Mackey's orbital analysis [10]. The unitary character $(\lambda, \mu) \longmapsto e^{2\pi\sqrt{-1}(m_1\lambda + m_2\mu)}$ of $\mathbb{R}^2$ is identified with ${}^t(m_1, m_2) \in \mathbb{R}^2$. Under this identification, the action of G_1 on the unitary dual $\widehat{\mathbb{R}}^2$ of $\mathbb{R}^2$ induced from the conjugate action of G_1 on $\mathbb{R}^2$ becomes ${}^t(m_1, m_2) \longmapsto g\,{}^t(m_1, m_2)$ for $g \in G_1$ and ${}^t(m_1, m_2) \in \mathbb{R}^2$. Then it is clear that $\widehat{\mathbb{R}}^2$ has two G_1-orbits; $\{{}^t(0,0)\}$ and $\mathbb{R}^2 \backslash \{{}^t(0,0)\}$. The stabilizer of ${}^t(0,0)$ in G_1 is obviously G_1 itself. If we take a representative $\sigma = {}^t(1,0)$ in the latter orbit, then the stabilizer of σ in G_1 becomes $G_\sigma = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{R} \right\} \simeq \mathbb{R}$. Mackey's method tells us that the unitary dual of $G_1 \ltimes \mathbb{R}^2$ exactly consists of the following two families:

(1) $\pi_1 \cdot 1_{\mathbb{R}^2}$ with an irreducible unitary representation π_1 of G_1,
(2) $\mathrm{Ind}_{G_\sigma \ltimes \mathbb{R}^2}^{G_1 \ltimes \mathbb{R}^2}(\gamma \cdot \sigma)$ with a unitary character γ of G_σ.

Therefore we have the following proposition.

PROPOSITION 2.6. *Let $(\rho, \mathcal{F}_\rho)$ be an irreducible unitary representation of R_J with trivial central character. Then $(\rho, \mathcal{F}_\rho)$ is unitary equivalent to one of the following representations.*

(1) *$\rho_{\pi_1,0}(\tilde{n}) = \pi_1(g) \otimes \tilde{\nu}_0(\tilde{n})$ for $\tilde{n} = g \cdot n \in M_J^\circ \ltimes N_J$, where $(\pi_1, \mathcal{W}_{\pi_1})$ is an irreducible unitary representation of M_J° and $(\tilde{\nu}_0, \mathcal{U}_0)$ is the trivial representaiton of R_J,*
(2) *$\rho_r = \mathrm{Ind}_{N_0}^{R_J} \eta_r$ with $r \in \mathbb{R}$, where*

$$N_0 = \left\{ n(n_0, n_1, n_2, n_3) = \left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array} \right) \cdot \left(\begin{array}{cc|cc} 1 & & n_1 & n_2 \\ & 1 & n_2 & n_3 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \middle| \, n_i \in \mathbb{R} \right\}$$

and $\eta_r \colon n(n_0, n_1, n_2, n_3) \longmapsto e^{2\pi\sqrt{-1}(n_0 + r n_3)}$.

An irreducible unitary representation $(\rho, \mathcal{F}_\rho)$ of R_J is called *of type m* if ρ is unitary equivalent to the tensor product representation $\rho_{\pi_1,m} = \pi_1 \otimes \tilde{\nu}_m$ in Proposition 2.5

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for $m \neq 0$ or in Proposition 2.6 for $m = 0$, that is, the stabilizer of the equivalent class of $\rho|_{N_J}$ in R_J is R_J itself.

Now we describe the action $\rho = \rho_{\pi_1, m}$ on the representation space $\mathcal{F}_\rho = \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m$ with respect to a suitable basis. Fix a basis $\{X_+, X_-, Z\}$ of $(\mathfrak{n}_J)_\mathbb{C}$ and $\{V_+, V_-, U\}$ of $(\mathfrak{m}_J)_\mathbb{C}$. Here

$$
\begin{aligned}
X_\pm &= \tfrac{1}{2}(E_{e_1-e_2} \pm \sqrt{-1}E_{e_1+e_2}), \qquad Z = -\sqrt{-1}E_{2e_1} \\
V_\pm &= \tfrac{1}{2}\left\{H_2 \pm \sqrt{-1}(E_{2e_2} - E_{-2e_2})\right\}, \qquad U = -\sqrt{-1}(E_{2e_2} + E_{-2e_2}).
\end{aligned} \tag{2.2}
$$

Then the multiplications are given by

$$
\begin{aligned}
&[V_+, V_-] = U, \quad [U, V_\pm] = \pm 2V_\pm, \quad [X_+, X_-] = Z, \quad [Z, X_\pm] = 0, \\
&[V_\pm, X_\pm] = 0, \quad [V_\pm, X_\mp] = -X_\pm, \quad [U, X_\pm] = \pm X_\pm, \quad [Z, V_\pm] = 0,
\end{aligned} \tag{2.3}
$$

and $[Z, U] = 0$. Taking account of the irreducible decomposition of the Weil representation W_m for $m \neq 0$, we can take the following basis of $\mathcal{U}_m$ (cf. Berndt and Schmidt [1; Chapter 3]): Let J be the set of indices defined by

$$
J = \begin{cases} \{\mathrm{sign}(m)(\tfrac{1}{2} + \mathbb{Z}_{\geqslant 0})\}, & \text{for } m \neq 0, \\ \{0\}, & \text{for } m = 0. \end{cases} \tag{2.4}
$$

Then $\mathcal{U}_m$ has a basis $\{u_j^m\}_{j \in J}$ such that the action of $(\mathfrak{r}_J)_\mathbb{C}$ is given by

$$
\begin{aligned}
&\tilde{\nu}_m(Z)u_j^m = 2\pi m u_j^m, \qquad \tilde{\nu}_m(U)u_j^m = j u_j^m, \\
&\tilde{\nu}_m(X_\pm)u_j^m = u_{j\pm 1}^m, \qquad \tilde{\nu}_m(V_\pm)u_j^m = \mp \tfrac{1}{4\pi m} u_{j\pm 2}^m, \\
&\tilde{\nu}_m(X_\mp)u_j^m = -2\pi m\left(j \mp \tfrac{1}{2}\right)u_{j\mp 1}^m, \qquad \tilde{\nu}_m(V_\mp)u_j^m = \pm \pi m\left(j \mp \tfrac{1}{2}\right)\left(j \mp \tfrac{3}{2}\right)u_{j\mp 2}^m.
\end{aligned} \tag{2.5}
$$

Here the double sign depends on either $m \geqslant 0$ or $m < 0$, and we understand that $u_j^m = 0$ if $j \notin J$ and that $1/m = 0$ for $m = 0$. On the other hand, let π_1 be an irreducible unitary representation of $\widetilde{M_J^\circ}$. As we explained in the subsequent paragraph of Proposition 2.3, there is a basis $\{w_l\}_{l \in L}$ of $\mathcal{W}_{\pi_1}$ such that the action of $(\mathfrak{m}_J)_\mathbb{C}$ is given by

$$
\pi_1(U)w_l = l w_l, \quad \pi_1(V_\pm)w_l = \tfrac{1}{2}(z_1 + 1 \pm l)w_{l\pm 2}, \tag{2.6}
$$

with the suitable parameter z_1. Using (2.2), (2.5) and (2.6), we can define the action

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$\rho = \rho_{\pi_1,m}$ of $(\mathfrak{r}_J)_{\mathbb{C}}$ on $\mathcal{F}_\rho = \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m$ with the basis $\{w_l \otimes u_j^m\}_{l\in L, j\in J}$ as follows:

$$\begin{aligned}
&\rho_{\pi_1,m}(E_{2e_1})(w_l \otimes u_j^m) = 2\pi m\sqrt{-1}(w_l \otimes u_j^m),\\
&\rho_{\pi_1,m}(E_{e_1-e_2})(w_l \otimes u_j^m) = (w_l \otimes u_{j\pm 1}^m) - 2\pi m\big(j \mp \tfrac{1}{2}\big)(w_l \otimes u_{j\mp 1}^m),\\
&\rho_{\pi_1,m}(E_{e_1+e_2})(w_l \otimes u_j^m) = \mp\sqrt{-1}\big\{(w_l \otimes u_{j\pm 1}^m) + 2\pi m\big(j \mp \tfrac{1}{2}\big)(w_l \otimes u_{j\mp 1}^m)\big\},\\
&\rho_{\pi_1,m}(H_2)(w_l \otimes u_j^m)\\
&\quad = \mp\frac{1}{4\pi m}(w_l \otimes u_{j\pm 2}^m) \pm \pi m\big(j \mp \tfrac{1}{2}\big)\big(j \mp \tfrac{3}{2}\big)(w_l \otimes u_{j\mp 2}^m) +\\
&\qquad + \tfrac{1}{2}(z_1 + l + 1)(w_{l+2} \otimes u_j^m) + \tfrac{1}{2}(z_1 - l + 1)(w_{l-2} \otimes u_j^m),\\
&\rho_{\pi_1,m}(E_{2e_2})(w_l \otimes u_j^m)\\
&\quad = -\frac{\sqrt{-1}}{2}\big\{-(j+l)(w_l \otimes u_j^m) -\\
&\qquad - \frac{1}{4\pi m}(w_l \otimes u_{j\pm 2}^m) - \pi m\big(j \mp \tfrac{1}{2}\big)\big(j \mp \tfrac{3}{2}\big)(w_l \otimes u_{j\mp 2}^m) +\\
&\qquad + \tfrac{1}{2}(z_1 + l + 1)(w_{l+2} \otimes u_j^m) - \tfrac{1}{2}(z_1 - l + 1)(w_{l-2} \otimes u_j^m)\big\},\\
&\rho_{\pi_1,m}(E_{-2e_2})(w_l \otimes u_j^m)\\
&\quad = -\frac{\sqrt{-1}}{2}\big\{-(j+l)(w_l \otimes u_j^m) +\\
&\qquad + \frac{1}{4\pi m}(w_l \otimes u_{j\pm 2}^m) + \pi m\big(j \mp \tfrac{1}{2}\big)\big(j \mp \tfrac{3}{2}\big)(w_l \otimes u_{j\mp 2}^m) -\\
&\qquad - \tfrac{1}{2}(z_1 + l + 1)(w_{l+2} \otimes u_j^m) + \tfrac{1}{2}(z_1 - l + 1)(w_{l-2} \otimes u_j^m)\big\}.
\end{aligned}$$

Here the double sign depends on either $m \geqslant 0$ or $m < 0$, and we understand that $w_l \otimes u_j^m = 0$ if $l \notin L$ or $j \notin J$ and that $1/m = 0$ for $m = 0$.

2.4. PARAMETRIZATION OF DISCRETE SERIES REPRESENTATIONS OF G

In this subsection, we recall the Harish-Chandra parametrization of the discrete series representations of $G = Sp(2, \mathbb{R})$. For the general case, see [8; Chapter IX].

Let $\mathfrak{h} = \mathbb{R}T_1 \oplus \mathbb{R}T_2$ be the compact Cartan subalgebra of $\mathfrak{g}$ defined in Section 2 and consider the corresponding Cartan subgroup

$$\exp\mathfrak{h} = \left\{ h_\theta = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \middle| \begin{array}{l} c_\theta = \mathrm{diag}(\cos\theta_1, \cos\theta_2) \\ s_\theta = \mathrm{diag}(\sin\theta_1, \sin\theta_2) \end{array}, \theta = (\theta_1, \theta_2) \in \mathbb{R}^2 \right\}$$

of G. Then the characters of this Cartan subgroup are given by

$$h_\theta \longmapsto \exp(\sqrt{-1}(m_1\theta_1 + m_2\theta_2)), \quad m_i \in \mathbb{Z},$$

and the derivations of these characters determine the weight lattice in $\mathfrak{h}_{\mathbb{C}}^*$.

In order to parametrize the discrete series representations of G, we enumerate all the positive root systems compatible to $\Delta_c^+ = \{(1, -1)\}$:

(I) $\quad \Delta_{\mathrm{I}}^+ = \{(1, -1), (2, 0), (1, 1), (0, 2)\}$,

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(II) $\Delta^+_{\mathrm{II}} = \{(1, -1), (2, 0), (1, 1), (0, -2)\}$,
(III) $\Delta^+_{\mathrm{III}} = \{(1, -1), (2, 0), (0, -2), (-1, -1)\}$,
(IV) $\Delta^+_{\mathrm{IV}} = \{(1, -1), (0, -2), (-1, -1), (-2, 0)\}$.

Let J be a variable running over the set of indices I, II, III, IV and let us denote the set of non-compact positive roots for the index J by $\Delta^+_{J,n} = \Delta^+_J - \Delta^+_c$. Let Ξ_J be the subset consisting of Δ^+_c-dominant weights such that $\langle \Lambda, \beta \rangle > 0$ for any $\beta \in \Delta^+_{J,n}$. Then the set $\bigcup_{J=\mathrm{I}}^{\mathrm{IV}} \Xi_J$ gives the Harish-Chandra parametrization of the discrete series representations of G. Let us write by π_Λ the discrete series representation with the Harish-Chandra parameter $\Lambda \in \bigcup_{J=\mathrm{I}}^{\mathrm{IV}} \Xi_J$. Then π_Λ is called *holomorphic* if $\Lambda \in \Xi_{\mathrm{I}}$ and *anti-holomorphic* if $\Lambda \in \Xi_{\mathrm{IV}}$. Moreover if $\Lambda \in \Xi_{\mathrm{II}} \cup \Xi_{\mathrm{III}}$, π_Λ is called *large* (in the sense of Vogan [16]).

The Blattner formula gives the description of the K-types of π_Λ. In particular, the minimal K-type $(\tau_\lambda, V_\lambda)$ of π_Λ is given by the formula $\lambda = \Lambda - \rho_c + \rho_n$, where ρ_c (resp. ρ_n) is the half sum of compact (resp. non-compact) positive roots in Δ^+_J. We call such λ *the Blattner parameter of* π_Λ.

3. Fourier–Jacobi Type Spherical Functions

3.1. RADIAL PART

Let $(\rho, \mathcal{F}_\rho)$ be an irreducible unitary representation of R_J and let (τ, V_τ) be a finite-dimensional K-module. We denote by $C^\infty_{\rho,\tau}(R_J \setminus G/K)$ the space of smooth functions $F: G \to \mathcal{F}_\rho \otimes V_\tau$ with the property

$$F(rgk) = \big(\rho(r) \otimes \tau(k)^{-1}\big)F(g), \qquad (r, g, k) \in R_J \times G \times K.$$

Put $C^\infty(A_J; \rho, \tau)$ the space of smooth functions $\varphi: A_J \to \mathcal{F}_\rho \otimes V_\tau$ satisfying

$$\big(\rho(m) \otimes \tau(m)\big)\varphi(a) = \varphi(a), \qquad m \in R_J \cap K = M_J^\circ \cap K, \ \ a \in A_J. \tag{3.1}$$

Here we remark that all elements in $M_J^\circ \cap K$ are commutative with $a \in A_J$. Then the restriction to A_J gives a linear map from $C^\infty_{\rho,\tau}(R_J \setminus G/K)$ to $C^\infty(A_J; \rho, \tau)$, which is injective. For each $f \in C^\infty_{\rho,\tau}(R_J \setminus G/K)$, we call $f|_{A_J} \in C^\infty(A_J; \rho, \tau)$ *the radial part of* f, where $|_{A_J}$ means the restriction to A_J.

Let $(\tau', V_{\tau'})$ be another finite-dimensional K-module. For each $\mathbb{C}$-linear map $u: C^\infty_{\rho,\tau}(R_J \setminus G/K) \to C^\infty_{\rho,\tau'}(R_J \setminus G/K)$, we have a unique $\mathbb{C}$-linear map $\mathcal{R}(u)$: $C^\infty(A_J; \rho, \tau) \to C^\infty(A_J; \rho, \tau')$ with the property $(uf)|_{A_J} = \mathcal{R}(u)(f|_{A_J})$ for $f \in C^\infty_{\rho,\tau}(R_J \setminus G/K)$. We call $\mathcal{R}(u)$ *the radial part of* u.

3.2. FOURIER–JACOBI TYPE SPHERICAL FUNCTIONS

Let $(\rho, \mathcal{F}_\rho)$ be an irreducible unitary representation of R_J and consider a C^∞-induced representation $C^\infty\mathrm{Ind}^G_{R_J}(\rho)$ with the representation space

$$C^\infty_\rho(R_J \setminus G) = \big\{F: G \to \mathcal{F}_\rho, \ \ C^\infty \ \big| \ F(rg) = \rho(r)F(g), (r, g) \in R_J \times G\big\}$$

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on which G acts by the right translation. Then $C^\infty_\rho(R_J \setminus G)$ has structure of $(\mathfrak{g}_\mathbb{C}, K)$-module. Moreover let (τ, V_τ) be an irreducible representation of K and take an irreducible admissible representation π of G with the K-type τ^*, where τ^* is the contragredient representation of τ. Now we consider the intertwining space

$$\mathcal{I}_{\rho,\pi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi, C^\infty\mathrm{Ind}^G_{R_J}(\rho))$$

between $(\mathfrak{g}_\mathbb{C}, K)$-modules. Let $i: \tau^* \to \pi|_K$ be a K-equivariant map and let i^* be the pullback via i. Then the map

$$\mathcal{I}_{\rho,\pi} \xrightarrow{i^*} \mathrm{Hom}_K(\tau^*, C^\infty\mathrm{Ind}^G_{R_J}(\rho)) \simeq C^\infty_{\rho,\tau}(R_J \setminus G/K)$$

gives a restriction of $T \in \mathcal{I}_{\rho,\pi}$ to the K-type τ^*. Here the last isomorphism is given by the correspondence between $\imath \in \mathrm{Hom}_K(\tau^*, C^\infty\mathrm{Ind}^G_{R_J}(\rho))$ and the function $F^{[\imath]} \in C^\infty_{\rho,\tau}(R_J \setminus G/K)$ such that $\imath(v^*)(g) = \langle v^*, F^{[\imath]}(g)\rangle$ for $v^* \in V_{\tau^*}$ and $g \in G$ with the canonical bilinear form $\langle\cdot,\cdot\rangle$ on $V_{\tau^*} \times (\mathcal{F}_\rho \otimes V_\tau)$. Now we denote the image of T in $C^\infty_{\rho,\tau}(R_J \setminus G/K)$ by T_i and define the space $\mathcal{J}_{\rho,\pi}(\tau)$ of *Fourier–Jacobi type spherical functions of type* $(\rho, \pi; \tau)$ *on* G by

$$\mathcal{J}_{\rho,\pi}(\tau) = \bigcup_{i\in \mathrm{Hom}_K(\tau^*,\pi|_K)} \{T_i \mid T \in \mathcal{I}_{\rho,\pi}\}.$$

If $\rho = \rho_{\pi_1,m}$ is a representation of type m and $a_r = \exp rH_1 = \mathrm{diag}(\xi, 1, \xi^{-1}, 1)$, $\xi = e^r$, then we put

$$\mathcal{J}^\circ_{\rho,\pi}(\tau) = \{f \in \mathcal{J}_{\rho,\pi}(\tau) \mid f|_{A_J}(a_r) \text{ is of moderate growth when } r \to \infty\},$$

and call $f \in \mathcal{J}^\circ_{\rho,\pi}(\tau)$ a Fourier–Jacobi type spherical function *with moderate growth*.

4. Differential Operators

4.1. SCHMID OPERATOR AND SHIFT OPERATOR

Let $C^\infty_{\rho,\tau}(R_J \setminus G/K)$ and $C^\infty(A_J; \rho, \tau)$ be as in Section 3.1. We introduce certain differential operators on $C^\infty_{\rho,\tau}(R_J \setminus G/K)$ and calculate their radial parts. First of all, the following two lemmas are obvious. (cf. Knapp [8; Chapter VIII])

LEMMA 4.1. *For* $a \in A_J$, *we have* $\mathfrak{g} = \mathrm{Ad}(a)\mathfrak{r}_J + \mathfrak{a}_J + \mathfrak{k}$.

LEMMA 4.2. *Let* $f \in C^\infty_{\rho,\tau}(R_J \setminus G/K)$. *For* $X \in U(\mathfrak{k}_\mathbb{C})$, $Y \in U((\mathfrak{r}_J)_\mathbb{C})$, $Z \in U((\mathfrak{a}_J)_\mathbb{C})$ *and* $a \in A_J$, *we have*

$$(\mathrm{Ad}(a^{-1})Y)ZXf(a) = \rho(Y) \otimes \tau(-X)(Zf)(a).$$

Moreover, H_1 *acts on* $C^\infty_{\rho,\tau}(R_J \setminus G/K)$ *by*

$$H_1 f(a_r) = \frac{\mathrm{d}}{\mathrm{d}r} f(a_r) = \xi\frac{\mathrm{d}}{\mathrm{d}\xi} f(a_r).$$

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Take an orthonormal basis $\{X_i\}$ of $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$. Now we define a first order gradient type differential operator $\nabla_{\rho,\tau}\colon C^\infty_{\rho,\tau}(R_J \setminus G/K) \to C^\infty_{\rho,\tau\otimes\mathrm{Ad}_{\mathfrak{p}_\mathbb{C}}}(R_J\backslash G/K)$ by

$$\nabla_{\rho,\tau}f = \sum_i R_{X_i} f \otimes X_i, \qquad f \in C^\infty_{\rho,\tau}(R_J \setminus G/K),$$

where

$$R_X f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(g\cdot \exp(tX))\bigg|_{t=0} \quad \text{for } X \in \mathfrak{g}_\mathbb{C}, g \in G.$$

This differential operator $\nabla_{\rho,\tau}$ is called *the Schmid operator*. Moreover, define $\nabla^\pm_{\rho,\tau}\colon C^\infty_{\rho,\tau}(R_J \setminus G/K) \to C^\infty_{\rho,\tau\otimes\mathrm{Ad}_{\mathfrak{p}_\pm}}(R_J\backslash G/K)$ by

$$\nabla^\pm_{\rho,\tau} f = \tfrac{1}{4}\sum_{\beta\in\Sigma_n^+} \|\beta\|^2 R_{X_{\mp\beta}} f \otimes X_{\pm\beta}, \quad f \in C^\infty_{\rho,\tau}(R_J \setminus G/K).$$

If we take the basis (2.1) of $\mathfrak{p}$ as $\{X_i\}$ then we have $\nabla_{\rho,\tau} f = 8c^2(\nabla^+_{\rho,\tau} f + \nabla^-_{\rho,\tau} f)$, and hence, $\nabla^+_{\rho,\tau} \oplus \nabla^-_{\rho,\tau}$ gives a decomposition of $\nabla_{\rho,\tau}$ corresponding to $\mathfrak{p}c = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ up to constant multiple. For each $\beta \in \Delta_n^+$ and $\lambda \in \Lambda$, *the shift operator* $\nabla^{\pm\beta}_{\rho,\tau_\lambda}\colon C^\infty_{\rho,\tau_\lambda}(R_J\backslash G/K) \to C^\infty_{\rho,\tau_{\lambda\pm\beta}}(R_J\backslash G/K)$ is defined as the composition of $\nabla^\pm_{\rho,\tau_\lambda}$ with the projector $P^{\pm\beta}$ from $V_{\tau_\lambda} \otimes \mathfrak{p}_\pm$ into the irreducible component $V_{\tau_{\lambda\pm\beta}}$: $\nabla^{\pm\beta}_{\rho,\tau_\lambda} = (1_{\mathcal{F}_\rho} \otimes P^{\pm\beta})\nabla^\pm_{\rho,\tau_\lambda}$.

PROPOSITION 4.3. *For $\varphi \in C^\infty(A_J; \rho, \tau)$, we have*

$$\begin{aligned}
\mathcal{R}(\nabla^\pm_{\rho,\tau})\varphi(a_r) = \Big[\xi\frac{\mathrm{d}}{\mathrm{d}\xi} &+ 1_{\mathcal{F}_\rho} \otimes (\tau\otimes \mathrm{Ad}_{\mathfrak{p}_\pm})\big(\pm\tfrac{1}{2}(Z+H')\big) - 4\mp \\
&\mp 2\sqrt{-1}\xi^2\rho(E_{2e_1}) \otimes (1_{V_\tau}\otimes 1_{\mathfrak{p}_\pm})\Big]\big(\varphi(a_r)\otimes X_{\pm(2,0)}\big)+ \\
&+ \Big[1_{\mathcal{F}_\rho} \otimes (\tau\otimes \mathrm{Ad}_{\mathfrak{p}_\pm})(\tfrac{1}{2}(X-\bar{X}) \pm \tfrac{1}{2}(X+\bar{X}))+ \\
&+ \xi\rho(E_{e_1-e_2} \mp \sqrt{-1}E_{e_1+e_2}) \otimes (1_{V_\tau}\otimes 1_{\mathfrak{p}_\pm})\Big]\big(\varphi(a_r)\otimes X_{\pm(1,1)}\big)+ \\
&+ \Big[1_{\mathcal{F}_\rho} \otimes (\tau\otimes \mathrm{Ad}_{\mathfrak{p}_\pm})\big(\pm\tfrac{1}{2}(Z-H')\big) - 2+ \\
&+ \rho(H_2 \mp 2\sqrt{-1}E_{2e_2}) \otimes (1_{V_\tau}\otimes 1_{\mathfrak{p}_\pm})\Big]\big(\varphi(a_r)\otimes X_{\pm(0,2)}\big),
\end{aligned}$$

Proof. Let $f \in C^\infty_{\rho,\tau}(R_J \setminus G/K)$. To prove the assertions, it suffices to compute the terms $\frac{1}{4}\|\beta\|^2 R_{X_{-\beta}} f \otimes X_\beta$ for every $\beta \in \Delta_n$. We note the obvious relations

$$\begin{aligned}
E_{2e_1} &= \xi^2\mathrm{Ad}(a_r^{-1})E_{2e_1}, & E_{e_1\pm e_2} &= \xi\mathrm{Ad}(a_r^{-1})E_{e_1\pm e_2},\\
E_{\pm 2e_2} &= \mathrm{Ad}(a_r^{-1})E_{\pm 2e_2}, & H_2 &= \mathrm{Ad}(a_r^{-1})H_2.
\end{aligned}$$

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By using Lemmas 2.1, 4.2 and the definition of the tensor product, we have

$$R_{X_{(-2,0)}} f(a_r) \otimes X_{(2,0)} = \Big\{ \xi \frac{\mathrm{d}}{\mathrm{d}\xi} + 1_{\mathcal{F}_\rho} \otimes (\tau \otimes \mathrm{Ad}_{\mathfrak{p}_+})\big(\tfrac{1}{2}(Z + H')\big) - 2 - \\ - 2\sqrt{-1}\xi^2 \rho(E_{2e_1}) \otimes (\tau) \Big\} \big(f(a_r) \otimes X_{(2,0)}\big).$$

Here we also use the relation $[Z + H', X_{(2,0)}] = 4X_{(2,0)}$. The similar calculation shows the equalities

$$\begin{aligned}
& R_{X_{(2,0)}} f(a_r) \otimes X_{(-2,0)} \\
&\quad = \Big\{ \xi \frac{\mathrm{d}}{\mathrm{d}\xi} + 1_{\mathcal{F}_\rho} \otimes (\tau \otimes \mathrm{Ad}_{\mathfrak{p}_-})\big(-\tfrac{1}{2}(Z + H')\big) - 2 + \\
&\qquad + 2\sqrt{-1}\xi^2 \rho(E_{2e_1}) \otimes (1_{V_\tau} \otimes 1_{\mathfrak{p}_-}) \Big\} \big(f(a_r) \otimes X_{(-2,0)}\big), \\
& \tfrac{1}{2} R_{X_{\mp(1,1)}} f(a_r) \otimes X_{\pm(1,1)} \\
&\quad = \big\{ 1_{\mathcal{F}_\rho} \otimes (\tau \otimes \mathrm{Ad}_{\mathfrak{p}_\pm})(\tfrac{1}{2}(X - \bar{X}) \pm \tfrac{1}{2}(X + \bar{X})) + \\
&\qquad + \xi \rho(E_{e_1 - e_2} \mp \sqrt{-1} E_{e_1 + e_2}) \otimes (1_{V_\tau} \otimes 1_{\mathfrak{p}_\pm}) \big\} \big(f(a_r) \otimes X_{\pm(1,1)}\big) - \\
&\qquad - 2 f(a_r) \otimes X_{\pm(2,0)}, \\
& R_{X_{\mp(0,2)}} f(a_r) \otimes X_{\pm(0,2)} \\
&\quad = \Big\{ 1_{\mathcal{F}_\rho} \otimes (\tau \otimes \mathrm{Ad}_{\mathfrak{p}_\pm})\big(\pm \tfrac{1}{2}(Z - H')\big) - 2 + \\
&\qquad + \rho(H_2 \mp 2\sqrt{-1} E_{2e_2}) \otimes (1_{V_\tau} \otimes 1_{\mathfrak{p}_\pm}) \Big\} \big(f(a_r) \otimes X_{\pm(0,2)}\big),
\end{aligned}$$

by using the relations

$$\begin{aligned}
&[2X, X_{(1,1)}] = 4X_{(2,0)}, \qquad [Z - H', X_{(0,\pm 2)}] = \pm 4X_{(0,\pm 2)}, \\
&[2\bar{X}, X_{(-1,-1)}] = -4X_{(-2,0)}, \qquad [Z + H', X_{(-2,0)}] = -4X_{(-2,0)}.
\end{aligned}$$

Combining these identities, we get the assertions immediately. □

4.2. ACTIONS OF THE DIFFERENTIAL OPERATORS

Let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1, m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ be an irreducible unitary representation of R_J of type m, and let $\{w_l \otimes u_j^m\}_{l \in L, j \in J}$ be the basis of $\mathcal{F}_\rho$ given in Section 2.3. Moreover let $(\tau_\lambda, V_\lambda)$ be an irreducible K-module with $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ and $\{v_k^\lambda\}_{0 \leqslant k \leqslant d_\lambda}$ be the standard basis of V_λ. We express a C^∞-function $\varphi\colon A_J \to \mathcal{F}_\rho \otimes V_\lambda$ as

$$\varphi(a) = \sum_{k=0}^{d_\lambda} \sum_{l \in L} \sum_{j \in J} c_{l,j,k}(a)(w_l \otimes u_j^m \otimes v_k^\lambda) \tag{4.1}$$

with C^∞-functions $c_{l,j,k}(a)$ on A_J.

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LEMMA 4.4. *If we write* $\varphi \in C^\infty(A_J; \rho, \tau)$ *as* (4.1), *then* $c_{l,j,k}(a) \equiv 0$ *for* $j + l \neq k - \lambda_1$. *Therefore* $\varphi \in C^\infty(A_J; \rho, \tau)$ *can be expressed as*

$$\sum_{\substack{k=0 \\ l=l(j,k)\in L}}^{d_\lambda} \sum_{j\in J} c_{j,k}(a)(w_l \otimes u_j^m \otimes v_k^\lambda), \tag{4.2}$$

with $l(j,k) = -j + k - \lambda_1$.

Proof. Since

$$R_J \cap K = M_J^\circ \cap K = \exp \mathbb{R}T_2 \simeq SO(2, \mathbb{R})$$

with

$$T_2 = \frac{\sqrt{-1}}{2}(Z - H') = E_{2e_2} + E_{-2e_2},$$

every $\varphi \in C^\infty(A_J; \rho_{\pi_1,m}, \tau_\lambda)$ must satisfy the condition

$$\big(\rho_{\pi_1,m}(\exp\theta T_2) \otimes \tau_\lambda(\exp\theta T_2)\big)\varphi(a) = \varphi(a), \quad \text{for any } \theta \in \mathbb{R}, \ \text{any } a \in A_J,$$

from (3.1). For φ expressed as (4.1), this condition is equivalent to

$$e^{\sqrt{-1}\theta(j+l)} \cdot e^{\sqrt{-1}\theta(\lambda_1-k)} c_{l,j,k}(a) = c_{l,j,k}(a),$$

for any $\theta \in \mathbb{R}$, any $a \in A_J$ and each $j \in J$, $l \in L$ and $0 \leqslant k \leqslant d_\lambda$ from the action of T_2 given in Section 2. Therefore we have $c_{l,j,k}(a) \equiv 0$ or $j + l = k - \lambda_1$. The expression (4.2) is obvious. □

Now we write down the actions of the radial parts of the shift operators $\nabla^{\pm}_{\rho,\tau_\lambda} \pm\beta$ on $C^\infty(A_J; \rho_{\pi_1}, m, \tau_\lambda)$ using the basis $\{w_l \otimes u_j^m \otimes v_k^\lambda\}_{l\in L, j\in J, 0\leqslant k\leqslant d_\lambda}$.

PROPOSITION 4.5. *Let us denote* $\varphi \in C^\infty(A_J; \rho_{\pi_1}, m, \tau_\lambda)$ *as* (4.2). *For each* $\beta \in \Delta_n^+$, *we have the formula*

$$\begin{aligned}
&\mathcal{R}(\nabla^{\beta}_{\rho,\tau_\lambda})\varphi(a_r) \\
&\quad = \sum_{\substack{k=0 \\ l=l(j,k)\in L}}^{d_\lambda} \sum_{j\in J} \Big[C_1^{\beta,\pm} c_{j,k}(a_r)(w_l \otimes u_j^m \otimes v_{k(\beta)}^{\lambda+\beta}) + C_2^{\beta,\pm} c_{j,k}(a_r)(w_l \otimes u_{j-1}^m \otimes v_{k(\beta)-1}^{\lambda+\beta}) + \\
&\qquad + C_3^{\beta,\pm} c_{j,k}(a_r)(w_l \otimes u_{j-2}^m \otimes v_{k(\beta)-2}^{\lambda+\beta}) + C_4^{\beta,\pm} c_{j,k}(a_r)(w_{l-2} \otimes u_j^m \otimes v_{k(\beta)-2}^{\lambda+\beta}) \Big]
\end{aligned}$$

with the coefficients $C_i^{\beta,\pm}$ *and the parameter* $k(\beta)$ *given below.*

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(1) *If* $\beta = (0,2)$, *then* $k(\beta) = k$ *and*

$$C_1^{\beta,+} = C_1^{\beta,-} = \xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (-k + \lambda_2 - 2) + 4\pi m\xi^2, \quad C_2^{\beta,+} = 8\pi m\big(j - \tfrac{1}{2}\big)\xi,$$
$$C_2^{\beta,-} = -4\xi, \quad C_3^{\beta,+} = 2\pi m\big(j-\tfrac{1}{2}\big)\big(j-\tfrac{3}{2}\big), \quad C_3^{\beta,-} = \frac{1}{2\pi m},$$
$$C_4^{\beta,+} = C_4^{\beta,-} = z_1 - l + 1.$$

(2) *If* $\beta = (1,1)$, *then* $k(\beta) = k + 1$ *and*

$$C_1^{\beta,+} = C_1^{\beta,-} = (k+1)\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (-k + \lambda_1 - 2) + 4\pi m\xi^2\right\},$$
$$C_2^{\beta,+} = -4\pi m\big(j-\tfrac{1}{2}\big)(d_\lambda - 2k)\xi, \quad C_2^{\beta,-} = 2(d_\lambda - 2k)\xi,$$
$$C_3^{\beta,+} = -2\pi m\big(j-\tfrac{1}{2}\big)\big(j-\tfrac{3}{2}\big)(d_\lambda + 1 - k), \quad C_3^{\beta,-} = -\frac{1}{2\pi m}(d_\lambda + 1 - k),$$
$$C_4^{\beta,+} = C_4^{\beta,-} = -(z_1 - l + 1)(d_\lambda + 1 - k).$$

(3) *If* $\beta = (2,0)$, *then* $k(\beta) = k + 2$ *and*

$$C_1^{\beta,+} = C_1^{\beta,-} = \tfrac{1}{2}(k+1)(k+2)\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (-k + \lambda_2 + 2d_\lambda) + 4\pi m\xi^2\right\},$$
$$C_2^{\beta,+} = -4\pi m\big(j-\tfrac{1}{2}\big)(k+1)(d_\lambda + 1 - k)\xi, \quad C_2^{\beta,-} = 2(k+1)(d_\lambda + 1 - k)\xi,$$
$$C_3^{\beta,+} = \pi m\big(j-\tfrac{1}{2}\big)\big(j-\tfrac{3}{2}\big)(d_\lambda + 1 - k)(d_\lambda + 2 - k),$$
$$C_3^{\beta,-} = \frac{1}{4\pi m}(d_\lambda + 1 - k)(d_\lambda + 2 - k),$$
$$C_4^{\beta,+} = C_4^{\beta,-} = \tfrac{1}{2}(z_1 - l + 1)(d_\lambda + 1 - k)(d_\lambda + 2 - k).$$

Similarly we have

$$\begin{aligned}
&\mathcal{R}(\nabla^{-\beta}_{\rho,\tau_\lambda})\varphi(a_r)\\
&= \sum_{k=0}^{d_\lambda}\sum_{\substack{j\in J\\ l=l(j,k)\in L}}\Big[C_1^{\prime\beta,\pm}c_{j,k}(a_r)(w_l\otimes u_j^m\otimes v^{\lambda-\beta}_{k'(\beta)}) + C_2^{\prime\beta,\pm}c_{j,k}(a_r)(w_l\otimes u_{j+1}^m\otimes v^{\lambda-\beta}_{k'(\beta)+1})\\
&\quad + C_3^{\prime\beta,\pm}c_{j,k}(a_r)(w_l\otimes u_{j+2}^m\otimes v^{\lambda-\beta}_{k'(\beta)+2}) + C_4^{\prime\beta,\pm}c_{j,k}(a_r)(w_{l+2}\otimes u_j^m\otimes v^{\lambda-\beta}_{k'(\beta)+2})\Big]
\end{aligned}$$

with

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(1) *If* $\beta = (2,0)$*, then* $k'(\beta) = k-2$ *and*

$$C_1^{\prime\beta,+} = C_1^{\prime\beta,-} = \xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (k - \lambda_2 - 2d_\lambda - 2) - 4\pi m\xi^2, \quad C_2^{\prime\beta,+} = 4\xi,$$
$$C_2^{\prime\beta,-} = -8\pi m\big(j+\tfrac{1}{2}\big)\xi, \quad C_3^{\prime\beta,+} = -\frac{1}{2\pi m}, \quad C_3^{\prime\beta,-} = -2\pi m\big(j+\tfrac{1}{2}\big)\big(j+\tfrac{3}{2}\big),$$
$$C_4^{\prime\beta,+} = C_4^{\prime\beta,-} = z_1 + l + 1.$$

(2) *If* $\beta = (1,1)$*, then* $k'(\beta) = k-1$ *and*

$$C_1^{\prime\beta,+} = C_1^{\prime\beta,-} = -(d_\lambda + 1 - k)\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (k - \lambda_1 - 2) - 4\pi m\xi^2\right\},$$
$$C_2^{\prime\beta,+} = -2(d_\lambda - 2k)\xi, \quad C_2^{\prime\beta,-} = 4\pi m\big(j+\tfrac{1}{2}\big)(d_\lambda - 2k)\xi,$$
$$C_3^{\prime\beta,+} = -\frac{1}{2\pi m}(k+1), \quad C_3^{\prime\beta,-} = -2\pi m\big(j+\tfrac{1}{2}\big)\big(j+\tfrac{3}{2}\big)(k+1),$$
$$C_4^{\prime\beta,+} = C_4^{\prime\beta,-} = (z_1 + l + 1)(k+1).$$

(3) *If* $\beta = (0,2)$*, then* $k'(\beta) = k$ *and*

$$C_1^{\prime\beta,+} = C_1^{\prime\beta,-} = \tfrac{1}{2}(d_\lambda + 1 - k)(d_\lambda + 2 - k)\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (k - \lambda_2) - 4\pi m\xi^2\right\},$$
$$C_2^{\prime\beta,+} = -2(k+1)(d_\lambda + 1 - k)\xi, \quad C_2^{\prime\beta,-} = 4\pi m\big(j+\tfrac{1}{2}\big)(k+1)(d_\lambda + 1 - k)\xi,$$
$$C_3^{\prime\beta,+} = -\frac{1}{4\pi m}(k+1)(k+2), \quad C_3^{\prime\beta,-} = -\pi m\big(j+\tfrac{1}{2}\big)\big(j+\tfrac{3}{2}\big)(k+1)(k+2),$$
$$C_4^{\prime\beta,+} = C_4^{\prime\beta,-} = \tfrac{1}{2}(z_1 + l + 1)(k+1)(k+2).$$

Here the double sign depends on either $m \geqslant 0$ *or* $m < 0$*, and we understand that* $w_l \otimes u_j^m \otimes v_k^\lambda = 0$ *if* $l \notin L$*,* $j \notin J$*,* $k < 0$*, or* $d_\lambda < k$*, and that* $1/m = 0$ *for* $m = 0$*.*

Proof. For each $\beta \in \Delta_n^+$, we calculate the composition of the radial parts $\mathcal{R}(\nabla^{\pm}_{\rho,\tau_\lambda})$ in Proposition 4.3 with the projectors $P^{\pm\beta}$. Then we have the equalities

$$\begin{aligned}
&\mathcal{R}(\nabla^{\pm\beta}_{\rho,\tau_\lambda})\varphi(a_r)\\
&\quad= \sum_{\substack{k=0\\ l=l(j,k)\in L}}^{d_\lambda}\sum_{j\in J}\Bigg[\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + 1_{\mathcal{F}_\rho}\otimes\tau_{\lambda\pm\beta}\big(\pm\tfrac{1}{2}(Z+H')\big) - 4 \mp 2\sqrt{-1}\xi^2\rho(E_{2e_1})\otimes 1_{V_{\tau_{\lambda\pm\beta}}}\right\}\\
&\qquad\times c_{j,k}(a_r)\big(w_l\otimes u_j^m\otimes P^{\pm\beta}(v_k^\lambda\otimes X_{\pm(2,0)})\big)+\\
&\qquad+\left\{1_{\mathcal{F}_\rho}\otimes\tau_{\lambda\pm\beta}(\tfrac{1}{2}(X-\bar{X})\pm\tfrac{1}{2}(X+\bar{X})) + \xi\rho(E_{e_1-e_2}\mp\sqrt{-1}E_{e_1+e_2})\otimes 1_{V_{\tau_{\lambda\pm\beta}}}\right\}\times\\
&\qquad\times c_{j,k}(a_r)\big(w_l\otimes u_j^m\otimes P^{\pm\beta}(v_k^\lambda\otimes X_{\pm(1,1)})\big)+\\
&\qquad+\left\{1_{\mathcal{F}_\rho}\otimes\tau_{\lambda\pm\beta}\big(\pm\tfrac{1}{2}(Z-H')\big) - 2 + \rho(H_2\mp 2\sqrt{-1}E_{2e_2})\otimes 1_{V_{\tau_{\lambda\pm\beta}}}\right\}\times\\
&\qquad\times c_{j,k}(a_r)\big(w_l\otimes u_j^m\otimes P^{\pm\beta}(v_k^\lambda\otimes X_{\pm(0,2)})\big)\Bigg].
\end{aligned}$$

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Thus the assertions can be proved immediately from the Clebsch–Gordan coefficients in Lemma 2.2 and the actions τ_λ of $\mathfrak{k}_{\mathbb{C}}$ and $\rho_{\pi_1,m}$ of $(\mathfrak{r}_J)_{\mathbb{C}}$ given in Sections 2.2 and 2.3. $\square$

5. Characterization of Spherical Functions

Let π be a discrete series representation of G and $(\rho, \mathcal{F}_\rho)$ be an irreducible unitary representation of R_J. Now we refer the following proposition which enables us to identify the intertwining space $\mathcal{I}_{\rho,\pi}$ with a solution space of some differential equations.

PROPOSITION 5.1. *Let $\pi = \pi_\Lambda$ be a discrete series representation of G with the Harish-Chandra parameter $\Lambda \in \Xi_J$ and $(\tau_\lambda^*, V_\lambda^*)$ be the minimal K-type of π. Then we have a linear isomorphism $\mathcal{I}_{\rho,\pi} \simeq \mathcal{J}_{\rho,\pi}(\tau_\lambda)$ and*

$$\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \left\{F \in C^\infty_{\rho,\tau_\lambda}(R_J\backslash G/K) \;\middle|\; \nabla^{-\beta}_{\rho,\tau_\lambda} F = 0, \quad \forall \beta \in \Delta^+_{J^*,n}\right\}$$

for any irreducible unitary representation $(\rho, \mathcal{F}_\rho)$ of R_J. Here the index J^* means IV, III, II *and* I *for $J =$* I, II, III *and* IV, *respectively.*

Proof. For holomorphic or anti-holomorphic discrete series representations, this assertion is reduced to the theorem of Cauchy–Riemann. For large discrete series representations π_Λ, it follows from the result of Yamashita [19; Theorem 2.4], since each of the Blattner parameters λ is far from the walls (see [19; Definition 1.7]). $\square$

6. The Holomorphic Case

In this section, we describe the spaces $\mathcal{J}_{\rho,\pi}(\tau)$ and $\mathcal{J}^\circ_{\rho,\pi}(\tau)$ for discrete series representations π of G of type I or IV, the minimal K-types τ^* of π, and irreducible unitary representations $\rho = \rho_{\pi_1,m}$ of R_J of type m, using the system of the differential equations in Proposition 5.1. The case of $m \neq 0$ is treated in Section 6.2, and that of $m = 0$ in Section 6.3. For simplicity, we discuss only for π of type I, because the 'symmetric' argument holds for π of type IV.

6.1. DIFFERENTIAL EQUATIONS

Let $\pi = \pi_\Lambda$ be a holomorphic discrete series representation of G with the Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2) \in \Xi_{\mathrm{I}}$ and τ_λ^* be the minimal K-type of π, i.e. $\lambda = (-\Lambda_2 - 2, -\Lambda_1 - 1)$ from the Blattner formula. For each $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$, we express $\varphi \in C^\infty(A_J; \rho, \tau)r$ as (4.2) with $l(j,k) = -j + k + \Lambda_2 + 2$.

PROPOSITION 6.1. *Let π, τ_λ, and ρ be as above. Then $\varphi \in C^\infty(A_J; \rho, \tau)r$ is in $\mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ iff each coefficient $c_{j,k}(a_r)$ of φ satisfies the following system of differential*

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equations

$$\begin{aligned} &A^{\pm}_{j,k}c_{j,k}(a_r)+B^{\pm}_{j+1,k+1}c_{j+1,k+1}(a_r)+\\ &\quad + C^{\pm}_{j+2,k+2}c_{j+2,k+2}(a_r)+D^{\pm}_{j,k+2}c_{j,k+2}(a_r)=0, \qquad (6.1)\\ &E^{\pm}_{j,k}c_{j,k}(a_r)+F^{\pm}_{j+1,k+1}c_{j+1,k+1}(a_r)+\\ &\quad + G^{\pm}_{j+2,k+2}c_{j+2,k+2}(a_r)+H^{\pm}_{j,k+2}c_{j,k+2}(a_r)=0, \qquad (6.2)\\ &I^{\pm}_{j,k}c_{j,k}(a_r)+J^{\pm}_{j+1,k+1}c_{j+1,k+1}(a_r)+\\ &\quad + K^{\pm}_{j+2,k+2}c_{j+2,k+2}(a_r)+L^{\pm}_{j,k+2}c_{j,k+2}(a_r)=0, \qquad (6.3) \end{aligned}$$

where

$$\begin{aligned} &A^{\pm}_{j,k}=\xi\frac{\mathrm{d}}{\mathrm{d}\xi}-(k+\Lambda_1+3)+4\pi m\xi^2,\quad B^{+}_{j,k}=8\pi m\big(j-\tfrac{1}{2}\big)\xi,\quad B^{-}_{j,k}=-4\xi,\\ &C^{+}_{j,k}=2\pi m\big(j-\tfrac{1}{2}\big)\big(j-\tfrac{3}{2}\big),\quad C^{-}_{j,k}=\frac{1}{2\pi m},\quad D^{\pm}_{j,k}=z_1-l(j,k)+1,\\ &E^{\pm}_{j,k}=(k+1)\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi}-(k+\Lambda_2+4)+4\pi m\xi^2\right\},\quad F^{+}_{j,k}=-4\pi m\big(j-\tfrac{1}{2}\big)(d_\lambda-2k)\xi,\\ &F^{-}_{j,k}=2(d_\lambda-2k)\xi,\quad G^{+}_{j,k}=-2\pi m\big(j-\tfrac{1}{2}\big)\big(j-\tfrac{3}{2}\big)(d_\lambda+1-k),\\ &G^{-}_{j,k}=-\frac{1}{2\pi m}(d_\lambda+1-k),\quad H^{\pm}_{j,k}=-(z_1-l(j,k)+1)(d_\lambda+1-k),\\ &I^{\pm}_{j,k}=\tfrac{1}{2}(k+1)(k+2)\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi}-(k-2d_\lambda+\Lambda_1+1)+4\pi m\xi^2\right\}\\ &J^{+}_{j,k}=-4\pi m\big(j-\tfrac{1}{2}\big)(k+1)(d_\lambda+1-k)\xi,\quad J^{-}_{j,k}=2(k+1)(d_\lambda+1-k)\xi,\\ &K^{+}_{j,k}=\pi m\big(j-\tfrac{1}{2}\big)\big(j-\tfrac{3}{2}\big)(d_\lambda+1-k)(d_\lambda+2-k),\\ &K^{-}_{j,k}=\frac{1}{4\pi m}(d_\lambda+1-k)(d_\lambda+2-k),\\ &L^{\pm}_{j,k}=\tfrac{1}{2}(z_1-l(j,k)+1)(d_\lambda+1-k)(d_\lambda+2-k). \end{aligned}$$

Here Equation (6.1) (resp. (6.2), (6.3)) is valid for $j\in J$, $0\leqslant k\leqslant d_\lambda-2$ (resp. $-1\leqslant k\leqslant d_\lambda-1$, $-2\leqslant k\leqslant d_\lambda$) and $l(j,k)\in L$, and the double sign depends on either $m\geqslant 0$ or $m<0$. Moreover, we understand $c_{j,k}=0$ if $j\notin J$, $k<0$, $d_\lambda<k$ or $l(j,k)\notin L$ and $z_1=s$ (resp. n_1-1, -1) for $\pi_1=\mathcal{P}^{\tau}_s$ and $\mathcal{C}^{\tau}_s$ (resp. $\mathcal{D}^{\pm}_{n_1}$, 1_{G_1}).

Proof. This assertion is obtained from Proposition 5.1 by comparing between the coefficients of $(w_l\otimes u^m_j\otimes v^\lambda_k)$ in both sides of the equations $\mathcal{R}(\nabla^{-\beta}_{\rho,\tau_\lambda})\varphi(a_r)=0$ for each $\beta\in\Delta^+_{n,\mathrm{IV}}$ and by making appropriate changes of the indices. □

6.2. THE CASE OF $m\neq 0$

In this subsection, let $(\rho,\mathcal{F}_\rho)=(\rho_{\pi_1,m},\mathcal{W}_{\pi_1}\otimes\mathcal{U}_m)$ be of type $m\neq 0$. For simplicity, we treat only the case of $m>0$, and a variable $x=4\pi m\xi^2$ is used frequently.

We discuss first the case that the Blattner parameter λ is not far from the wall, that is, $d_\lambda=0$ and 1. Let $d_\lambda=0$, i.e. $\Lambda_1-\Lambda_2=1$. From (6.3), it is easy to see that

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$(z_1, j) = (\Lambda_2 + \frac{1}{2}, \frac{1}{2})$ if $c_{j,0}(a_r) \neq 0$. Since $\Lambda_2 + \frac{3}{2} > 0$, then we have $\pi_1 = \mathcal{D}^+_{\Lambda_2+\frac{3}{2}}$ if $\mathcal{J}_{\rho,\pi}(\tau_\lambda) \neq \{0\}$. For such π_1, the coefficient function $c_{\frac{1}{2},0}(a_r)$ of $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ is $x^{\frac{1}{2}(\Lambda_1+1)}\mathrm{e}^{-\frac{1}{2}x}$ up to constants. Let $d_\lambda = 1$, i.e. $\Lambda_1 - \Lambda_2 = 2$. Then we find that if $c_{j,0}(a_r) \neq 0$ (resp. $c_{j,1}(a_r) \neq 0$) then $(z_1, j) = (\Lambda_2 + \frac{1}{2}, \frac{1}{2})$ (resp. $(\Lambda_2 + \frac{3}{2}, \frac{1}{2})$ or $(\Lambda_2 + \frac{1}{2}, \frac{3}{2})$), from (6.2) and (6.3). Hence we have $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \{0\}$ except for the cases of $\pi_1 = \mathcal{D}^+_{\Lambda_2+\frac{3}{2}+k_0}$ with $k_0 = 0$ and 1. For $\pi_1 = \mathcal{D}^+_{\Lambda_2+\frac{3}{2}}$, the pair of the coefficient functions $(c_{\frac{1}{2},0}(a_r), c_{\frac{3}{2},1}(a_r))$ of $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ is $(\sqrt{4\pi m}\, x^{\frac{1}{2}(\Lambda_1+1)}\mathrm{e}^{-\frac{1}{2}x}, x^{\frac{1}{2}\Lambda_1}\mathrm{e}^{-\frac{1}{2}x})$ up to constants. Similarly the coefficient function $c_{\frac{1}{2},1}(a_r)$ of φ for $\pi_1 = \mathcal{D}^+_{\Lambda_2+\frac{5}{2}}$ is $x^{\frac{1}{2}\Lambda_1}\mathrm{e}^{-\frac{1}{2}x}$ up to constants.

Next we consider the case of $d_\lambda \geqslant 2$. The following lemma is a consequence of Proposition 6.1.

LEMMA 6.2. *Let $d_\lambda \geqslant 2$. Then each coefficient $c_{j,k}(a_r)$ of $\varphi \in \mathcal{J}_{\rho,\pi}(\tau)|_{A_J}$ satisfies the following relations.*

$$4\pi m\xi\big(j + \tfrac{1}{2}\big)c_{j+1,k+1}(a_r) = (k+1)c_{j,k}(a_r), \tag{6.4}$$

$$8\pi m\xi^2(z_1 - l(j,k) - 1)c_{j,k+2}(a_r) = -(k+1)(k+2)c_{j,k}(a_r), \tag{6.5}$$

Here (6.4) (resp. (6.5)) is valid for $j \in J$, $-1 \leqslant k \leqslant d_\lambda - 1$ (resp. $-1 \leqslant k \leqslant d_\lambda - 2$) and $l(j,k) \in L$.

Proof. From Proposition 6.1, the relations

$$\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} - (\Lambda_1 + 2) + 4\pi m\xi^2\right\}c_{j,k}(a_r) + 4\pi m\xi\big(j + \tfrac{1}{2}\big)c_{j+1,k+1}(a_r) = 0, \tag{6.6}$$

$$\begin{aligned}(k+1)&\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} - (\Lambda_2 + 2) + 4\pi m\xi^2\right\}c_{j,k}(a_r) - \\ &- 4\pi m\xi\big(j + \tfrac{1}{2}\big)(d_\lambda - k)c_{j+1,k+1}(a_r) = 0\end{aligned} \tag{6.7}$$

hold. Here (6.6) (resp. (6.7)) which is valid for $j \in J$, $0 \leqslant k \leqslant d_\lambda - 1$ (resp. $-1 \leqslant k \leqslant d_\lambda$) and $l(j,k) \in L$ is obtained from (6.1) (resp. (6.3)) and (6.2). We obtain the relation (6.4) immediately from (6.6) and (6.7).

Next we prove (6.5). From (6.2) and (6.3), the relations

$$\begin{aligned}(k+1)(k+2)&c_{j,k}(a_r) - 4\pi m\xi\big(j + \tfrac{1}{2}\big)(k+2)c_{j+1,k+1}(a_r) + \\ &+ 2\pi m\big(j + \tfrac{1}{2}\big)\big(j + \tfrac{3}{2}\big)(d_\lambda - 1 - k)c_{j+2,k+2}(a_r) + \\ &+ (z_1 - l(j,k) - 1)(d_\lambda - 1 - k)c_{j,k+2}(a_r) = 0\end{aligned}$$

holds for $j \in J$, $-1 \leqslant k \leqslant d_\lambda - 1$ and $l(j,k) \in L$. Combining this with (6.4), we obtain Equation (6.5). □

From (6.4), we find that $c_{j,k}(a_r) = 0$ except for the indices $j \in J$ and $0 \leqslant k \leqslant d_\lambda$ such that $l(\frac{1}{2}, 0) = \Lambda_2 + \frac{3}{2} \leqslant l(j,k) \leqslant \Lambda_1 + \frac{1}{2} = l(\frac{1}{2}, d_\lambda)$. Hence, we have $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \{0\}$

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except for the case of $\pi_1 = \mathcal{D}^+_{n_1}$ with $\Lambda_2 + \frac{3}{2} \leqslant n_1 \leqslant \Lambda_1 + \frac{1}{2}$. Here we use the relation (6.3) for $k = -2$ and (6.5). If we put $k_0 = n_1 - \Lambda_2 - \frac{3}{2}$ for such n_1, then $l(\frac{1}{2}, k_0) = n_1$ and the coefficient $c_{\frac{1}{2},k_0}(a_r)$ is given by a solution of the differential equation $\{2x(\mathrm{d}/\mathrm{d}x) - (\Lambda_1 + 1 - k_0) + x\}c_{\frac{1}{2},k_0}(a_r) = 0$ from Proposition 6.1. Thus we have $c_{\frac{1}{2},k_0}(a_r) = c \cdot x^{\frac{1}{2}(\Lambda_1+1-k_0)}\mathrm{e}^{-\frac{1}{2}x}$ with some constant c.

Now we can state the following result.

THEOREM 6.3. *Let $\pi = \pi_\Lambda$ be a holomorphic discrete series representation of G with the Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2) \in \Xi_{\mathrm{I}}$ and τ^*_λ be the minimal K-type of π. Moreover let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ be an irreducible unitary representation of R_J of type $m > 0$. Then the space $\mathcal{J}_{\rho,\pi}(\tau_\lambda)$ is not zero iff $\pi_1 = \mathcal{D}^+_{n_1}$ with $\Lambda_2 + \frac{3}{2} \leqslant n_1 \leqslant \Lambda_1 + \frac{1}{2}$. For such π_1, we have $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = \dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$ and the coefficient function $c_{\frac{1}{2},k_0}(a_r)$ with $k_0 = n_1 - \Lambda_2 - \frac{3}{2}$ of $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ is $x^{\frac{1}{2}(\Lambda_1+1-k_0)}\mathrm{e}^{-\frac{1}{2}x}$ up to constants. The other coefficients are given by the formulas (6.4) and (6.5) inductively from $c_{\frac{1}{2},k_0}(a_r)$.*

Similarly, we can prove the following theorem for $m < 0$.

THEOREM 6.4. *Let π and τ_λ be as in Theorem 6.3, and let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ be of type $m < 0$. Then $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.*

The 'symmetric' argument shows the following results.

THEOREM 6.5. *Let $\pi = \pi_\Lambda$ be an anti-holomorphic discrete series representation of G with the Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2) \in \Xi_{\mathrm{IV}}$ and τ^*_λ be the minimal K-type of π with $\lambda = (-\Lambda_2 + 1, -\Lambda_1 + 2)$. Moreover, let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ be an irreducible unitary representation of R_J of type $m < 0$. Then the space $\mathcal{J}_{\rho,\pi}(\tau_\lambda)$ is not zero iff $\pi_1 = \mathcal{D}^-_{n_1}$ with $-\Lambda_1 + \frac{3}{2} \leqslant n_1 \leqslant -\Lambda_2 + \frac{1}{2}$. For such π_1, we have $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = \dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$ and the coefficient function $c_{-\frac{1}{2},k_0}(a_r)$ with $k_0 = -n_1 - \Lambda_2 + \frac{1}{2}$ of $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ expressed as (4.2) with $l(j,k) = -j + k + \Lambda_2 - 1$ is $x'^{\frac{1}{2}(-\Lambda_1+2+k_0)}\mathrm{e}^{-\frac{1}{2}x'}$ up to constants, where $x' = -4\pi m\xi^2$. The other coefficients are given from $c_{-\frac{1}{2},k_0}(a_r)$ inductively by the formulas*

$$4\pi m\xi\left(j - \tfrac{1}{2}\right)c_{j-1,k-1}(a_r) = -(d_\lambda - k + 1)c_{j,k}(a_r),$$
$$8\pi m\xi^2(z_1 + l(j,k) - 1)c_{j,k-2}(a_r) = (d_\lambda - k + 1)(d_\lambda - k + 2)c_{j,k}(a_r),$$

Here the first (resp. the second) formula is valid for $j \in J$, $1 \leqslant k \leqslant d_\lambda + 1$ (resp. $2 \leqslant k \leqslant d_\lambda + 1$) and $l(j,k) \in L$.

THEOREM 6.6. *Let π and τ_λ be as in Theorem 6.5. Moreover let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ be of type $m > 0$. Then $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.*

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6.3. THE CASE OF $m = 0$

In this subsection, let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,0}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_0)$ be of type 0. In this case, we can write $\varphi \in C^\infty(A_J; \rho, \tau)r$ as

$$\varphi(a_r) = \sum_{\substack{k=0 \\ l=l(0,k)\in L}}^{d_\lambda} c_{0,k}(a_r)(w_l \otimes u_0^0 \otimes v_k^\lambda). \tag{6.8}$$

First, we consider the case of $\pi_1 \neq 1_{G_1}$. Let $d_\lambda = 0$, i.e. $\Lambda_1 - \Lambda_2 = 1$. From (6.3), it is easy to see that if $c_{0,0}(a_r) \neq 0$ then $z_1 = \Lambda_1 = \Lambda_2 + 1$. Hence we have $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \{0\}$ except for the case of $\pi_1 = \mathcal{D}^+_{\Lambda_1+1}$. For such π_1, we have $c_{0,0}(a_r) = c \cdot \xi^{\Lambda_2+2}$ with some constant c. Next, let $d_\lambda = 1$, i.e. $\Lambda_1 - \Lambda_2 = 2$. From (6.2) and (6.3), we can find immediately that $c_{0,0}(a_r) = 0$ and that if $c_{0,1}(a_r) \neq 0$ then $z_1 = \Lambda_1$. Therefore, if $\mathcal{J}_{\rho,\pi}(\tau_\lambda) \neq \{0\}$ then $\pi_1 = \mathcal{D}^+_{\Lambda_1+1}$. For such π_1, $c_{0,1}(a_r)$ is ξ^{Λ_2+2} up to constants. Now let us consider the case that the Blattner parameter λ is far from the wall, that is $d_\lambda \geqslant 2$. Proposition 6.1 induces the relations

$$\begin{aligned}
&(k+1)c_{0,k}(a_r) - (z_1 - k - \Lambda_2 - 3)c_{0,k+2}(a_r) = 0,\\
&(k+1)(k+2)c_{0,k}(a_r) - (z_1 - k - \Lambda_2 - 3)(d_\lambda - 1 - k)c_{0,k+2}(a_r) = 0,
\end{aligned}$$

of $c_{0,k}(a_r)$ and $c_{0,k+2}(a_r)$. Here the first relation is valid for $0 \leqslant k \leqslant d_\lambda - 2$ and the second for $-1 \leqslant k \leqslant d_\lambda - 1$. From these relations, we find that $c_{0,k}(a_r) = 0$ for $0 \leqslant k \leqslant d_\lambda - 1$. Then we have $z_1 = \Lambda_1$ if $c_{0,d_\lambda}(a_r) \neq 0$ from (6.1) and, hence, $\pi_1 = \mathcal{D}^+_{\Lambda_1+1}$ if $\mathcal{J}_{\rho,\pi}(\tau_\lambda) \neq \{0\}$. The coefficient $c_{0,d_\lambda}(a_r)$ is ξ^{Λ_2+2} up to constants.

Next let $\pi_1 = 1_{G_1}$. Then $\rho = \rho_{1_{G_1},0}$ is the trivial representation 1_{R_J} of R_J, and $\varphi \in C^\infty(A_J; \rho, \tau_\lambda)$ can be expressed as

$$\varphi(a) = \begin{cases} 0, & \text{for } \lambda_1 < 0 \text{ or } \lambda_2 > 0,\\ c_{0,\lambda_1}(a)(w_0 \otimes u_0^0 \otimes v_{\lambda_1}^\lambda), & \text{for } \lambda_1 \geqslant 0 \text{ and } \lambda_2 \leqslant 0. \end{cases} \tag{6.9}$$

Taking acount of the inequality $\Lambda_2 > 0$, we have $\lambda_2 = -\Lambda_2 - 2 < 0$ and, hence, $C^\infty(A_J; 1_{R_J}, \tau_\lambda) = \{0\}$ from (6.9).

Now we obtain the following result.

THEOREM 6.7. *Let π and τ_λ^* be as in Theorem 6.3. Moreover, let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,0}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_0)$ be an irreducible unitary representation of R_J of type 0. Then the space $\mathcal{J}_{\rho,\pi}(\tau_\lambda)$ is not zero iff $\pi_1 = \mathcal{D}^+_{\Lambda_1+1}$. For such π_1, we have $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = \dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$, and $\varphi \in \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda)|_{A_J}$ can be written as $c_{0,d_\lambda}(a_r)$ $(w_{\Lambda_1+1} \otimes u_0^0 \otimes v_{d_\lambda}^\lambda)$ by the coefficient function $c_{0,d_\lambda}(a_r) = c \cdot \xi^{\Lambda_2+2}$ with some constant c.*

The 'symmetric' argument shows the following result.

THEOREM 6.8. *Let π and τ_λ^* be as in Theorem 6.5. Moreover, let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,0}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_0)$ be an irreducible unitary representation of R_J of type 0. Then*

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the space $\mathcal{J}_{\rho,\pi}(\tau_\lambda)$ *is not zero iff* $\pi_1 = \mathcal{D}^-_{-\Lambda_2+1}$. *For such* π_1, *we have* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = \dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$, *and* $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ *can be written as* $c_{0,0}(a_r)$ $(w_{\Lambda_2-1} \otimes u_0^0 \otimes v_0^\lambda)$ *by the coefficient function* $c_{0,0}(a_r) = c \cdot \xi^{-\Lambda_1+2}$ *with some constant* c.

7. The Large Case

As Section 6, we describe the spaces $\mathcal{J}_{\rho,\pi}(\tau)$ and $\mathcal{J}^\circ_{\rho,\pi}(\tau)$ for discrete series representations π of G of type II or III, the minimal K-types τ^* of π, and irreducible unitary representations $\rho = \rho_{\pi_1,m}$ of R_J of type m, using Proposition 5.1. The case of $m \neq 0$ is treated in Section 7.2, and that of $m = 0$ in Section 7.3. For simplicity, we discuss only for π of type II, because the 'symmetric' argument holds for the case π of type III.

7.1. DIFFERENTIAL EQUATIONS

Let $\pi = \pi_\Lambda$ be a large discrete series representation of G with the Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2) \in \Xi_{\mathrm{II}}$ and τ^*_λ be the minimal K-type of π, i.e. $\lambda = (-\Lambda_2, -\Lambda_1 - 1)$ from the Blattner formula. For each $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$, we express $\varphi \in C^\infty(A_J; \rho, \tau_\lambda)$ as (4.2) with $l(j,k) = -j + k + \Lambda_2$.

PROPOSITION 7.1. *Let* π, τ_λ, *and* ρ *be as above. Then* $\varphi \in C^\infty(A_J; \rho, \tau_\lambda)$ *is in* $\mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ *iff each coefficient* $c_{j,k}(a_r)$ *of* φ *satisfies the following system of differential equations.*

$$\begin{aligned} &A^\pm_{j,k} c_{j,k}(a_r) + B^\pm_{j+1,k+1} c_{j+1,k+1}(a_r) + \\ &\quad + C^\pm_{j+2,k+2} c_{j+2,k+2}(a_r) + D^\pm_{j,k+2} c_{j,k+2}(a_r) = 0, \end{aligned} \tag{7.1}$$

$$\begin{aligned} &E^\pm_{j,k} c_{j,k}(a_r) + F^\pm_{j+1,k+1} c_{j+1,k+1}(a_r) + \\ &\quad + G^\pm_{j+2,k+2} c_{j+2,k+2}(a_r) + H^\pm_{j,k+2} c_{j,k+2}(a_r) = 0, \end{aligned} \tag{7.2}$$

$$\begin{aligned} &I^\pm_{j+2,k+2} c_{j+2,k+2}(a_r) + J^\pm_{j+1,k+1} c_{j+1,k+1}(a_r) + \\ &\quad + K^\pm_{j,k} c_{j,k}(a_r) + L^\pm_{j+2,k} c_{j+2,k}(a_r) = 0, \end{aligned} \tag{7.3}$$

where $A^\pm_{j,k}$, $B^\pm_{j,k}$, $C^\pm_{j,k}$, $D^\pm_{j,k}$, $F^\pm_{j,k}$, $G^\pm_{j,k}$ *and* $H^\pm_{j,k}$ *are given in Proposition 6.1, and*

$$E^\pm_{j,k} = (k+1)\left\{\xi \frac{\mathrm{d}}{\mathrm{d}\xi} - (k + \Lambda_2 + 2) + 4\pi m \xi^2\right\},$$

$$I^\pm_{j,k} = \xi \frac{\mathrm{d}}{\mathrm{d}\xi} + (k - 2d_\lambda + \Lambda_1 - 1) - 4\pi m \xi^2, \quad J^+_{j,k} = 4\xi, \quad J^-_{j,k} = -8\pi m\left(j + \tfrac{1}{2}\right)\xi,$$

$$K^+_{j,k} = -\frac{1}{2\pi m}, \quad K^-_{j,k} = -2\pi m\left(j + \tfrac{1}{2}\right)\left(j + \tfrac{3}{2}\right), \quad L^\pm_{j,k} = z_1 + l(j,k) + 1.$$

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Here Equation (7.1) (resp. (7.2), (7.3)) is valid for $j \in J$ *(resp.* $j \in J, j+2 \in J$*),* $0 \leqslant k \leqslant d_\lambda - 2$ *(resp.* $-1 \leqslant k \leqslant d_\lambda - 1$, $0 \leqslant k \leqslant d_\lambda - 2$*), and* $l(j,k) \in L$*, and the double sign depends on either* $m \geqslant 0$ *or* $m < 0$*. Moreover we understand* $c_{j,k} = 0$ *if* $j \notin J$, $k < 0$, $k > d_\lambda$ *or* $l(j,k) \notin L$ *and* $1/m = 0$ *for* $m = 0$*. The parameter* z_1 *means* s *(resp.* $n_1 - 1$, -1*) for* $\pi_1 = \mathcal{P}_s^\tau$ *and* $\mathcal{C}_s^\tau$ *(resp.* $\mathcal{D}_{n_1}^{\pm}$, 1_{G_1}*).*

Proof. This assertion can be obtained similarly to Proposition 6.1. □

7.2. THE CASE OF $m \neq 0$

In this subsection, let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ be an irreducible unitary representation of R_J of type $m \neq 0$. For simplicity, we treat only the case of $m > 0$, and a variable $x = 4\pi m\xi^2$ is used frequently as before.

LEMMA 7.2. *Each coefficient* $c_{j,k}(a_r)$ *of* $\varphi(a_r) \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ *satisfies the following relations.*

$$\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} - (\Lambda_1 + 2) + 4\pi m\xi^2\right\} c_{j,k}(a_r) + 4\pi m\xi\left(j + \tfrac{1}{2}\right) c_{j+1,k+1}(a_r) = 0, \tag{7.4}$$

$$\begin{aligned}\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (k - \Lambda_1 - 1) + 4\pi m\xi^2\right\} c_{j,k}(a_r) - 2\pi m\left(j + \tfrac{1}{2}\right)\left(j + \tfrac{3}{2}\right) c_{j+2,k+2}(a_r) - {}\\ - (z_1 - l(j,k) - 1) c_{j,k+2}(a_r) = 0,\end{aligned} \tag{7.5}$$

$$\begin{aligned}\left\{\xi^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - 2(\Lambda_1 + 2)\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + \alpha_0(\xi)\right\} c_{j,k}(a_r) + {}\\ + 8\pi m\xi^2(z_1 - l(j,k) - 1) c_{j,k+2}(a_r) = 0,\end{aligned} \tag{7.6}$$

where

$$\alpha_0(\xi) = (\Lambda_1 + 2)(\Lambda_1 + 3) - (2k+1)4\pi m\xi^2 - 16\pi^2 m^2 \xi^4.$$

Here (7.4) (resp. (7.5) and (7.6)) is valid for $j \in J$, $0 \leqslant k \leqslant d_\lambda - 1$ *(resp.* $0 \leqslant k \leqslant d_\lambda - 2$*) and* $l(j,k) \in L$*.*

Proof. For $0 \leqslant k \leqslant d_\lambda - 2$, the relations (7.4) and (7.5) are obtained from (7.1) and (7.2). For $k = d_\lambda - 1$, the identity (7.4) coincides with (7.2).

By using (7.4), the equation

$$\left\{\xi^2\frac{\mathrm{d}}{\mathrm{d}\xi^2} + \beta_1(\xi)\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + \beta_0(\xi)\right\} c_{j,k}(a_r) - 16\pi^2 m^2 \xi^2\left(j + \tfrac{1}{2}\right)\left(j + \tfrac{3}{2}\right) c_{j+2,k+2}(a_r) = 0$$

holds for $j \in J$, $0 \leqslant k \leqslant d_\lambda - 2$ and $l(j,k) \in L$, where

$$\beta_1(\xi) = -(2\Lambda_1 + 4 - 8\pi m\xi^2),$$
$$\beta_0(\xi) = (\Lambda_1 + 2)(\Lambda_1 + 3) - (2\Lambda_1 + 3)4\pi m\xi^2 + 16\pi^2 m^2 \xi^4.$$

Combining this with (7.5), we have (7.6). □

The next two lemmas assert that we may decide only suitable finite numbers of the coefficient functions to solve the system in Proposition 7.1.

LEMMA7.3. *Fix $j_0 \in J$ and $0 \leqslant k_0 \leqslant d_\lambda - 3$ such that $l_0 = l(j_0, k_0) \in L$. Take a number N such that $2 \leqslant N \leqslant d_\lambda - 1 - k_0$. Let us assume that $l_0 + 2 \in L$ and the functions $c_{j,k}(a_r)$ with $l(j,k) = l_0$, $l_0 + 2$ and $k_0 \leqslant k \leqslant k_0 + N$ satisfies the system of equations in Proposition 7.1. If we define two functions $c_{j_0+N\pm 1, k_0+N+1}(a_r)$ by (7.4) from $c_{j_0+N-1\pm 1, k_0+N}(a_r)$, then these satisfy the system in Proposition 7.1.*

Proof. In order to get the assertion, it suffices to verify the relations (7.5) for $j = j_0 + N - 1$ and $k = k_0 + N - 1$ and (7.3) for $j = j_0 + N - 3$ and $k = k_0 + N - 1$. Both of these relations can be seen by using the definition by (7.4) and the assumption of the lemma through direct computation. First we prove the relation (7.5) for $j = j_N = j_0 + N - 1$ and $k = k_N = k_0 + N - 1$. By using (7.4), the left hand side of (7.5) for $j = j_N$ and $k = k_N$ becomes

$$\begin{aligned} &-\frac{1}{4\pi m\xi(j_N - \frac{1}{2})}\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} - (\Lambda_1 + 2) + 4\pi m\xi^2\right\}\times \\ &\quad\times\left[\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (k_N - 2 - \Lambda_1) + 4\pi m\xi^2\right\}c_{j_N-1,k_N-1}(a_r) - \right. \\ &\quad\left. - 2\pi m(j_N + \tfrac{1}{2})(j_N - \tfrac{1}{2})c_{j_N+1,k_N+1}(a_r) - (z_1 - l(j_N, k_N) - 1)c_{j_N-1,k_N+2}(a_r)\right]. \end{aligned}$$

This equals to zero from the relation (7.5) for $j = j_N - 1$ and $k = k_N - 1$. Next we show (7.3) for $j = j'_N = j_0 + N - 3$ and $k = k_N$. Since

$$\begin{aligned} &L^+_{j'_N+2,k_N} c_{j'_N+2,k_N}(a_r) \\ &\quad = -L^+_{j'_N+1,k_N-1}\cdot\frac{1}{4\pi m\xi(j'_N + \frac{3}{2})}\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} - (\Lambda_1 + 2) + 4\pi m\xi^2\right\}c_{j'_N+1,k_N-1}(a_r) \\ &\quad = \frac{1}{4\pi m\xi(j'_N + \frac{3}{2})}\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} - (\Lambda_1 + 2) + 4\pi m\xi^2\right\}\times \\ &\qquad\times\left[I^+_{j'_N+1,k_N+1}c_{j'_N+1,k_N+1}(a_r) + J^+_{j'_N,k_N}c_{j'_N,k_N}(a_r) + K^+_{j'_N-1,k_N-1}c_{j'_N-1,k_N-1}(a_r)\right] \end{aligned}$$

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from (7.4) and (7.3), the left hand side of (7.3) for $j = j'_N$ and $k = k_N$ becomes

$$\begin{aligned}
&\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (k_N + 1 - 2d_\lambda + \Lambda_1) - 4\pi m\xi^2\right\} c_{j'_N+2,k_N+2}(a_r) + \\
&\quad + \left[4\xi + \frac{1}{4\pi m\xi(j'_N + \frac{3}{2})}\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} - (\Lambda_1 + 2) + 4\pi m\xi^2\right\} I^+_{j'_N+1,k_N+1}\right] c_{j'_N+1,k_N+1}(a_r) + \\
&\quad + \left[-\frac{1}{2\pi m} + \frac{1}{4\pi m\xi(j'_N + \frac{3}{2})}\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} - (\Lambda_1 + 2) + 4\pi m\xi^2\right\} J^+_{j'_N,k_N}\right] c_{j'_N,k_N}(a_r) + \\
&\quad + \frac{1}{4\pi m\xi(j'_N + \frac{3}{2})}\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} - (\Lambda_1 + 2) + 4\pi m\xi^2\right\} K^+_{j'_N-1,k_N-1} c_{j'_N-1,k_N-1}(a_r).
\end{aligned}$$

Using the relation (7.4), we find that this is zero. □

LEMMA 7.4. *Fix $l \in L$.*

(1) *Suppose that $l - 2 \in L$ and that the functions $c_{j,k}(a_r)$ with $l(j,k) = l, l+2, j \in J$ and $0 \leqslant k \leqslant d_\lambda$ satisfy the system of equations in Proposition 7.1. Let us define $c_{j,k}(a_r)$ with $l(j,k) = l - 2$ and $j \in J$ by (7.3) for $0 \leqslant k \leqslant d_\lambda - 2$ and by (7.4) for $k = d_\lambda - 1$, d_λ. Then the functions $c_{j,k}(a_r)$ with $l(j,k) = l$, $l - 2$, $j \in J$ and $0 \leqslant k \leqslant d_\lambda$ satisfy the system in Proposition 7.1.*

(2) *Suppose that $l > 0$ and that the functions $c_{j,k}(a_r)$ with $l(j,k) = l$, $l - 2$, $j \in J$ and $0 \leqslant k \leqslant d_\lambda$ satisfy the system of equations in Proposition 7.1. If $l - 2$ is not in L, we assume further that the functions $c_{j,k}(a_r)$ with $l(j,k) = l$ satisfy (7.4). Moreover, define $c_{j,k}(a_r)$ with $l(j,k) = l + 2$ by (7.1). Then the functions $c_{j,k}(a_r)$ with $l(j,k) = l$, $l + 2$, $j \in J$ and $0 \leqslant k \leqslant d_\lambda$ satisfy the system in Proposition 7.1.*

Proof. To see the assertion (1), it suffices to prove that the relations (7.4) for $j \in J$ and $0 \leqslant k \leqslant d_\lambda - 3$ such that $l(j,k) = l - 2$, (7.1) for $j \in J$ and $0 \leqslant k \leqslant d_\lambda - 2$ such that $l(j,k) = l - 2$, and (7.2) for $j \in J$ and $k = -1$ such that $l(j,-1) = l - 2$ hold. For the assertion (2), we need to demonstrate that the relations (7.2) and (7.3) hold. In place of (7.2), we may verify that the functions $c_{j,k}(a_r)$ and $c_{j+1,k+1}(a_r)$ with the indecies $j \in J$ and $0 \leqslant k \leqslant d_\lambda - 1$ such that $l(j,k) = l \in L$ satisfy the relation (7.4). In analogy to the proof of Lemma 7.3, the above mentioned relations can be shown from the definition of each function $c_{j,k}(a_r)$ and the assumption of the lemma through direct computation. We leave the detail to the reader. □

Let $\pi_1 = \mathcal{P}^\tau_s$ $(s \in \sqrt{-1}\mathbb{R}, \tau = \pm\frac{1}{2})$ or $\mathcal{C}^\tau_s$ $(0 < s < \frac{1}{2}, \tau = \pm\frac{1}{2})$ and fix an index k_0 such that $l(\frac{1}{2}, k_0) \in L$ and $0 \leqslant k_0 \leqslant d_\lambda - 2$. Then the coefficients $c_{\frac{1}{2},k_0}(a_r)$ and $c_{\frac{1}{2},k_0+2}(a_r)$ satisfy the equation

$$\begin{aligned}
&\left\{\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + (k_0 - 2d_\lambda + \Lambda_1 + 1) - 4\pi m\xi^2\right\} c_{\frac{1}{2},k_0+2}(a_r) + \\
&\quad + \left(z_1 + l(\tfrac{1}{2}, k_0) + 1\right) c_{\frac{1}{2},k_0}(a_r) = 0
\end{aligned}$$

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from (7.3) for $j = -\frac{3}{2}$. This equation together with the relation (7.6) for $j = \frac{1}{2}$ and $k = k_0$ leads the following differential equation for $c_{\frac{1}{2},k_0}(a_r)$;

$$\left\{8x^3 \frac{\mathrm{d}^3}{\mathrm{d}x^3} + \gamma_2(x)4x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \gamma_1(x)2x\frac{\mathrm{d}}{\mathrm{d}x} + \gamma_0(x)\right\} c_{\frac{1}{2},k_0}(a_r) = 0, \tag{7.8}$$

where

$$\begin{aligned}
\gamma_2(x) &= k_0 - 2d_\lambda - \Lambda_1 - x,\\
\gamma_1(x) &= (\Lambda_1 + 2)(\Lambda_1 + 3) - (2\Lambda_1 + 3)(k_0 - 2d_\lambda + \Lambda_1 + 1) - 2(k_0 - \Lambda_1 - 1)x - x^2,\\
\gamma_0(x) &= (\Lambda_1 + 2)(\Lambda_1 + 3)(k_0 - 2d_\lambda + \Lambda_1 - 1) -\\
&\quad - \Big\{(2k_0 + 1)(k_0 - 2d_\lambda + \Lambda_1 + 1) + (\Lambda_1 + 2)(\Lambda_1 + 3) +\\
&\quad + 2\big(z_1 + l(\tfrac{1}{2}, k_0) + 1\big)\big(z_1 - l(\tfrac{1}{2}, k_0) - 1\big)\Big\}x +\\
&\quad + (k_0 + 2d_\lambda - \Lambda_1 - 2)x^2 + x^3.
\end{aligned}$$

This is also true for the indices $k_0 = d_\lambda - 1$ and d_λ. Putting $c_{\frac{1}{2},k_0}(a_r) = e^{\frac{1}{2}x}y$, we find that y satisfies the differential equation of Meijer [3; 5.4(1), p. 210] for $(p, q, m, n) = (2, 3, 3, 0)$ (cf. (A.17) in Appendix) with the parameters

$$\begin{aligned}
a_1 &= \frac{2z_1 + 5 + 2d_\lambda}{4}, \quad a_2 = \frac{-2z_1 + 5 + 2d_\lambda}{4},\\
b_1 &= \frac{\Lambda_1 + 2}{2}, \quad b_2 = \frac{\Lambda_1 + 3}{2}, \quad b_3 = \frac{-k_0 + 2d_\lambda - \Lambda_1 + 1}{2}.
\end{aligned}$$

Therefore, up to constant multiple, the solution $c_{\frac{1}{2},k_0}(a_r)$ of (7.8) satisfying the moderate growth condition is $\mathrm{e}^{\frac{1}{2}x} G^{3,0}_{2,3}\left(x \,\middle|\, \begin{matrix} a_1, a_2 \\ b_1, b_2, b_3 \end{matrix}\right)$ with the above parameters. If we determine $c_{\frac{1}{2},k_0+2}(a_r)$ (resp. $c_{\frac{1}{2},k_0-2}(a_r)$) by the relation (7.6) (resp. (7.7)) and $c_{\frac{1}{2}+i,k_0+i}(a_r)$ (resp. $c_{\frac{1}{2}+i,k_0-2+i}(a_r)$) for $i = 1, 2$ by (7.4) from each solution of (7.8) for $0 \leqslant k_0 \leqslant d_\lambda - 2$ (resp. $k_0 = d_\lambda - 1$ and d_λ), then these functions satisfy the system in Proposition 7.1 obviously.

Next let $\pi_1 = \mathcal{D}^-_{n_1}$ with $n_1 \leqslant -\Lambda_2 + \frac{1}{2}$. Then $k_0 = -n_1 + \frac{1}{2} - \Lambda_2$ is the maximum value in the set of numbers k such that $l(\frac{1}{2}, k) \in L$ and $0 \leqslant k \leqslant -\Lambda_2 < d_\lambda$. Since $c_{\frac{1}{2},k_0+2}(a_r) = 0$, we have a differential equation

$$\left\{4x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - (2\Lambda_1 + 3)2x\frac{\mathrm{d}}{\mathrm{d}x} + \alpha_0(x)\right\} c_{\frac{1}{2},k_0}(a_r) = 0, \tag{7.9}$$

from (7.6) for $j = \frac{1}{2}$ and $k = k_0$, where

$$\alpha_0(x) = (\Lambda_1 + 2)(\Lambda_1 + 3) - (2k_0 + 1)x - x^2.$$

Putting $c_{\frac{1}{2},k_0}(a_r) = \mathrm{e}^{\frac{1}{2}x}y$, we find that y satisfies the differential equation of Meijer for

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$(p, q, m, n) = (1, 2, 2, 0)$ (cf. (A.18) in Appendix) with the parameters

$$a_1 = \frac{\Lambda_1 + 4 + k_0}{2}, \quad b_1 = \frac{\Lambda_1 + 2}{2}, \quad b_2 = \frac{\Lambda_1 + 3}{2}.$$

Therefore the solution $c_{\frac{1}{2},k_0}(a_r)$ of (7.9) with the moderate growth property is

$$e^{\frac{1}{2}x} G^{2,0}_{1,2}\left(x \,\middle|\, \begin{matrix} a_1, a_2 \\ b_1, b_2, b_3 \end{matrix} \right) = x^{\frac{3+2\Lambda_1}{4}} W_{\kappa,\mu}(x)$$

up to constants, where the parameters $(a_1,\ b_1,\ b_2)$ are as above,

$$\kappa = -\frac{1 + 2k_0}{4} \quad \text{and} \quad \mu = -\tfrac{1}{4}.$$

Of course, if we determine $c_{\frac{1}{2}+i,k_0+i}(a_r)$ $(i = 1,\ \ 2)$ by (7.4) from each solution of (7.9), then these coefficients satisfy the system in Proposition 7.1 clearly.

Let $\pi_1 = \mathcal{D}^-_{n_1}$ with $n_1 > -\Lambda_2 + \frac{1}{2}$. In this case, there exists no number k such that $l(\frac{1}{2}, k) \in L$ and $0 \leqslant k \leqslant d_\lambda$. Let j_0 be the minimum value of $j \in J$ such that $l(j, 0) \in L$, that is $j_0 = n_1 + \Lambda_2$. Since $j_0 \neq \frac{1}{2}$ and $d_\lambda \neq 0$, we have the relation $2\xi c_{j_0,0}(a_r) + (j_0 + \frac{1}{2})c_{j_0+1,1}(a_r) = 0$ from (7.2) for $j = j_0 - 1$ and $k = -1$. Using this relation and (7.4), we can see that $c_{j_0,0}(a_r) = c \cdot e^{\frac{1}{2}x} x^{\frac{1}{2}(\Lambda_1+2)}$ with some constant c. If we determine the coefficients $c_{j_0+i,i}(a_r)$ $(i = 1,\ \ 2)$ by (7.4), then these satisfy the system in Proposition 7.1, as we can see through direct computation.

Let $\pi_1 = \mathcal{D}^+_{n_1}$. If $n_1 > \Lambda_1 + \frac{1}{2}$, there is no $j \in J$ and $0 \leqslant k \leqslant d_\lambda$ such that $l(j, k) \in L$. Hence $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \{0\}$. When $n_1 \leqslant \Lambda_1 + \frac{1}{2}$, there exists a number k such that $l(\frac{1}{2}, k) \in L$ and $-\Lambda_2 \leqslant k \leqslant d_\lambda$. Let k_0 be the minimum value of k such that $l(\frac{1}{2}, k) \in L$, i.e. $k_0 = n_1 + \frac{1}{2} - \Lambda_2$. From (7.3) for $j = -\frac{3}{2}$ and $k = k_0 - 2$ we can see that

$$c_{\frac{1}{2},k_0}(a_r) = c \cdot e^{\frac{1}{2}x} x^{-\frac{k_0 - 2d_\lambda + \Lambda_1 - 1}{2}}$$

with some constant c. Since $z_1 = n_1 - 1$ and $l(\frac{1}{2}, k_0) = n_1$, the relations (7.1) and (7.2) for the coefficients $c_{\frac{1}{2},k_0}(a_r)$ and $c_{\frac{1}{2}+i,k_0-2+i}(a_r) = 0$ $(0 \leqslant i \leqslant 2)$ hold trivially.

From Lemma 7.3, 7.4 and the above computation for each π_1, we get the following result.

THEOREM 7.5. *Let $\pi = \pi_\Lambda$ be a large discrete series representation of G with the Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2) \in \Xi_{\mathrm{II}}$ and τ^*_λ be the minimal K-type of π. Moreover let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ be an irreducible unitary representation of R_J of type $m > 0$. Then;*

(1) *If $\pi_1 = \mathcal{P}^\tau_s$ $(\tau = \pm\frac{1}{2}, s \in \sqrt{-1}\mathbb{R})$ or $\mathcal{C}^\tau_s$ $(\tau = \pm\frac{1}{2}, 0 < s < \frac{1}{2})$, then* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 3$ *and* $\dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$. *The coefficient function $c_{\frac{1}{2},k_0}(a_r)$ of $\varphi \in \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda)|_{A_J}$ with*

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the index $0 \leqslant k_0 \leqslant d_\lambda$ *such that* $l(\frac{1}{2}, k_0) \in L$ *is*

$$e^{\frac{1}{2}x} G^{3,0}_{2,3}\left(x \,\middle|\, \begin{matrix} \frac{2s+5+2d_\lambda}{4}, \; \frac{-2s+5+2d_\lambda}{4} \\ \frac{\Lambda_1+2}{2}, \; \frac{\Lambda_1+3}{2}, \; \frac{-k_0+d_\lambda-\Lambda_2+2}{2} \end{matrix}\right),$$

up to constants, and the other coefficients are given by (7.1), (7.3), (7.4) and (7.6) inductively from $c_{\frac{1}{2},k_0}(a_r)$.

(2) *If* $\pi_1 = \mathcal{D}^+_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3}\backslash\mathbb{Z},\ n_1 > \Lambda_1 + \frac{1}{2})$, *then* $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.

(3) *If* $\pi_1 = \mathcal{D}^+_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3}\backslash\mathbb{Z},\ n_1 \leqslant \Lambda_1 + \frac{1}{2})$, *then* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 1$ *and* $\mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.

(4) *If* $\pi_1 = \mathcal{D}^-_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3}\backslash\mathbb{Z},\ n_1 \leqslant -\Lambda_2 + \frac{1}{2})$, *then* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 2$ *and* $\dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$. *The coefficient function* $c_{\frac{1}{2},k_0}(a_r)$ *of* $\varphi \in \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda)|_{A_J}$ *with* $k_0 = -n_1 + \frac{1}{2} - \Lambda_2$ *is*

$$e^{\frac{1}{2}x} G^{2,0}_{1,2}\left(x \,\middle|\, \begin{matrix} \frac{\Lambda_1+4+k_0}{2} \\ \frac{\Lambda_1+2}{2}, \; \frac{\Lambda_1+3}{2} \end{matrix}\right) = x^{\frac{3+2\Lambda_1}{4}} W_{\kappa,\mu}(x),$$

up to constants, where $\kappa = -\frac{1}{4}(1+2k_0)$, $\mu = -\frac{1}{4}$. *The other coefficients are given by (7.1) and (7.3) inductively from* $c_{\frac{1}{2},k_0}(a_r)$.

(5) *If* $\pi_1 = \mathcal{D}^-_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3}\backslash\mathbb{Z},\ n_1 > -\Lambda_2 + \frac{1}{2})$, *then* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 1$ *and* $\mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.

Similarly, we can prove the following theorem for $m < 0$.

THEOREM 7.6. *Let* π *and* τ_λ *be as in Theorem 7.5. Moreover, let* $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ *be of type* $m < 0$. *Then;*

(1) *If* $\pi_1 = \mathcal{P}^\tau_s$ $(\tau = \pm\frac{1}{2},\ s \in \sqrt{-1}\mathbb{R})$ *or* $\mathcal{C}^\tau_s$ $(\tau = \pm\frac{1}{2},\ 0 < s < \frac{1}{2})$, *then* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 1$ *if* $\tau + \frac{1}{2} \equiv \Lambda_2 + 1 \pmod 2$, $= 0$ *if* $\tau + \frac{1}{2} \equiv \Lambda_2 \pmod 2$ *and* $\mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.

(2) *If* $\pi_1 = \mathcal{D}^+_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3}\backslash\mathbb{Z},\ n_1 > \Lambda_1 + \frac{1}{2})$, *then* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 3$ *and* $\dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$. *The coefficient function* $c_{j_0,k_0}(a_r)$ *of* $\varphi \in \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda)|_{A_J}$ *with the indices* $j_0 \in J$ *and* $0 \leqslant k_0 \leqslant d_\lambda$ *such that* $l(j_0, k_0) = n_1$ *is*

$$e^{\frac{1}{2}x'} G^{3,0}_{2,3}\left(x' \,\middle|\, \begin{matrix} \frac{2\Lambda_1+5-2j_0}{4}, \; \frac{2\Lambda_1+7-2j_0}{4} \\ \frac{\Lambda_1+2}{2}, \; \frac{\Lambda_1+3}{2}, \; \frac{-k_0+d_\lambda-\Lambda_2+2}{2} \end{matrix}\right), \quad x' = -4\pi m\xi^2,$$

up to constants, and the other coefficients are given from $c_{j_0,k_0}(a_r)$ *inductively.*

(3) *If* $\pi_1 = \mathcal{D}^+_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3}\backslash\mathbb{Z},\ n_1 \leqslant \Lambda_1 + \frac{1}{2})$, *then* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 1$ *and* $\mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.

(4) *If* $\pi_1 = \mathcal{D}^-_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3}\backslash\mathbb{Z})$, *then* $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.

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By the 'symmetric' argument, the following results for discrete series representations of type III hold.

THEOREM 7.7. *Let $\pi = \pi_\Lambda$ be a large discrete series representation of G with the Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2) \in \Xi_{\mathrm{III}}$ and τ_λ^* be the minimal K-type of π with $\lambda = (-\Lambda_2 + 1, -\Lambda_1)$. Moreover, let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ be an irreducible unitary representation of R_J of type $m < 0$. Then;*

(1) *If $\pi_1 = \mathcal{P}_s^\tau$ $(\tau = \pm\frac{1}{2}, s \in \sqrt{-1}\mathbb{R})$ or $\mathcal{C}_s^\tau$ $(\tau = \pm\frac{1}{2}, 0 < s < \frac{1}{2})$, then $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 3$ and $\dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$. For $\varphi \in \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda)|_{A_J}$ expressed as (4.2) with $l(j,k) = -j + k + \Lambda_2 - 1$, the coefficient function $c_{-\frac{1}{2},k_0}(a_r)$ of φ with the index $0 \leqslant k_0 \leqslant d_\lambda$ such that $l(-\frac{1}{2}, k_0) \in L$ is*

$$e^{\frac{1}{2}x'} G^{3,0}_{2,3}\left(x' \,\middle|\, \begin{matrix} \dfrac{2s+5+2d_\lambda}{4}, \dfrac{-2s+5+2d_\lambda}{4} \\ \dfrac{-\Lambda_2+2}{2}, \dfrac{-\Lambda_2+3}{2}, \dfrac{k_0+\Lambda_1+2}{2} \end{matrix} \right), \quad x' = -4\pi m \xi^2,$$

up to constants, and the other coefficients are given from $c_{-\frac{1}{2},k_0}(a_r)$ inductively.

(2) *If $\pi_1 = \mathcal{D}^-_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3} \backslash \mathbb{Z},\ n_1 > -\Lambda_2 + \frac{1}{2})$, then $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.*

(3) *If $\pi_1 = \mathcal{D}^-_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3} \backslash \mathbb{Z},\ n_1 \leqslant -\Lambda_2 + \frac{1}{2})$, then $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 1$ and $\mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.*

(4) *If $\pi_1 = \mathcal{D}^+_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3} \backslash \mathbb{Z},\ n_1 \leqslant \Lambda_1 + \frac{1}{2})$, then $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 2$ and $\dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$. The coefficient function $c_{-\frac{1}{2},k_0}(a_r)$ of $\varphi \in \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda)|_{A_J}$ with $k_0 = n_1 + \frac{1}{2} - \Lambda_2$ is*

$$e^{\frac{1}{2}x'} G^{2,0}_{1,2}\left(x' \,\middle|\, \begin{matrix} \dfrac{-\Lambda_2+4+d_\lambda-k_0}{2} \\ \dfrac{-\Lambda_2+2}{2}, \dfrac{-\Lambda_2+3}{2} \end{matrix} \right) = x^{\frac{3-2\Lambda_2}{4}} W_{\kappa,\mu}(x), \quad x' = -4\pi m \xi^2.$$

up to constants, where $\kappa = -\frac{1}{4}(1 + 2(d_\lambda - k_0))$, $\mu = -\frac{1}{4}$. The other coefficients are given from $c_{-\frac{1}{2},k_0}(a_r)$ inductively.

(5) *If $\pi_1 = \mathcal{D}^+_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3} \backslash \mathbb{Z},\ n_1 > \Lambda_1 + \frac{1}{2})$, then $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 1$ and $\mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.*

THEOREM 7.8. *Let π and τ_λ be as in Theorem 7.7. Moreover, let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,m}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ be of type $m > 0$. Then;*

(1) *If $\pi_1 = \mathcal{P}_s^\tau$ $(\tau = \pm\frac{1}{2}, s \in \sqrt{-1}\mathbb{R})$ or $\mathcal{C}_s^\tau$ $(\tau = \pm\frac{1}{2}, 0 < s < \frac{1}{2})$, then $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 1$ if $\tau + \frac{1}{2} \equiv \Lambda_1 \pmod 2$, $= 0$ if $\tau + \frac{1}{2} \equiv \Lambda_1 + 1 \pmod 2$ and $\mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.*

(2) *If $\pi_1 = \mathcal{D}^-_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3} \backslash \mathbb{Z},\ n_1 + > -\Lambda_2 + \frac{1}{2})$, then $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 3$ and $\dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$. For $\varphi \in \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda)|_{A_J}$ expressed as (4.2) with $l(j,k) = -j + k + \Lambda_2 - 1$, the coefficient function $c_{j_0,k_0}(a_r)$ of φ with the indices $j_0 \in J$ and*

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$0 \leqslant k_0 \leqslant d_\lambda$ *such that* $l(j_0, k_0) = -n_1$ *is*

$$e^{\frac{1}{2}x} G^{3,0}_{2,3}\left(x \left| \begin{matrix} \dfrac{-2\Lambda_2+5+2j_0}{4}, & \dfrac{-2\Lambda_2+7+2j_0}{4} \\ \dfrac{-\Lambda_2+2}{2}, \quad \dfrac{-\Lambda_2+3}{2}, & \dfrac{k_0+\Lambda_1+2}{2} \end{matrix} \right. \right),$$

up to constants, and the other coefficients are given from $c_{j_0,k_0}(a_r)$ *inductively.*

(3) *If* $\pi_1 = \mathcal{D}^-_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3} \backslash \mathbb{Z},\ n_1 \leqslant -\Lambda_2 + \frac{1}{2})$, *then* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = 1$ *and* $\mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.

(4) *If* $\pi_1 = \mathcal{D}^+_{n_1}$ $(n_1 \in \frac{1}{2}\mathbb{Z}_{\geqslant 3} \backslash \mathbb{Z})$, *then* $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = \{0\}$.

7.3. THE CASE OF $m = 0$

In this subsection, let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,0}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_0)$ be of type 0. Then, as before, we can write $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ as (6.8).

First we consider the case of $\pi_1 \neq 1_{G_1}$.

LEMMA 7.9. *In the above setting, the system of the diferential equations in Proposition 7.1 is equivalent to the following system.*

$$(k+1)c_{0,k}(a_r) = (z_1 - k - \Lambda_2 - 1)c_{0,k+2}(a_r), \quad \textit{for } -1 \leqslant k \leqslant d_\lambda - 2, \tag{7.10}$$

$$\left\{\xi\frac{d}{d\xi} - (\Lambda_1+2)\right\}c_{0,k}(a_r) = 0, \quad \textit{for } 0 \leqslant k \leqslant d_\lambda - 1, \tag{7.11}$$

$$(z_1^2 - \Lambda_2^2)c_{0,k}(a_r) = 0, \quad \textit{for } 0 \leqslant k \leqslant d_\lambda - 1, \tag{7.12}$$

$$(z_1 - \Lambda_1)\left\{\xi\frac{d}{d\xi} - (\Lambda_1+2)\right\}c_{0,d_\lambda}(a_r) = 0, \tag{7.13}$$

$$\left[(z_1^2 - \Lambda_2^2) + (\Lambda_1 - \Lambda_2)\left\{\xi\frac{d}{d\xi} - (\Lambda_1+2)\right\}\right]c_{0,d_\lambda}(a_r) = 0. \tag{7.14}$$

Proof. Combining Equations (7.1) with (7.2) to cancel out the differential terms, we obtain (7.10). Equations (7.11) and (7.13) are obtained immediately from (7.1) and (7.10). By Equations (7.1), (7.3) and (7.10), we have

$$\left[(z_1^2 - \Lambda_2^2) + (k+1)\left\{\xi\frac{d}{d\xi} - (\Lambda_1+2)\right\}\right]c_{0,k}(a_r) = 0, \quad \text{for } 0 \leqslant k \leqslant d_\lambda - 2,$$
$$\left[(z_1^2 - \Lambda_2^2) + (k-1)\left\{\xi\frac{d}{d\xi} - (\Lambda_1+2)\right\}\right]c_{0,k}(a_r) = 0, \quad \text{for } 2 \leqslant k \leqslant d_\lambda.$$

Therefore we have (7.12) and (7.14) from these identities together with (7.11). □

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For a non-zero element $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)$ let k_0 be the minimum index such that $c_{0,k}(a_r) \neq 0$. If $k_0 = d_\lambda$, Equation (7.10) for $k = d_\lambda - 2$ shows $z_1 = \Lambda_1$ and, hence, $\pi_1 = \mathcal{D}^+_{\Lambda_1+1}$, for $l(0, d_\lambda) = \Lambda_1 + 1 > 0$. Since $\Lambda_1 \neq \Lambda_2$, Equation (7.14) shows that $c_{0,d_\lambda}(a_r)$ is $c \cdot \xi^{-\Lambda_2+2}$ with some constant c. If $k_0 < d_\lambda$, Equation (7.12) for $k = k_0$ shows $z_1 = -\Lambda_2$ because $\Lambda_2 < 0$. Then we have $\pi_1 = \mathcal{D}^\pm_{-\Lambda_2+1}$. We remark that an index k with $c_{0,k}(a_r) \neq 0$ satisfies the inequality $\Lambda_2 \leqslant k_0 + \Lambda_2 \leqslant l(0,k) = k + \Lambda_2 \leqslant d_\lambda + \Lambda_2$. Since $c_{0,k}(a_r) = 0$ if $l(0,k)$ is not in L, π_1 must be $\mathcal{D}^+_{-\Lambda_2+1}$. The fact $l(0,k_0) \in L$ means the inequality $-2\Lambda_2 + 1 \leqslant k_0$, in particular, $3 \leqslant k_0$. From Equation (7.10) for $k = k_0 - 2$, we have $k_0 = -2\Lambda_2 + 1$. Then $c_{0,k_0}(a_r)$ is $c \cdot \xi^{\Lambda_1+2}$ with some constant c from (7.11). Moreover the other functions $c_{0,k}(a_r)$ with $k > k_0$, $k \equiv k_0 (\mathrm{mod}\ 2)$ are determined by the formula (7.10) inductively. Of course, we have $c_{0,k}(a_r) = 0$ if k is even.

Next let $\pi_1 = 1_{G_1}$. Then $\rho = \rho_{1_{G_1},0}$ is the trivial representation 1_{R_J} of R_J. For $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ expressed as (6.9), the system of differential equations in Proposition 7.1 becomes

$$\left\{\xi \frac{\mathrm{d}}{\mathrm{d}\xi} + (-d_\lambda - 2)\right\} c_{0,\lambda_1}(a_r) = \left\{\xi \frac{\mathrm{d}}{\mathrm{d}\xi} - 2\right\} c_{0,\lambda_1}(a_r) = 0.$$

Since $d_\lambda \neq 0$, we have $c_{0,\lambda_1}(a_r) = 0$ and, hence, $\mathcal{J}_{\rho,\pi}(\tau_\lambda) = \{0\}$.

Now we can state the following theorem.

THEOREM 7.10. *Let π and τ^*_λ be as in Theorem 7.5, and let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,0}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_0)$ be an irreducible unitary representation of R_J of type 0. Then the space $\mathcal{J}_{\rho,\pi}(\tau_\lambda)$ is not zero iff $\pi_1 = \mathcal{D}^+_{\Lambda_1+1}$ or $\mathcal{D}^+_{-\Lambda_2+1}$, and for such π_1 we have* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = \dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$. *When $\pi_1 = \mathcal{D}^+_{\Lambda_1+1}$, $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ can be written as $c_{0,d_\lambda}(a_r)(w_{\Lambda_1+1} \otimes u^0_0 \otimes v^\lambda_{d_\lambda})$ by the coefficient function $c_{0,d_\lambda}(a_r) = c \cdot \xi^{-\Lambda_2+2}$ with some constant c. When $\pi_1 = \mathcal{D}^+_{-\Lambda_2+1}$, the coefficients $c_{0,k}(a_r)$ of $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ are zero except for $k \geqslant k_0 = -2\Lambda_2 + 1$ and $k \equiv k(\mathrm{mod}\ 2)$. The coefficient function $c_{0,k_0}(a_r)$ is ξ^{Λ_1+2} up to constants and the other coefficients are given by the formula (7.10) for $z_1 = -\Lambda_2$ inductively.*

The 'symmetric' argument shows the following result.

THEOREM 7.11. *Let π and τ^*_λ be as in Theorem 7.6, and let $(\rho, \mathcal{F}_\rho) = (\rho_{\pi_1,0}, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_0)$ be an irreducible unitary representation of R_J of type 0. Then the space $\mathcal{J}_{\rho,\pi}(\tau_\lambda)$ is not zero iff $\mathcal{D}^-_{-\Lambda_2+1}$* or *$\mathcal{D}^-_{\Lambda_1+1}$, and for such π_1 we have* $\dim \mathcal{J}_{\rho,\pi}(\tau_\lambda) = \dim \mathcal{J}^\circ_{\rho,\pi}(\tau_\lambda) = 1$. *When $\pi_1 = \mathcal{D}^-_{-\Lambda_2+1}$, $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ can be written as $c_{0,0}(a_r)(w_{\Lambda_2-1} \otimes u^0_0 \otimes v^\lambda_0)$ by the coefficient function $c_{0,0}(a_r) = c \cdot \xi^{\Lambda_1+2}$ with some constant c. When $\pi_1 = \mathcal{D}^-_{\Lambda_1+1}$, the coefficients $c_{0,k}(a_r)$ of $\varphi \in \mathcal{J}_{\rho,\pi}(\tau_\lambda)|_{A_J}$ are zero except for $k \leqslant k_0 = -\Lambda_1 - \Lambda_2$ and $k \equiv k_0(\mathrm{mod}\ 2)$. The coefficient function $c_{0,k_0}(a_r)$ is*

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$\xi^{-\Lambda_2+2}$ *up to constants and the other coefficients are given by the formula*

$$(d_\lambda - k + 1)c_{0,k}(a_r) = (\Lambda_1 + \Lambda_2 + k - 2)c_{0,k-2}(a_r)$$

inductively.

8. The Case of $\rho = \mathrm{Ind}\,_{N_0}^{R_J}\eta_r$

Let $(\rho, \mathcal{F}_\rho) = (\mathrm{Ind}_{N_0}^{R_J}\eta_r, \mathcal{F}_\rho)$ be an irreducible unitary representation which is not of type m. Then, for the intertwining space $\mathcal{I}_{\rho,\pi}$, we have

$$\begin{aligned}\mathcal{I}_{\rho,\pi} &= \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi, C^\infty\mathrm{Ind}_{R_J}^{G}(\mathrm{Ind}_{N_0}^{R_J}\eta_r))\\ &= \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi, C^\infty\mathrm{Ind}_{R_J}^{G}(\mathrm{Ind}_{N_0}^{R_J}\eta_r)_\infty)\\ &\subseteq \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi, C^\infty\mathrm{Ind}_{R_J}^{G}(C^\infty\mathrm{Ind}_{N_0}^{R_J}\eta_r))\\ &\simeq \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi, C^\infty\mathrm{Ind}_{N_0}^{G}\eta_r) =: \mathcal{I}_{r,\pi},\end{aligned}$$

where $(\Phi)_\infty$ is the smooth representation associated with a representation Φ on the subspace of all smooth vectors for Φ. We consider the space $\mathcal{I}_{r,\pi}$ instead of $\mathcal{I}_{\rho,\pi}$.

Let $r \neq 0$. Then η_r is a non-degenerate character of the maximal unipotent subgroup N_0 of G and, hence, the space $\mathcal{I}_{r,\pi}$ becomes the space of ordinary algebraic Whittaker models for π. Denote the subspace of $\mathcal{I}_{r,\pi}$ with moderate growth (cf. [14; (8.1)]) by $\mathcal{I}^\circ_{r,\pi}$. The following multiplicity theorem was proved by Shalika [15], Kostant [9], Wallach [17], and Oda [13].

THEOREM 8.1. *Let* $r \neq 0$.

(1) *For a holomorphic or an anti-holomorphic deiscrete series representation* π, *we have* $\mathcal{I}_{r,\pi} = \mathcal{I}^\circ_{r,\pi} = \{0\}$.
(2) *For a large discrete series representation* π, *we have* $\dim\mathcal{I}_{r,\pi} = 4$ *and* $\dim\mathcal{I}^\circ_{r,\pi} \leqslant 1$. *Moreover,* $\dim\mathcal{I}^\circ_{r,\pi} = 1$ *iff* π *is of type II and* $r > 0$ *or* π *is of type III and* $r < 0$.

For $r = 0$, the character η_0 is a degenerate one for which we cannot apply the multiplicity theorem of Shalika and Kostant.

THEOREM 8.2.

(1) *For a holomorphic or an anti-holomorphic deiscrete series representation* π, *we have* $\mathcal{I}_{0,\pi} = \mathcal{I}^\circ_{0,\pi} = \{0\}$.
(2) *For a large discrete series representation* π, *we have* $\dim\mathcal{I}_{0,\pi} = 4$ *and* $\dim\mathcal{I}^\circ_{0,\pi} = 2$.

Proof. In analogy to the paper of Oda [13], this theorem can be shown by solving systems of differential equations that characterizes the space $\mathcal{I}_{0,\pi}$. These systems are induced from the Cauchy–Riemann theorem (for (1)) and Yamashita [19; Theorem 2.4] (for (2)). □

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Appendix: Meijer's G-Functions

In this appendix we recall some basic facts of the G-functions of Meijer briefly. Our main references are the original papers of Meijer [11; I, II], and also we refer to the famous table in [3].

A.1. DEFINITION AND BASIC PROPERTIES

Suppose that m, n, p and q are integers such that

$$0 \leqslant m \leqslant q, \quad 0 \leqslant n \leqslant p \leqslant q, \quad 1 \leqslant q. \tag{A.1}$$

Moreover, suppose that a variable x satisfies the inequalities

$$x \neq 0, \quad |x| < 1 \quad \text{if } p = q; \qquad x \neq 0 \quad \text{if } p < q. \tag{A.2}$$

DEFINITION A.1. Let (m, n, p, q) be as (A.1). *The Meijer's G-function* $G^{m,n}_{p,q}(x)$ *with the parameters* a_i *and* b_j $(1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q)$ is the function

$$\begin{aligned} G^{m,n}_{p,q}(x) &= G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \\ &= \frac{1}{2\pi\sqrt{-1}} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - t) \prod_{i=1}^{n} \Gamma(1 - a_i + t)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + t) \prod_{i=n+1}^{p} \Gamma(a_i - t)} x^t \, dt, \end{aligned}$$

where the contour L is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_j - t)$ $(1 \leqslant j \leqslant m)$ once in the negative direction, but none of the poles of $\Gamma(1 - a_i + t)$ $(1 \leqslant i \leqslant n)$. Here an empty product is interpreted as 1, and the parameters satisfy

$$a_i - b_j \neq 1, 2, 3, \dots \quad (1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant m), \tag{A.3}$$

that is, no pole of $\Gamma(b_j - t)$ $(1 \leqslant j \leqslant m)$ coincides with any pole of $\Gamma(1 - a_i + t)$ $(1 \leqslant i \leqslant n)$.

The integral in Definition A.1 converges for x in (A.2) from the asymptotic expansion of the gamma function. If

$$p + q < 2(m + n) \quad \text{and} \quad |\arg x| < \left(m + n - \frac{p+q}{2}\right)\pi,$$

we may exchange the contour L for another one which runs from $-i\infty$ to $+i\infty$ so that all poles of $\Gamma(b_j - t)$ $(1 \leqslant j \leqslant m)$ are to the right, and all the poles of $\Gamma(1 - a_i + t)$ $(1 \leqslant i \leqslant n)$ to the left.

Clearly $G^{m,n}_{p,q}(x)$ is a symmetric function of $a_1, \dots a_n$, of $a_{n+1}, \dots a_p$, of $b_1, \dots b_m$, and of $b_{m+1}, \dots b_q$.

Assume that the parameters b_j satisfy

$$b_j - b_h \neq 0, \pm 1, \pm 2, \dots \quad (1 \leqslant j \leqslant m, \quad 1 \leqslant h \leqslant m, \quad j \neq h). \tag{A.4}$$

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Then the function $G^{m,n}_{p,q}(x)$ has an expression by the generalized hypergeometric function ${}_pF_{q-1}$:

$$
\begin{aligned}
&G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)\\
&\quad=\sum_{h=1}^{m}\frac{\prod_{\substack{j=1\\ j\neq h}}^{m}\Gamma(b_j-b_h)\prod_{i=1}^{n}\Gamma(1+b_h-a_i)}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{i=n+1}^{p}\Gamma(a_i-b_h)}x^{b_h}\times\\
&\quad\times{}_pF_{q-1}\left(\begin{matrix}1+b_h-a_1,\dots,1+b_h-a_p\\ 1+b_h-b_1,\overset{*}{\dots},1+b_h-b_q\end{matrix};(-1)^{p-m-n}x\right),
\end{aligned}
\tag{A.5}
$$

where ${}_pF_{q-1}$ is defined by

$$
{}_pF_{q-1}\left(\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_{q-1}\end{matrix};x\right)=\sum_{n=1}^{\infty}\left(\prod_{i=1}^{p}\frac{\Gamma(\alpha_i+n)}{\Gamma(\alpha_i)}\prod_{j=1}^{q-1}\frac{\Gamma(\beta_j)}{\Gamma(\beta_j+n)}\right)\cdot\frac{x^n}{n!},
$$

for x in (A.2), and the asterisk means that the number $1+b_h-b_h$ is to be omitted in the sequence $1+b_h-b_1,\dots,1+b_h-b_q$. In particular, we have the relation

$$
G^{2,0}_{1,2}\left(x\,\middle|\,\begin{matrix}a\\ b,c\end{matrix}\right)=x^{\frac{b+c-1}{2}}\mathrm{e}^{-\frac{1}{2}x}W_{\kappa,\mu}(x),\quad \kappa=\frac{b+c+1}{2}-a,\quad \mu=\frac{b-c}{2},
\tag{A.6}
$$

with the Whittaker function $W_{\kappa,\mu}$ (cf. [3; p. 264]) from (A.5) and the transformation theorem of Kummer; $\mathrm{e}^{-x}{}_1F_1(a;b;x)={}_1F_1(b-a;b;-x)$.

The following formulas are immediately deduced from the definition:

$$
G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_{q-1},a_1\end{matrix}\right)=G^{m,n-1}_{p-1,q-1}\left(x\,\middle|\,\begin{matrix}a_2,\dots,a_p\\ b_1,\dots,b_{q-1}\end{matrix}\right),\quad n,p,q\geqslant 1,\quad m<q,
\tag{A.7}
$$

$$
G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ a_p,b_2,\dots,b_q\end{matrix}\right)=G^{m-1,n}_{p-1,q-1}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_{p-1}\\ b_2,\dots,b_q\end{matrix}\right),\quad m,p,q\geqslant 1,\quad n<p,
\tag{A.8}
$$

$$
x^{\sigma}G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)=G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1+\sigma,\dots,a_p+\sigma\\ b_1+\sigma,\dots,b_q+\sigma\end{matrix}\right).
\tag{A.9}
$$

A.2. DIFFERENTIAL EQUATIONS

Let (m,n,p,q) be as (A.1) and let us assume that the parameters a_i and b_j $(1\leqslant i\leqslant p,$ $1\leqslant j\leqslant q)$ satisfy (A.3). Now we consider the following homogeneous linear

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differential equation of q-th order;

$$\left\{(-1)^{p-m-n}x\prod_{i=1}^{p}(\partial-a_i+1)-\prod_{j=1}^{q}(\partial-b_j)\right\}y=0. \tag{A.10}$$

If $p<q$, the only singularities of (A.10) are $x=0$ and ∞, where $x=0$ is a regular singularity and $x=\infty$ is an irregular one. On the other hand, if $p=q$, then $x=0,\infty$ and $(-1)^{p-m-n}$ are the only singularities of (A.10) and these are all regular. Suppose further that the parameters b_j satisfy (A.4). We find that the q functions

$$\mathrm{e}^{(m+n-p-1)\pi\sqrt{-1}b_h}G^{1,p}_{p,q}\left(x\mathrm{e}^{(p+1-m-n)\pi\sqrt{-1}}\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_h,b_1,\overset{*}{\dots},b_q\end{matrix}\right),\quad 1\leqslant h\leqslant q, \tag{A.11}$$

satisfy the differential equation (A.10), where the asterisk means that the number b_h is to be omitted in the sequence $b_1,\dots,b_q$. Obviously the q functions (A.11) are mutually linearly independent from the expression (A.5), and thus they form a fundamental system of solutions for the neighborhood of $x=0$. If the parameters b_j do not satisfy (A.4), then some of (A.11) must be replaced by expressions involving logarithmic terms.

Next we determine a fundamental system of solutions of (A.10) for the neighborhood of $x=\infty$. We treat only the case $p<q$ since the case $p=q$ does not appear throughout this paper. Because $x=\infty$ is an irregular singularity, x must be restricted to a sector for the neighborhood of $x=\infty$. To every value of $|\arg x|$, we can take two integers λ and ω such that

$$|\arg x+(q-m-n-2\lambda+1)\pi|<\frac{q-p+2}{2}\pi, \tag{A.12}$$

$$|\arg x+(q-m-n-2\psi)\pi|<(q-p+\varepsilon)\pi \tag{A.13}$$

for $\psi=\omega,\ \ \omega+1,\dots,\omega+q-p-1$. Here $\varepsilon=\frac{1}{2}$ if $q=p+1$, and $\varepsilon=1$ if $q\geqslant p+2$. Let us assume that the parameters a_j satisfy

$$a_j-a_h\neq 0,\pm1,\pm2,\dots\quad (1\leqslant j\leqslant p,\ \ 1\leqslant h\leqslant p,\ \ j\neq h). \tag{A.14}$$

We consider the p functions

$$G^{q,1}_{p,q}\left(x\mathrm{e}^{(q-m-n-2\lambda+1)\pi\sqrt{-1}}\,\middle|\,\begin{matrix}a_t,a_1,\overset{*}{\dots},a_p\\ b_1,\dots,b_q\end{matrix}\right),\quad 1\leqslant t\leqslant p, \tag{A.15}$$

and $q-p$ functions

$$G^{q,0}_{p,q}\left(x\mathrm{e}^{(q-m-n-2\psi)\pi\sqrt{-1}}\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right),\quad \omega\leqslant\psi\leqslant\omega+q-p-1, \tag{A.16}$$

where the asterisk means that the number a_t is to be omitted in the sequence $a_1,\dots,a_p$. The following lemma is proved by Barnes.

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LEMMA A.2.

(1) *Under the condition (A.3) for* $i = t$*, the function*

$$G^{q,1}_{p,q}\left(x \middle| \begin{matrix} a_t, a_1, \overset{*}{\ldots}, a_p \\ b_1, \ldots, b_q \end{matrix}\right), \quad 1 \leqslant t \leqslant p,$$

has the asymptotic expansion

$$G^{q,1}_{p,q}\left(x \middle| \begin{matrix} a_t, a_1, \overset{*}{\ldots}, a_p \\ b_1, \ldots, b_q \end{matrix}\right) \sim \frac{\prod_{j=1}^{q} \Gamma(1 + b_j - a_t)}{\prod_{\substack{i=1 \\ i \neq t}}^{p} \Gamma(1 + a_i - a_t)} x^{-1+a_t} \times$$
$$\times\, {}_qF_{p-1}\left(\begin{matrix} 1 + b_1 - a_t, \ldots, 1 + b_q - a_t \\ 1 + a_1 - a_t, \overset{*}{\ldots}, 1 + a_p - a_t \end{matrix}; -\frac{1}{x}\right),$$

as $|x| \to \infty$ *in* $|\arg x| < \frac{1}{2}(q - p + 2)\pi$*, where the asterisk means that the number* $1 + a_t - a_t$ *is to be omitted in the sequence* $1 + a_1 - a_t, \ldots, 1 + a_p - a_t$*.*

(2) *The function* $G^{q,0}_{p,q}\left(x \middle| \begin{smallmatrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{smallmatrix}\right)$ *has the asymptotic expansion*

$$G^{q,0}_{p,q}\left(x \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) \sim \mathrm{e}^{(p-q)x^{\frac{1}{q-p}}} x^{\vartheta} \left\{ \frac{(2\pi)^{\frac{q-p-1}{2}}}{\sqrt{q-p}} + \frac{M_1}{x^{\frac{1}{q-p}}} + \frac{M_2}{x^{\frac{2}{q-p}}} + \ldots \right\}$$

as $|x| \to \infty$ *in* $|\arg x| < (q - p + \varepsilon)\pi$*, where the parameter* ϑ *is given by*

$$\vartheta = \frac{1}{q-p}\left(\frac{p-q+1}{2} + \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right)$$

and the coefficients M_j *do not depend on* x *but on the parameters* a_i *and* b_j*.*

From Lemma A.2, the p functions (A.15) and $q - p$ functions (A.16) are mutually linearly independent. Therefore these q functions forms a fundamental system of solutions of (A.10) for the neighborhood of $x = \infty$ under the conditions (A.12), (A.13) and (A.14).

In this paper, we need the above results for the cases of $(p, q) = (2, 3)$ and $(1, 2)$ especially.

LEMMA A.3. *Suppose that* $a_i - b_j \neq 1, 2, 3, \ldots$ $(1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant 3)$*,* $a_1 - a_2 \neq 0, \pm 1, \pm 2, \ldots,$ $-\frac{3}{2}\pi < \arg x < \frac{1}{2}\pi$ *and* $x \neq 0$*. Then the linear differential equation of 3-rd order*

$$\left\{ x^3 \frac{\mathrm{d}^3}{\mathrm{d}x^3} + \alpha_2(x) x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha_1(x) x \frac{\mathrm{d}}{\mathrm{d}x} + \alpha_0(x) \right\} y = 0 \tag{A.17}$$

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with

$$\alpha_2(x) = 3 - b_1 - b_2 - b_3 + x,$$
$$\alpha_1(x) = (1 - b_1)(1 - b_2)(1 - b_3) + b_1 b_2 b_3 + (3 - a_1 - a_2)x,$$
$$\alpha_0(x) = -b_1 b_2 b_3 + (1 - a_1)(1 - a_2)x$$

has a fundamental system of three linearly independent solutions in the neighborhood of $x = \infty$ *given by*

$$G^{3,1}_{2,3}\left(x\mathrm{e}^{\pi\sqrt{-1}}\,\middle|\,\begin{matrix} a_1, a_2 \\ b_1, b_2, b_3\end{matrix}\right), \qquad G^{3,1}_{2,3}\left(x\mathrm{e}^{\pi\sqrt{-1}}\,\middle|\,\begin{matrix} a_2, a_1 \\ b_1, b_2, b_3\end{matrix}\right) \quad \text{and}$$
$$G^{3,0}_{2,3}\left(x\,\middle|\,\begin{matrix} a_1, a_2 \\ b_1, b_2, b_3\end{matrix}\right).$$

The asymptotic expansions of these three functions are given in Lemma A.2 for $(p, q) = (2, 3)$.

LEMMA A.4. *Suppose that* $a_1 - b_j \neq 1, 2, 3, \ldots \ (1 \leqslant j \leqslant 2)$, $-\frac{3}{2}\pi < \arg x < \frac{1}{2}\pi$ *and* $x \neq 0$. *Then the linear differential equation of 2-nd order*

$$\left\{x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta_1(x) x \frac{\mathrm{d}}{\mathrm{d}x} + \beta_0(x)\right\} y = 0 \tag{A.18}$$

with

$$\beta_1(x) = 1 - b_1 - b_2 + x, \quad \beta_0(x) = b_1 b_2 + (1 - a_1)x$$

has a fundamental system of two linearly independent solutions for the neighborhood of $x = \infty$ *given by*

$$G^{2,1}_{1,2}\left(x\mathrm{e}^{\pi\sqrt{-1}}\,\middle|\,\begin{matrix} a_1 \\ b_1, b_2\end{matrix}\right) \quad \text{and} \quad G^{2,0}_{1,2}\left(x\,\middle|\,\begin{matrix} a_1 \\ b_1, b_2\end{matrix}\right).$$

The asymptotic expansions of these two functions are given in Lemma A.2 for $(p, q) = (1, 2)$. *Here the function* $G^{2,0}_{1,2}\left(x\,\middle|\,\begin{smallmatrix} a_1 \\ b_1, b_2\end{smallmatrix}\right)$ *has the expression (A.6) by the Whittaker function* $W_{\kappa,\mu}(x)$.

Remark. Even if some of the differences $a_i - b_j$ take the positive integer values, the unique solution of the differential equation (A.17) (resp. (A.18)) with exponential asymptotics is given by the function $G^{3,0}_{2,3}(x)$ (resp. $G^{2,0}_{1,2}(x)$) up to constant multiple.

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