LATTICE ISOMORPHISMS OF ASSOCIATIVE ALGEBRAS

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1. Introduction and notation

Let A be an associative algebra over the field F. We denote by $\mathscr{L}(A)$ the lattice of all subalgebras of A. By an $\mathscr{L}$-isomorphism (lattice isomorphism) of the algebra A onto an algebra B over the same field, we mean an isomorphism

$$\phi : \mathscr{L}(A) \to \mathscr{L}(B)$$

of $\mathscr{L}(A)$ onto $\mathscr{L}(B)$. We investigate the extent to which the algebra B is determined by the assumption that it is $\mathscr{L}$-isomorphic to a given algebra A. In this paper, we are mainly concerned with the case in which A is a finite-dimensional semi-simple algebra.

The one-to-one map $\sigma : A \to B$ of an algebra A over the field F onto an algebra B over F is called a semi-isomorphism[1] if

(i) σ is semi-linear (that is, for some automorphism α of F,

$$(\lambda_1 a_1 + \lambda_2 a_2)^\sigma = \lambda_1^\alpha a_1^\sigma + \lambda_2^\alpha a_2^\sigma$$

for all $a_1, a_2 \in A$ and all $\lambda_1, \lambda_2 \in F$), and

(ii) σ is multiplicative or anti-multiplicative (that is, either $(xy)^\sigma = x^\sigma y^\sigma$ for all $x, y \in A$, or $(xy)^\sigma = y^\sigma x^\sigma$ for all $x, y \in A$).

We remark that, for maps $\sigma : A \to B$ of not necessarily associative rings, such that $(x+y)^\sigma = x^\sigma + y^\sigma$ for all $x, y \in A$, the apparently weaker condition

(ii′) for each pair x, y of elements of A, either $(xy)^\sigma = x^\sigma y^\sigma$ or $(xy)^\sigma = y^\sigma x^\sigma$, in fact implies (ii).[2]

Since any semi-isomorphism of an algebra A onto an algebra B induces an $\mathscr{L}$-isomorphism, from the assumption that A is $\mathscr{L}$-isomorphic to B, we cannot in general hope to prove any stronger relationship between A and B than semi-isomorphism. However the algebra $M_n(F)$ of all $n \times n$ matrices over the ground field F has the property that any algebra semi-isomorphic

[1] Closely related concepts are discussed in Ancochea [1], Hua [4] and Kaplansky [6].

[2] Jacobson, N.: Lectures on abstract algebra, vol. I, p. 74, exercise 6.

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to $M_n(F)$ is in fact isomorphic to it. In § 4, we prove that any algebra $\mathscr{L}$-isomorphic to $M_n(F)$, $n \geqq 2$, is isomorphic to $M_n(F)$. In § 5, we show that, if an algebra A is $\mathscr{L}$-isomorphic to the algebra $M_n(\Delta)$ where $n \geqq 3$ and Δ is a division algebra over F, then A is semi-isomorphic to $M_n(\Delta)$. In § 6, we show that, apart from certain special cases, if ϕ is an $\mathscr{L}$-isomorphism of a finite-dimensional semi-simple algebra A onto an algebra B, then B is also semi-simple and the images under ϕ of the simple direct summands of A of dimension greater than one are simple direct summands of B.

By "algebra" we mean "associative algebra over the field F", and "$A \simeq B$" means that A and B are isomorphic as algebras over F. We write mappings exponentially; thus the image of A under the map ϕ will be denoted by A^ϕ. If $a_1, \ldots, a_n$ are elements of an algebra A, we denote by $\langle a_1, \ldots a_n\rangle$ the *subspace* of A spanned by $a_1, \ldots, a_n$. If A is a finite-dimensional algebra, we denote the radical of A by $R(A)$. For any algebra A, we put

$$l(A) = \text{length of the longest chain in } \mathscr{L}(A),$$
$$d(A) = \text{dimension of } A.$$

Clearly $d(A) \geqq l(A)$. If A is a nilpotent algebra, then the factors A^i/A^{i+1} of the series of ideals

$$A > A^2 > \ldots > A^n > A^{n+1} = 0$$

are all null. Every subspace of A^i/A^{i+1} is a subalgebra, and so

$$l(A^i/A^{i+1}) = d(A^i/A^{i+1}).$$

Since

$$l(A) \geqq \sum_{i=1}^{n} l(A^i/A^{i+1})$$

and

$$d(A) = \sum_{i=1}^{n} d(A^i/A^{i+1}) = \sum_{i=1}^{n} l(A^i/A^{i+1}),$$

it follows that $l(A) = d(A)$ for any (not necessarily finite-dimensional) nilpotent algebra A.

2. Condition for finite dimension

If the algebra A is finite-dimensional, then $l(A)$ is finite. Conversely, we have

Theorem 1. *Let A be an associative algebra and suppose that $l(A)$ is finite. Then $d(A)$ is finite.*

Proof. Since $l(A)$ is finite, the sum of all nilpotent left ideals of A is the sum of a finite set of nilpotent left ideals. It follows as in the usual theory of rings with minimum condition, that the radical $R(A)$, defined as the sum of all nilpotent left ideals of A, is a nilpotent two-sided ideal and that $A/R(A)$ has radical 0. Since $R = R(A)$ is nilpotent, $d(R) = l(R)$ which is finite. Thus we need only consider the case $R(A) = 0$.

From $R(A) = 0$, it follows as in Artin, Nesbitt and Thrall [2], p. 29, Corollary 4.3B, that A has an identity element 1. The field F can be identified with the subalgebra $F1$, and it follows that A *regarded as a ring* satisfies both chain conditions for left ideals, every ring left ideal being a subalgebra of A. Therefore A is a finite direct sum of simple algebras. Each of these simple algebras is a total matrix algebra $M_n(D)$ over a division algebra D, and $l(D)$ is finite. It remains to prove $d(D)$ finite.

Suppose K is a commutative subalgebra of D. Then K is an extension field of F. Let t be any element of K and let $P = F[t]$ be the algebra of polynomials in t. If t is transcendental over F, then $l(P)$ is infinite. Therefore K is algebraic over F. Since $l(K)$ is finite, K is finitely generated over F. Therefore K is finite-dimensional over F.

Let Z be the centre of D and let K be a maximal subfield of D. Then K is its own centraliser in D, the dimension of K over Z is finite and therefore the dimension of D over K is finite.[3] Therefore the dimension of D over F is finite.

3. Algebras A with $l(A)$ small

LEMMA 1. *Suppose* $l(A) = 1$. *Then* $d(A) = 1$.

PROOF. If A is nilpotent, then $d(A) = l(A) = 1$. If A is not nilpotent, then A contains an idempotent e. But $\langle e \rangle$ is a subalgebra and therefore $A = \langle e \rangle$.

Every minimal subalgebra of an algebra A is either spanned by an idempotent or is null. Since a division algebra has no nilpotent elements and its identity is its only idempotent, a division algebra has a unique minimal subalgebra.

LEMMA 2. *If the finite-dimensional algebra A has a unique minimal subalgebra, then A is either nilpotent or a division algebra.*

PROOF. If A is not nilpotent, then it contains an idempotent e which spans the unique minimal subalgebra of A. In this case, $R(A) = 0$ since otherwise $R(A)$ would contain the minimal subalgebra. Thus A is a direct sum of simple algebras. But each summand contains a minimal subalgebra and therefore A is simple.

[3] See Jacobson [5], p. 165, Corollary to the "fundamental theorem of finite Galois theory."

Therefore $A \simeq M_n(D)$ for some n and some division algebra D. If $n > 1$, then $M_n(D)$ has more than one minimal subalgebra. Therefore A is a division algebra.

LEMMA 3. $$l(M_2(F)) = 4.$$

PROOF. Let e_{ij} be the matrix with 1 in the ij position and all other entries 0. Then

$$0 < \langle e_{11} \rangle < \langle e_{11}, e_{22} \rangle < \langle e_{11}, e_{12}, e_{22} \rangle < M_2(F)$$

is a chain of length 4. Therefore $l(M_2(F)) \geqq 4$. But

$$l(M_2(F)) \leqq d(M_2(F)) = 4.$$

Therefore $l(M_2(F)) = 4$.

LEMMA 4. *Suppose A is a semi-simple algebra and $l(A) \leqq 3$. Then A is a direct sum of division algebras.*

PROOF. A is a direct sum of simple algebras. Since $l(M_2(F)) = 4$, each summand must be a division algebra.

LEMMA 5. *Suppose $l(A) = 2$ and that A has at least two minimal subalgebras. Then $d(A) = 2$.*

PROOF. If A is nilpotent, then $d(A) = l(A) = 2$. If A is not nilpotent, then $l(R(A)) = 0$ or $l(R(A)) = 1$. If $l(R) = 1$, then also $l(A/R) = 1$ and by Lemma 1, $d(R) = d(A/R) = 1$. If $l(R) = 0$, then $R = 0$, A is semi-simple and by Lemma 4, A is a direct sum of division algebras. Since A has at least two minimal subalgebras, A is not a division algebra. It follows that A is the direct sum of two division algebras $A = D_1 \oplus D_2$. Since $l(A) = 2$, $l(D_1) = l(D_2) = 1$, which implies by lemma 1, that $d(D_1) = d(D_2) = 1$; and so $d(A) = 2$.

LEMMA 6. *Let k be the cardinal of F. Suppose $l(A) = 2$. Then A is isomorphic to one of the algebras listed in the following table:*

Type	Defining relations	Number of minimal subalgebras
I	Extension field K of F with F as a maximal subfield	1
II	$\langle a, a^2 \rangle$, $a^3 = 0$.	1
III(a)	$\langle e, r \rangle$, $e^2 = e$, $r^2 = 0$, $er = re = 0$	2
III(b)	$\langle e, r \rangle$, $e^2 = e$, $r^2 = 0$, $er = re = r$	2
IV	$F \oplus F$	3
V	$\langle a_1, a_2 \rangle$, $a_i a_j = 0$ for all i, j.	$k+1$
VI(a)	$\langle e, r \rangle$, $e^2 = e$, $r^2 = 0$, $er = r$, $re = 0$	$k+1$
VI(b)	The opposed algebra of VI(a).	$k+1$

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PROOF. By Lemmas 2 and 5, either A is a division algebra or $d(A) = 2$. If $d(A) = 2$, then $d(R) = 0,1$ or 2.

(i) Suppose A is a division algebra with identity 1. Then $F1$ is the only minimal subalgebra of A. There exists $t \in A$, $t \notin F1$. Since $l(A) = 2$, $A = F[t]$, the algebra of all polynomials in t, and is therefore commutative. Thus A is an extension field of F.

(ii) Suppose A is semi-simple, but not a division algebra. Then it follows from Lemma 4 that $A \simeq F \oplus F$. If e_1, e_2 are the identities of the two direct summands of A, it is easily seen that $\langle e_1 \rangle$, $\langle e_2 \rangle$, $\langle e_1+e_2 \rangle$ are all the minimal subalgebras of A.

(iii) Suppose $d(R) = 1$. Then $R = \langle r \rangle$ for some r and $r^2 = 0$. Since A is not nilpotent, A contains an idempotent e and $A = \langle e, r \rangle$. Since $\langle r \rangle$ is an ideal, $er = \lambda r$ and $re = \mu r$ for some $\lambda, \mu \in F$. But

$$\begin{aligned} e(er) &= \lambda er = \lambda^2 r \\ &= (ee)\, r = er = \lambda r. \end{aligned}$$

Therefore $\lambda = 0,1$ and similarly $\mu = 0,1$. We thus have the four types III(a), III(b), VI(a), VI(b). It remains to verify that these have the numbers of minimal subalgebras given in the table.

A has the $k+1$ one-dimensional subspaces $\langle e+\theta r \rangle$, $(\theta \in F)$ and $\langle r \rangle$. The subspace $\langle e+\theta r \rangle$ is a subalgebra if $(e+\theta r)^2 \in \langle e+\theta r \rangle$. But

$$\begin{aligned} (e+\theta r)^2 &= e+\theta(er+re) \\ &= e+\theta(\lambda+\mu)r. \end{aligned}$$

Thus $\langle e+\theta r \rangle$ is a subalgebra if and only if

$$\theta(\lambda+\mu) = \theta$$

that is, if $\theta = 0$ or if $\lambda+\mu = 1$.

If A is of type III (whether III(a) or III(b)), then $\lambda+\mu \neq 1$ and the only minimal subalgebras of A are $\langle e \rangle$, $\langle r \rangle$. If A is of type VI, then $\lambda+\mu = 1$, $\langle e+\theta r \rangle$ is subalgebras for all $\theta \in F$ and A has $k+1$ minimal subalgebras.

(iv) Suppose A is nilpotent. Either A is null in which case every subspace of A is a subalgebra, or $A^2 = \langle b \rangle$ for some $b \neq 0$, and $A = \langle a, b \rangle$, $A^3 < A^2$ and therefore $A^3 = 0$. Thus $ab = ba = b^2 = 0$. Since A is not null, $a^2 \neq 0$ and therefore $A^2 = \langle a^2 \rangle$, A is of type II and clearly has only one minimal subalgebra. This completes the proof of the lemma.

4. Lemmas on matrix algebras

Let $M = M_n(\Delta)$ be the algebra of all $n \times n$ matrices over the finite-dimensional division algebra Δ. We denote by η_{ij} the matrix with 1 in the ij position and all other entries 0.

The subalgebra $\langle \eta_{11}, \eta_{22} \rangle$ is an algebra of type IV and has exactly three minimal subalgebras. An $\mathscr{L}$-isomorphism ϕ of M onto another algebra takes $\langle \eta_{11}, \eta_{22} \rangle$ to an algebra $\langle \eta_{11}, \eta_{22} \rangle^{\phi}$ with exactly three minimal subalgebras. We observe from the table in Lemma 6, that an algebra A with $l(A) = 2$ and exactly three minimal subalgebras is determined to within isomorphism by these properties except when $F = GF(2)$, the field of two elements.

LEMMA 7. *Suppose* $F = GF(2)$ *and* $M = M_2(F)$. *Then*

$$\langle \eta_{11}+\eta_{22}, \eta_{11}+\eta_{12}+\eta_{21} \rangle$$

is a maximal subalgebra of M *and has only one minimal subalgebra.*

PROOF. Put $1 = \eta_{11}+\eta_{22}$, $a = \eta_{11}+\eta_{12}+\eta_{21}$. Then $a^2+a+1 = 0$ and the minimum polynomial of a over the field $F1$ is x^2+x+1, which is irreducible. Therefore $F[a]$ is a field of dimension 2 over F. Therefore $K = \langle \eta_{11}+\eta_{22}, \eta_{11}+\eta_{12}+\eta_{21} \rangle$ has only one minimal subalgebra. If N is any subalgebra of M containing K, then N can be regarded as a left vector space over K. It follows that the dimension of K over F divides the dimension of N over F. Thus $d(N) = 2$ and $N = K$ or $d(N) = 4$ and $N = M$. Thus K is a maximal subalgebra of M.

We now suppose that $\phi : \mathscr{L}(M) \to \mathscr{L}(A)$ is an $\mathscr{L}$-isomorphism of M onto an algebra A. We put $E_{ij} = \langle \eta_{ij} \rangle^{\phi}$. Then $d(E_{ij}) = 1$. We take e_{ij} such that $E_{ij} = \langle e_{ij} \rangle$.

LEMMA 8. *Let* $M = M_2(F)$, *that is* $n = 2$, $\Delta = F$. *Put* $I = \langle \eta_{11}+\eta_{22} \rangle^{\phi}$. *Then* I *is in the centre of* A, $I^2 = I$ *and* $E_{12}^2 = E_{21}^2 = 0$.

PROOF. Since $I \cup E_{12}$ has exactly two minimal subalgebras, $I \cup E_{12}$ is commutative. Since I is in the centre of $I \cup E_{12}$ and of $I \cup E_{21}$, I is in the centre of $I \cup E_{12} \cup E_{21} = A$. Since $I \cup E_{12}$ is of type III, we have either $I^2 = I$, $E_{12}^2 = 0$ or $I^2 = 0$, $E_{12}^2 = E_{12}$. We show that the latter is not possible.

Since $E_{11} \cup E_{22}$ has exactly three minimal subalgebras, $E_{11} \cup E_{22}$ is of type IV and $I^2 = I$ if $F \neq GF(2)$. Suppose $F = GF(2)$ and $I^2 = 0$. By Lemma 7, $K = \langle \eta_{11}+\eta_{22}, \eta_{11}+\eta_{12}+\eta_{21} \rangle$ is a maximal subalgebra of M with $\langle \eta_{11}+\eta_{22} \rangle$ as its only minimal subalgebra. Therefore I is the only minimal subalgebra of K^{ϕ}. Since $I^2 = 0$, K^{ϕ} is nilpotent. $I = R(I \cup E_{12}) = R(I \cup E_{21})$ since $I \cup E_{12}$ and $I \cup E_{21}$ are of type III. Therefore I is an ideal of $A = E_{12} \cup E_{21}$. Therefore $l(A/R(A)) \leqq 3$. By Lemma 4. $A/R(A)$ is a direct sum of division algebras and so has no nilpotent elements. All nilpotent elements of A are thus in $R(A)$. Therefore $K^{\phi} \leqq R(A)$. But A is not nilpotent since $I \cup E_{12}$ is not nilpotent. Therefore $R(A) = K^{\phi}$ since K^{ϕ} is maximal in A. But this implies $d(A/R(A)) = 1$, $d(R(A)) = l(K^{\phi}) = 2$ and therefore $d(A) = 3$ contrary to $l(A) = l(M) = 4$. Therefore $I^2 = I$.

LEMMA 9. *Under the assumptions of Lemma* 8, $E_{11}^2 = E_{11}$.

PROOF. $E_{11} \cup I$ has exactly three minimal subalgebras. By Lemma 8, it is commutative and non-nilpotent. By Lemma 6, $E_{11} \cup I$ must be of type IV even if $F = GF(2)$. Therefore $E_{11}^2 = E_{11}$. Similarly $E_{22}^2 = E_{22}$.

LEMMA 10. *Suppose $M = M_2(F)$. Then $A \simeq M$ and, for suitable choice of the e_{ij}, the e_{ij} have either the same multiplication as the η_{ij} or the opposed multiplication.*

PROOF. By Lemmas 8 and 9, we have

$$E_{12} = R(E_{11} \cup E_{12}) = R(E_{22} \cup E_{12})$$

and therefore $E_{12} \leqq R(E_{11} \cup E_{12} \cup E_{22})$. Since $E_{11} \cup E_{22}$ is semi-simple, $(E_{11} \cup E_{12} \cup E_{22})/R(E_{11} \cup E_{12} \cup E_{22})$ has a subalgebra isomorphic to $E_{11} \cup E_{22}$, and it follows that $R(E_{11} \cup E_{12} \cup E_{22}) = E_{12}$.

Suppose $R = R(A) \neq 0$. Then $R \cap (E_{11} \cup E_{12} \cup E_{22}) \leqq E_{12}$. If $R \cap (E_{11} \cup E_{12} \cup E_{22}) = 0$, then $R \cup (E_{11} \cup E_{12} \cup E_{22}) = A$ and $A/R \simeq E_{11} \cup E_{12} \cup E_{22}$ which is impossible as $E_{11} \cup E_{12} \cup E_{22}$ has non-zero radical. Therefore $R \cap (E_{11} \cup E_{12} \cup E_{22}) = E_{12}$. Similarly $R \geqq E_{21}$ and therefore $A = E_{12} \cup E_{21} \leqq R$. But A is not nilpotent. Therefore $R = 0$. Since any simple algebra which is not a division algebra contains a subalgebra isomorphic to $M_2(F)$, either A is a direct sum of division algebras or $A \simeq M_2(F)$. Since A contains nilpotent elements, A is not a direct sum of division algebras.

We now prove that the e_{ij} may be chosen as asserted. Since $A \simeq M_2(F)$, A has an identity element 1 and $\langle 1 \rangle$ is the centre of A. By Lemma 8, I is in the centre of A and therefore $I = \langle 1 \rangle$.

Since $E_{11}^2 = E_{11}$, we may take e_{11} idempotent. Similarly we may take e_{22} idempotent. But e_{11}, 1, $1-e_{11}$ are idempotents in $E_{11} \cup E_{22}$ which has only three idempotents. Therefore $1-e_{11} = e_{22}$ and $e_{11}e_{22} = e_{22}e_{11} = 0$.

However e_{12} is chosen, we have either $e_{11}e_{12} = e_{12}$, $e_{12}e_{11} = 0$ or $e_{11}e_{12} = 0$, $e_{12}e_{11} = e_{12}$. We consider the first case, the same argument applying to the second with the order of all products reversed. Since $(e_{11}+e_{22})e_{12} = e_{12}$, we have $e_{22}e_{12} = 0$, $e_{12}e_{22} = e_{12}$. If $e_{21}e_{11} = 0$, then we must also have $e_{22}e_{21} = 0$. This implies

$$e_{12}e_{21} = (e_{12}e_{22})e_{21} = e_{12}(e_{22}e_{21}) = 0$$
$$e_{21}e_{12} = e_{21}(e_{11}e_{12}) = (e_{21}e_{11})e_{12} = 0$$

contrary to $A = E_{12} \cup E_{21}$. Therefore

$$e_{22}e_{21} = e_{21} = e_{21}e_{11},\ e_{11}e_{21} = 0 = e_{21}e_{22}.$$

For any $a = \alpha e_{11}+\beta e_{12}+\gamma e_{21}+\delta e_{22} \in A$, $\alpha, \beta, \gamma, \delta \in F$, we have $e_{11}ae_{11} = \alpha e_{11}$. But

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$$e_{12}e_{21} = (e_{11}e_{12})(e_{21}e_{11}) = e_{11}(e_{12}e_{21})e_{11}.$$

Therefore $e_{12}e_{21} = \lambda e_{11}$. Similarly $e_{21}e_{12} = \mu e_{22}$. But

$$\lambda e_{12} = (e_{12}e_{21})e_{12} = e_{12}(e_{21}e_{12}) = \mu e_{12}$$

and therefore $\lambda = \mu$. Since $A = E_{12} \cup E_{21}$, $\lambda \neq 0$. We replace e_{12} by $e'_{12} = e_{12}/\lambda$. Then $e'_{12}e_{21} = e_{11}$. Thus we may choose the e_{ij} so that $\lambda = 1$ and the e_{ij} have the same multiplication as the η_{ij}.

LEMMA 11. *Let Δ be a finite-dimensional division algebra, $M = M_n(\Delta)$, $n \geqq 2$ and let N be a nilpotent subalgebra of M. Then N^ϕ is nilpotent and $d(N^\phi) = d(N)$.*

PROOF. If $d(N) = 1$, then for some subalgebra U of M containing N, there exists an isomorphism $\alpha : U \to M_2(F)$ of U onto $M_2(F)$ such that $N^\alpha = \langle \eta_{12} \rangle$. This follows from consideration of the similarity invariants of a matrix $\eta \in N$. By Lemma 8, N^ϕ is nilpotent.

For general N, every one-dimensional subalgebra of N is nilpotent. Hence every minimal subalgebra of N^ϕ is nilpotent and therefore N^ϕ is nilpotent. We then have

$$d(N^\phi) = l(N^\phi) = l(N) = d(N).$$

LEMMA 12. *Let $M = M_n(F)$, $n \geqq 2$. Then $A \simeq M$ and, for suitable choice of the e_{ij}, the e_{ij} have either the same multiplication as the η_{ij} or the opposed multiplication.*

PROOF. Since $\langle \eta_{ii}, \eta_{ij}, \eta_{ji}, \eta_{jj} \rangle$ for $i \neq j$ is isomorphic to $M_2(F)$, by Lemma 9, $E_{ii}^2 = E_{ii}$ for all i. If we choose for e_{ii} the unique idempotent in E_{ii}, then by Lemma 10 applied to $\langle \eta_{ii}, \eta_{ij}, \eta_{ji}, \eta_{jj} \rangle$, we have $e_{ii}e_{jj} = 0$ for $i \neq j$. However e_{ij} is chosen $(i \neq j)$, we have either $e_{ii}e_{ij} = e_{ij} = e_{ij}e_{jj}$, $e_{ij}e_{ii} = 0 = e_{jj}e_{ij}$ or $e_{ii}e_{ij} = 0 = e_{ij}e_{jj}$, $e_{ij}e_{ii} = e_{ij} = e_{jj}e_{ij}$.

By Lemma 11, $\langle e_{ij}, e_{kl} \rangle$ is nilpotent if i, j, k, l are distinct. Since it has $k+1$ minimal subalgebras, it is null and therefore $e_{ij}e_{kl} = 0$. Similarly $e_{ij}e_{ik} = 0$ and $e_{ij}e_{kj} = 0$ if i, j, k are distinct. Since $e_{rr}e_{jk} = e_{jk}$ either for $r = j$ or for $r = k$, by taking the appropriate value for r, we obtain in either case

$$e_{ii}e_{jk} = e_{ii}e_{rr}e_{jk} = 0$$

if i, j, k are distinct, since then $e_{ii}e_{rr} = 0$. Similarly $e_{jk}e_{ii} = 0$.

By Lemma 11, if i, j, k are distinct, then $E_{ij} \cup E_{jk}$ is a three-dimensional nilpotent subalgebra. Therefore $e_{ij}e_{jk}$ and $e_{jk}e_{ij}$ are not both 0. If $e_{ii}e_{ij} = e_{ij}$, then

$$e_{jk}e_{ij} = e_{jk}(e_{ii}e_{ij}) = (e_{jk}e_{ii})e_{ij} = 0$$

and so $e_{ij}e_{jk} \neq 0$, whence $(e_{ij}e_{jj})e_{jk} \neq 0$ and therefore $e_{jj}e_{jk} = e_{jk}$. By

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repeated application of this argument, we have that, if $e_{11}e_{12} = e_{12}$, then $e_{ii}e_{ij} = e_{ij}$ for all i, j. We suppose $e_{11}e_{12} = e_{12}$, and prove that the e_{ij} may be chosen so that they have the same multiplication as the η_{ij}. The same argument applies with the order of all products reversed if $e_{11}e_{12} = 0$, giving e_{ij} with the opposed multiplication.

Since $d(E_{ij} \cup E_{jk}) = 3$ and $d(E_{ij} \cup E_{ik}) = 2$, e_{ij}, e_{jk}, e_{ik} is a basis of $E_{ij} \cup E_{jk}$ and therefore

$$e_{ij}e_{jk} = \alpha e_{ij} + \beta e_{jk} + \gamma e_{ik}$$

for some $\alpha, \beta, \gamma \in F$. But

$$\begin{aligned} e_{ij}e_{jk} &= (e_{ii}e_{ij})(e_{jk}e_{kk}) = e_{ii}(e_{ij}e_{jk})e_{kk} \\ &= \gamma e_{ik}. \end{aligned}$$

It remains to prove that the e_{ij} can be so chosen that $e_{ij}e_{jk} = e_{ik}$ for all i, j, k.

We choose $e_{12}, e_{13}, \ldots, e_{1n}$ arbitrarily. By Lemma 10, we can choose e_{i1} such that $e_{1i}e_{i1} = e_{11}$. The e_{i1} are uniquely determined by this condition and satisfy $e_{i1}e_{1i} = e_{ii}$. For i, j distinct and not equal to 1, we can choose e_{ij} such that $e_{1i}e_{ij} = e_{1j}$. This determines e_{ij} uniquely. We then have

$$e_{1k} = e_{1j}e_{jk} = (e_{1i}e_{ij})e_{jk} = e_{1i}(e_{ij}e_{jk})$$

and therefore $e_{ij}e_{jk} = e_{ik}$ for all i, j, k.

Lemma 13. *Let η be an idempotent of $M = M_n(\Delta)$, $n \geqq 2$. Then $\langle \eta \rangle^\phi$ is not nilpotent, $\langle \eta \rangle^\phi = \langle e \rangle$ for some idempotent e.*

Proof. Let r be the rank of η. Then for some inner automorphism α of M, $(\eta_{11} + \eta_{22} + \ldots + \eta_{rr})^\alpha = \eta$. Let N be the subalgebra $M_n(F)$ of M. By Lemma 12 applied to N^α, $\langle \eta \rangle^\phi$ is not nilpotent. Since $d(\langle \eta \rangle^\phi) = 1$ and $\langle \eta \rangle^\phi$ is not nilpotent, there exists a unique idempotent e such that $\langle e \rangle = \langle \eta \rangle^\phi$.

5. Simple algebras

Theorem 2. *Let $S = M_n(\Delta)$ where $n \geqq 2$ and Δ is a finite dimensional division algebra. Let $\phi : \mathscr{L}(S) \to \mathscr{L}(A)$ be an $\mathscr{L}$-isomorphism of S onto A. Then $A \simeq M_n(D)$ for some division algebra D which is $\mathscr{L}$-isomorphic to Δ and $d(D) = d(\Delta)$.*

Proof. For any subalgebra U of S, we have be Lemmas 11, 13, that U^ϕ is nilpotent if and only if U is nilpotent. Thus the maximal nilpotent subalgebras of A are the images under ϕ of the maximal nilpotent subalgebras of S. By Barnes [3], $R(A)$ is the intersection of the maximal nilpotent subalgebras of A. Since $R(S) = 0$, $R(A) = 0$ and A is semi-simple.

Let N be the subalgebra $M_n(F)$ of S and let ξ be the identity of S. We may identify Δ with the subalgebra $\Delta\xi$ of S. Then $S = N \cup \Delta$, $N \cap \Delta = \langle \xi \rangle$.

Let B be any simple direct summand of A. Then B contains an idempotent e. Let $U = \langle e \rangle^{\phi^{-1}}$. If U is nilpotent, then by lemma 11, $U\phi = \langle e \rangle$ is nilpotent, contrary to e being idempotent. Therefore U is non-nilpotent and so contains an idempotent η. Clearly $U = \langle \eta \rangle$ and $\langle e \rangle = \langle \eta \rangle^\phi$. But $\eta \in N^\alpha$ for some inner automorphism α of S. Since $N^{\alpha\phi} \simeq M_n(F)$ and $B \cap N^{\alpha\phi} \geqq \langle e \rangle \neq 0$, we have $B \geqq N^{\alpha\phi}$ and therefore $\langle \xi \rangle^\phi \leqq B$. Since $\langle \xi \rangle^{\alpha\phi}$ is the only minimal subalgebra of $\Delta^{\alpha\phi}$ and is not nilpotent, $\Delta^{\alpha\phi}$ is a division algebra. Since $B \cap \Delta^{\alpha\phi} \geqq \langle \xi \rangle^{\alpha\phi} \neq 0$ and B is an ideal, $B \geqq \Delta^{\alpha\phi}$. Thus $B \geqq N^{\alpha\phi} \cup \Delta^{\alpha\phi} = (N \cup \Delta)^{\alpha\phi} = S^{\alpha\phi} = A$. Therefore A is simple.

Since A simple, $A \simeq M_m(D)$ for some division algebra D and some m. If $U \simeq M_r(F)$ is a subalgebra of $M_m(D)$, then $r \leqq m$. Since $N^\phi \simeq M_n(F)$ is a subalgebra of A, we have $n \leqq m$. By the same argument applied to the $\mathscr{L}$-isomorphism ϕ^{-1}, we have $n \geqq m$. We therefore have $A \simeq M_n(D)$. But $\Delta_1 = \Delta\eta_{11} = \eta_{11}S\eta_{11}$ is the unique maximal division subalgebra of S containing η_{11}. It follows that Δ_1^ϕ is the unique maximal division subalgebra of A containing e_{11} and therefore $\Delta_1^\phi = e_{11}Ae_{11} = De_{11}$. Thus $D \simeq \Delta_1^\phi$ and it follows that D is $\mathscr{L}$-isomorphic to Δ.

Consider the maximal nilpotent subalgebra U of S consisting of all upper triangular matrices $\sum_{i<j} \delta_{ij}\eta_{ij}$. This is the unique maximal nilpotent subalgebra of S containing the η_{ij} with $i < j$. It follows that U^ϕ is the subalgebra of A consisting of all elements of the form $\sum_{i<j} d_{ij}e_{ij}$ where $d_{ij} \in D$. Since U and U^ϕ are nilpotent, $d(U) = d(U^\phi)$. But $d(U) = \frac{1}{2}n(n-1)d(\Delta)$ and $d(U^\phi) = \frac{1}{2}n(n-1)d(D)$. Therefore $d(D) = d(\Delta)$.

COROLLARY. *Let S be a finite- dimensional simple algebra over the finite field F. Suppose S is not a field. Let $\phi : \mathscr{L}(S) \to \mathscr{L}(A)$ be an $\mathscr{L}$-isomorphism of S onto A. Then $A \simeq S$.*

PROOF. A finite-dimensional division algebra over a finite field is an extension field and is determined up to isomorphism by its dimension.

THEOREM 3. *Let $S = M_n(\Delta)$ for some finite-dimensional division algebra Δ. Suppose $n \geqq 3$. Let $\phi : \mathscr{L}(S) \to \mathscr{L}(A)$ be an $\mathscr{L}$-isomorphism of S onto A. Then S is semi-isomorphic to A.*

PROOF. (i) The subalgebra $\Delta_1 = \Delta\eta_{11}$ of S is a division algebra isomorphic to Δ. The subalgebra $V = \Delta\eta_{12} + \Delta\eta_{13}$ is null and every subspace of V is a subalgebra of S. V can be considered as a left vector space over Δ_1. We show that the Δ_1-subspaces of V are the subalgebras $P \leqq V$ with the property $P = V \cap (\Delta_1 \cup P)$.

Suppose P is a Δ_1-subspace of V. Then $\Delta_1 \cup P = \Delta_1 \oplus P$ (vector space direct sum). Let $\delta\eta_{11} + p \in V$, $\delta \in \Delta$, $p \in P$. Then $\delta\eta_{11} \in V$ which implies $\delta = 0$. Thus $(\Delta_1 \cup P) \cap V \leqq P$ and hence $(\Delta_1 \cup P) \cap V = P$.

Conversely, suppose $V \cap (\Delta_1 \cup P) = P$. Since V is an ideal in $\Delta_1 \cup V$,

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$P = V \cap (\Delta_1 \cup P)$ is an ideal in $\Delta_1 \cup P$. Therefore $\Delta_1 P \leqq P$ and P is a Δ_1-subspace of V.

(ii) By theorem 2, $A \simeq M_n(D)$ for some division algebra D. Consider the subalgebra $S' = M_n(F)$ of $M_n(\Delta)$. Put $A' = S'^{\phi}$. Then ϕ induces an $\mathscr{L}$-isomorphism of S' onto A', and by lemma 12, A' has a basis $e_{ij}(i, j = 1, 2, \ldots, n)$, where $\langle e_{ij} \rangle = \langle \eta_{ij} \rangle^{\phi}$, with either the same multiplication as the η_{ij} or the opposed multiplication. By dimension considerations as in the proof of theorem 2, it follows that the e_{ij} are a basis of A as a left vector space over D.

We now suppose that the e_{ij} have the same multiplication as the η_{ij} and prove that there exists a semi-linear map $\sigma : \Delta \to D$ such that $(\delta\delta')^{\sigma} = \delta^{\sigma}\delta'^{\sigma}$ for all $\delta, \delta' \in \Delta$. In the case in which the e_{ij} have the opposed multiplication, the same argument with the order of all products in A reversed proves the existence of a semi-linear map $\sigma : \Delta \to D$ such that $(\delta\delta')^{\sigma} = \delta'^{\sigma}\delta^{\sigma}$ for all $\delta, \delta' \in \Delta$. In either case, we then have S semi-isomorphic to A.

Put $D_1 = De_{11}$. Then it follows as in the proof of Theorem 2, that $\Delta_1^{\phi} = D_1$. Put $W = D_1 e_{12} + D_1 e_{13}$. We prove that $W = V^{\phi}$. For any subalgebra U of S, it follows from Lemmas 11, 13 and Barnes [3], that $R(U^{\phi}) = (R(U))^{\phi}$. But $V = R(\Delta_1 \cup V)$, $W = R(D_1 \cup W)$ and

$$\begin{aligned}(\Delta_1 \cup V)^{\phi} &= (\Delta_1 \cup \langle \eta_{12} \rangle \cup \langle \eta_{13} \rangle)^{\phi} \\ &= D_1 \cup E_{12} \cup E_{13} \\ &= D_1 \cup W.\end{aligned}$$

Therefore $V^{\phi} = W$.

W is a left vector space over D_1 and by (i), the D_1-subspaces of W are the subalgebras Q such that $Q = W \cap (D_1 \cup Q)$. Thus P^{ϕ} is a D_1-subspace of W if and only if P is a Δ_1-subspace of V. Thus we have an isomorphism ϕ of the lattice of F-subspaces of V onto the lattice of F-subspaces of W which takes Δ_1-subspaces of V to D_1-subspaces of W.

If $\Delta = F$, the result holds trivially. Suppose $\Delta \neq F$. Then $d(V) = 2d(\Delta) > 3$. By the "fundamental theorem of projective geometry", there exists a semi-linear map $\sigma : V \to W$ which induces ϕ (restricted to V).

The elements e_{12}, e_{13} of E_{12}, E_{13} are chosen arbitrarily in the proof of Lemma 12. We may therefore take $e_{12} = \eta_{12}^{\sigma}$ and $e_{13} = \eta_{13}^{\sigma}$. For any $\delta \in \Delta_1$, $(\delta\eta_{12})^{\sigma} \in D_1 e_{12}$ since σ maps Δ_1-subspaces of V to D_1-subspaces of W. Therefore there exists a unique $d \in D_1$ such that $de_{12} = (\delta\eta_{12})^{\sigma}$. Put $\delta^{\sigma} = d$. This defines a semi-linear map $\delta \to \delta^{\sigma}$ of Δ_1 onto D_1.

Since σ is semi-linear, $(\eta_{12}+\eta_{13})^{\sigma} = \eta_{12}^{\sigma}+\eta_{13}^{\sigma} = e_{12}+e_{13}$. For any $\delta \in \Delta_1$,

$$\begin{aligned}(\delta(\eta_{12}+\eta_{13}))^{\sigma} &= (\delta\eta_{12})^{\sigma}+(\delta\eta_{13})^{\sigma} \\ &= \delta^{\sigma} e_{12}+d^{*} e_{13}\end{aligned}$$

for some $d^* \in D_1$. But $(\delta(\eta_{12}+\eta_{13}))^\sigma \in D_1(e_{12}+e_{13})$ and therefore $\delta^* = \delta^\sigma$. Therefore, for any $\delta, \delta' \in \Delta_1$,

$$(\delta(\eta_{12}+\delta'\eta_{13}))^\sigma = (\delta\eta_{12})^\sigma + (\delta\delta'\eta_{13})^\sigma$$
$$= \delta^\sigma e_{12} + (\delta\delta')^\sigma e_{13}.$$

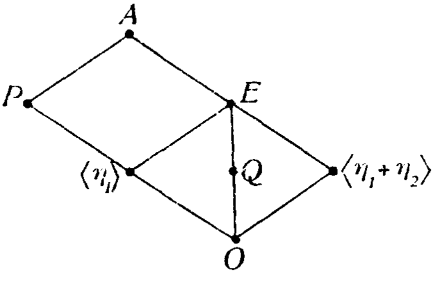
But $(\delta(\eta_{12}+\delta'\eta_{13}))^\sigma \in D_1(e_{12}+\delta'^\sigma e_{13})$ and therefore $(\delta\delta')^\sigma = \delta^\sigma\delta'^\sigma$.

6. Semi-simple algebras

LEMMA 14. *Let $F = GF(2)$ and let $A = P \oplus Q$ where P is a proper extension field of F of finite dimension and $Q \simeq F$. Let $\phi : \mathscr{L}(A) \to \mathscr{L}(B)$ be an $\mathscr{L}$-isomorphism of A onto B. Then $B = P^\phi \oplus S$ where $S \simeq F$ and P^ϕ is an extension field of F*

PROOF. (i) We need only consider the case $l(P) = 2$, for if $l(P) > 2$, we take $K \leqq P$ such that $l(K) = 2$. If the result holds for $K \oplus Q$ and η_1 is the identity of P, then $\langle \eta_1 \rangle^\phi$ is not nilpotent, P^ϕ has a unique minimal subalgebra $\langle \eta_1 \rangle^\phi$ and so is a field. By hypothesis, $(K \oplus Q)^\phi = K^\phi \oplus S$ where $S \simeq F$. Take e_1 the identity of P^ϕ and e_2 the identity of S. Then $e_1 e_2 = e_2 e_1 = 0$. For all $x \in P^\phi$, $xe_2 = (xe_1)e_2 = 0$, $e_2 x = e_2(e_1 x) = 0$. Therefore P^ϕ is an ideal in B and $B = P^\phi \oplus S$.

(ii) Let η_1, η_2 be the identitites of P, Q. Put $E = \langle \eta_1, \eta_2 \rangle$. Then $\langle \eta_1 \rangle$, $\langle \eta_2 \rangle = Q$, $\langle \eta_1 + \eta_2 \rangle$ are all the minimal subalgebras of A and $\mathscr{L}(A)$ is



(iii) Suppose B is nilpotent. Then $d(B) = 3$. If $b \in B$ and $b^3 \neq 0$, then $\langle b, b^2, b^3 \rangle = B$ and $\langle b^2, b^3 \rangle$ is the only maximal subalgebra of B. Therefore $b^3 = 0$ for all $b \in B$. If $b^2 \neq 0$, then $\langle b, b^2 \rangle$ is a subalgebra $\mathscr{L}$-isomorphic to P, and therefore $b \in P^\phi$.

We have $P^\phi = \langle u, u^2 \rangle$, $Q^\phi = \langle v \rangle$ for some u, v. Since E^ϕ is nilpotent and has three minimal subalgebras, E^ϕ is null. Therefore the three minimal subalgebras of B are $\langle u^2 \rangle$, $\langle v \rangle$, $\langle u^2 + v \rangle$.

Consider the element $u+v$. Since $u+v \notin P^\phi$, we have $(u+v)^2 = 0$. Therefore $\langle u+v \rangle$ is another minimal subalgebra of B. Therefore B is not nilpotent.

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(iv) Suppose P^ϕ is nilpotent. Then $P^\phi = \langle u, u^2 \rangle$ for some u, and $Q^\phi = \langle e \rangle$ for some idempotent e. Since $l(B) = 3$, every nilpotent element of B is in $R = R(B)$. Therefore $P^\phi \leqq R$. But B is not nilpotent. Therefore $R = P^\phi$ and $\langle \eta_1 \rangle^\phi = R^2$ is an ideal in B. Since B/R^2 has only two minimal subalgebras, it is commutative and $eu - ue \in \langle u^2 \rangle$. Therefore $ueu - u^2e = 0$, $eu^2 - ueu = 0$ and $eu^2 = u^2e$. This implies that E^ϕ has only two minimal subalgebras. Therefore P^ϕ is not nilpotent, and so must be an extension field of F.

(v) Suppose $R = R(B) \neq 0$. Then $d(R) = 1$, $R = \langle r \rangle$ for some r, and either $R = Q^\phi$ or $R = \langle \eta_1 + \eta_2 \rangle^\phi$. Let e be the identity of P^ϕ. Then E^ϕ is of type VI and without loss of generality, we may suppose $er = r$, $re = 0$. Then $d(P^\phi r) = d(P^\phi)$ since P^ϕ is a field and $er = r \neq 0$. But $r \in R$ which is an ideal, and therefore $d(P^\phi) = 1$ contrary to $l(P) = 2$. Therefore $R(B) = 0$.

(vi) Since $R(B) = 0$, B is a direct sum of fields. Since P^ϕ is a proper extension of F, one of the direct summands of B is a proper extension. But P^ϕ is the only subalgebra of B with lattice of length 2 and only one minimal subalgebra. Therefore P^ϕ is a direct summand of B. Clearly the other direct summand is isomorphic to F.

It is clear from the lattice diagram that a lattice automorphism of A may map Q to $\langle \eta_1 + \eta_2 \rangle$. Thus Q^ϕ need not be a direct summand of B.

LEMMA 15. *Let $F = GF(2)$ and let A be a finite-dimensional semi-simple algebra over F, not a field or direct sum of one-dimensional algebras. Let $\phi : \mathscr{L}(A) \to \mathscr{L}(B)$ be an $\mathscr{L}$-isomorphism of A onto B, and let η be an idempotent in A. Then $\langle \eta \rangle^\phi$ is not nilpotent.*

PROOF. Let $S_1, \ldots, S_m$ be the simple direct summands of A. We may suppose $d(S_1) \geqq 2$.

Suppose $\eta \in S_1$. If $S_1 \simeq M_n(K)$, $n \geqq 2$ for some extension field K of F, then the result holds by Lemma 13. If S_1 is a field, then by hypothesis, $A \neq S_1$. Let η_2 be the identity of S_2. By Lemma 14 applied to $S_1 \cup \langle \eta_2 \rangle$, $\langle \eta \rangle^\phi$ is not nilpotent.

Suppose $\eta \notin S_1$. S_1 contains a field K which is a proper extension of F since, if S_1 is not itself a field, then it has a subalgebra isomorphic to $M_2(F)$ which has a subalgebra isomorphic to $GF(4)$. We take some such subfield K of S_1. Then $K \cup \langle \eta \rangle \simeq K \oplus F$ and by Lemma 14, $\langle \eta \rangle^\phi$ is not nilpotent.

LEMMA 16. *Let A be a finite-dimensional semi-simple algebra over the field F, and let $\phi : L(A) \to \mathscr{L}(B)$ be an $\mathscr{L}$-isomorphism of A onto an algebra B. Suppose A is not a division algebra. If $F = GF(2)$, suppose further that A is not a direct sum of one-dimensional algebras. Let U be a subalgebra of A. Then U^ϕ is nilpotent if and only if U is nilpotent.*

PROOF. (i) Let $N = \langle n \rangle$ be a one-dimensional nilpotent subalgebra of A. We prove N^ϕ nilpotent. Let $S_1, \ldots, S_r$ be the simple direct summands of A which are not division algebras. (If there are no such summands, then A has no nilpotent subalgebras and we have nothing to prove.) Then $N \leqq S_1 \oplus \cdots \oplus S_r$. If $N \leqq S_i$ for some i, then N^ϕ is nilpotent by Lemma 11. We use induction over r. Suppose the result holds for subalgebras of $S_1 \oplus \ldots \oplus S_{r-1}$. Then $n = u+v$, $u \in S_1 \oplus \cdots \oplus S_{r-1}$, $v \in S_r$, and we may suppose $u \neq 0$, $v \neq 0$. Then $\langle u \rangle^\phi$, $\langle v \rangle^\phi$ are nilpotent and therefore $\langle u, v \rangle^\phi$ is nilpotent. But $N^\phi \leqq \langle u, v \rangle^\phi$ and so is nilpotent.

(ii) Let η be an idempotent in A. We prove that $\langle \eta \rangle^\phi$ is not nilpotent. In the case $F = GF(2)$, this holds by Lemma 15. Suppose $F \neq GF(2)$. There exists an idempotent $\eta' \neq \eta$ which commutes with η since the identity of A is not the only idempotent in A. For this η', $\langle \eta, \eta' \rangle$ is of type IV and by Lemma 6, $\langle \eta, \eta' \rangle^\phi$ is of type IV and $\langle \eta \rangle^\phi$ is not nilpotent.

(iii) We have now proved the result for one-dimensional subalgebras of A. But a subalgebra is nilpotent if and only if all its one-dimensional subalgebras are nilpotent. Hence the result holds for all subalgebras of A.

LEMMA 17. *Suppose S_1, S_2 are $\mathscr{L}$-isomorphic finite-dimensional simple algebras and $A = S_1 \oplus S_2$. Let η be the identity of A. Then there exists a subalgebra S of A $\mathscr{L}$-isomorphic to S_1 and containing η if and only if $S_1 \simeq S_2$.*

PROOF. (i) Suppose $\alpha : S_1 \to S_2$ is an isomorphism. Put

$$S_\alpha = \{s+s^\alpha | s \in S_1\}.$$

Then $S_\alpha \simeq S_1$. If η_i is the identity of S_i, then $\eta_1^\alpha = \eta_2$ and $\eta = \eta_1+\eta_2 \in S_\alpha$.

(ii) Suppose S is $\mathscr{L}$-isomorphic to S_1 and $\eta \in S$. If $S_1 \simeq M_n(\Delta)$, $n \geqq 2$, then S is simple by Theorem 2. If S_1 is a division algebra, then A has no nilpotent elements. Since S has only one minimal subalgebra, S is a division algebra. Thus in either case, S is simple. Each element $s \in S$ is uniquely expressible in the form $s = s_1+s_2$, $s_i \in S_i$. The map $\alpha_i : S \to S_i$ given by $s^{\alpha_i} = s_i$ is a homomorphism. Since S is simple and $\eta^{\alpha_i} = \eta_i \neq 0$, α_i is a monomorphism. Since $l(S) = l(S_i)$ and $l(S_i)$ is finite, α_i is onto and therefore $S_1 \simeq S \simeq S_2$.

THEOREM 4. *Let A be a finite-dimensional semi-simple algebra over the field F, and let $\phi : \mathscr{L}(A) \to \mathscr{L}(B)$ be an $\mathscr{L}$-isomorhism of A onto an algebra B. Let $S_1, \ldots, S_r$ be the simple direct summands of A. Suppose A is not a division algebra and, in the case $F = GF(2)$, that not all the S_i are one-dimensional. Then B is semi-simple. For each S_i of dimension greater than one, S_i^ϕ is a simple direct summand of B. If $S_i \simeq S_j$, then $S_i^\phi \simeq S_j^\phi$.*

PROOF. By Lemma 16, ϕ maps the maximal nilpotent subalgebras of A to the maximal nilpotent subalgebras of B. By Barnes [3], it follows

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that $(R(A))^\phi = R(B)$ and therefore $R(B) = 0$.

Suppose S is a simple subalgebra of A. If S is not a division algebra, then S^ϕ is simple by Theorem 2. If S is a division algebra, then by Lemmas 2, 16, S^ϕ is a division algebra. In either case, S^ϕ is simple.

Let e be any idempotent of B. There exists an idempotent $\eta \in A$ such that $\langle \eta \rangle^\phi = \langle e \rangle$. Let S_i be a simple direct summand of A of dimension greater than one. Then $\langle S_i, \eta \rangle$ has only one maximal simple subalgebra of dimension greater than one. Therefore $\langle S_i, \eta \rangle^\phi$ is semi-simple algebra with only one maximal simple subalgebra of dimension greater than one, and therefore this subalgebra S_i^ϕ is a direct summand of $\langle S_i, \eta \rangle^\phi$. Therefore $eS_i^\phi \leqq S_i^\phi$, $S_i^\phi e \leqq S_i^\phi$. Let $T_1, \ldots, T_s$ be the simple direct summands of B, let e_k be the identity of T_k and ε the identity of S_i^ϕ. Since $\varepsilon e_j \in S_i^\phi \cap T_j$, either $\varepsilon e_j = 0$ or $S_i^\phi \leqq T_j$. Since $e_1 + \ldots + e_s$ is the identity of B, for some $j, \varepsilon e_j \neq 0$ and $S_i^\phi \leqq T_j$. But similarly $T_j^{\phi^{-1}} \leqq S_k$ for some k. Therefore $S_i^\phi = T_j$ and S_i^ϕ is a direct summand of B.

Suppose $S_i \simeq S_j$, $i \neq j$. If $d(S_i) = 1$, then trivially $S_i^\phi \simeq S_j^\phi$. Suppose $d(S_i) \geqq 2$. Then S_i^ϕ, S_j^ϕ are direct summands of $(S_i \oplus S_j)^\phi$. There exists $S \simeq S_i$ contained in $S_i \oplus S_j$ and containing the identity η of $S_i \oplus S_j$. Let η_i, η_j be the identities of S_i, S_j and e_i, e_j those of S_i^ϕ, S_j^ϕ. Since $\langle \eta_i \rangle^\phi = \langle e_i \rangle$, $\langle \eta_j \rangle^\phi = \langle e_j \rangle$ and $\langle \eta_i, \eta_j \rangle$ has only three minimal subalgebras, $\langle \eta \rangle^\phi = \langle e \rangle$ where e is the identity of $S_i^\phi + S_j^\phi$. Thus S^ϕ is a simple subalgebra of $S_i^\phi + S_j^\phi$ $\mathscr{L}$-isomorphic to S_i^ϕ and containing e. By Lemma 17, $S_i^\phi \simeq S_j^\phi$.

We remark that the method of proof of Theorem 3 can be extended to show that, if $S_i \simeq S_j \simeq M_n(\Delta)$, $n \geqq 2$, Δ a finite dimensional division algebra, then $S_i \oplus S_j$ is semi-isomorphic to $(S_i \oplus S_j)^\phi$. We need only consider the case $n = 2$. If $\eta_{rs} \in S_i$ have the usual meaning and η'_{rs} are the corresponding elements of S_j, we consider $\Delta\eta_{12} + \Delta\eta'_{12}$ as a left vector space over $\Delta(\eta_{11} + \eta'_{11})$ and the result follows as before. If one of the direct summands $S_i \simeq M_n(F)$, $n \geqq 2$, then we have $S_i \simeq S_i^\phi$. Thus we have

COROLLARY. *Let A be a finite-dimensional semi-simple algebra over an algebraically closed field F. Suppose A has dimension greater than one. Let $\phi : \mathscr{L}(A) \to \mathscr{L}(B)$ be an $\mathscr{L}$-isomorphism of A onto an algebra B over F. Then $A \simeq B$.*

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