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RANDOMIZATION MODULI OF CONTINUITY FOR ℓ^2-NORM SQUARED ORNSTEIN-UHLENBECK PROCESSES

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ABSTRACT We establish exact randomized moduli of continuity for ℓ^2-norm squared independent Ornstein-Uhlenbeck processes

1. **Introduction.** Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k, *i.e.*, $X_k(\cdot)$ are stationary Gaussian processes with $EX_k(t) = 0$ and

$$EX_k(s)X_k(t) = \frac{\gamma_k}{\lambda_k}\exp(-\lambda_k|t-s|), \quad k = 1, 2, \ldots,$$

where $\gamma_k \geq 0$, $\lambda_k > 0$.

The process $Y(\cdot)$ was introduced by Dawson (1972) as the stationary solution of an infinite array of stochastic differential equations

$$dX_k(t) = -\lambda_k X_k(t)\,dt + (2\gamma_k)^{\frac{1}{2}}\,dW_k(t), \quad k = 1, 2, \ldots,$$

where $\{W_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ are independent Wiener processes. Since $EX_t^2(t) = \frac{\gamma_k}{\lambda_k}$, it is easy to see that for every fixed t, $Y(t)$ is almost surely in ℓ^2 if and only if

$$\Gamma_0 = E\Big(\sum_{k=1}^{\infty} X_k^2(t)\Big) = \sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k} < \infty. \tag{1.1}$$

The main purpose of this paper is to study the behavior of the real valued process

$$\chi^2(t) = \sum_{k=1}^{\infty} X_k^2(t), \quad -\infty < t < \infty. \tag{1.2}$$

We introduce first the following notations:

$$\sigma_k^2(h) = E\big(X_k(t+h) - X_k(t)\big)^2 = \frac{2\gamma_k}{\lambda_k}(1 - e^{-\lambda_k h}), \tag{1.3}$$

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(1.4) $$\tilde\sigma_k^2(h) = E\big(X_k^2(t+h) - X_k^2(t)\big)^2 = 4\Big(\frac{\gamma_k}{\lambda_k}\Big)^2(1-e^{-2\lambda_k h}),$$

(1.5) $$\sigma^*(h) = \max_{k\ge 1}\sigma_k(h), \quad \tilde\sigma^*(h) = \max_{k\ge 1}\tilde\sigma_k(h),$$

(1.6) $$\sigma^2(h) = \sum_{k=1}^{\infty}\sigma_k^2(h) = 2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h}),$$

(1.7) $$\tilde\sigma^2(h) = \sum_{k=1}^{\infty}\tilde\sigma_k^2(h) = 4\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2(1-e^{-2\lambda_k h}).$$

We will need also the notion of a quasi-increasing function. A function $f(x)$ defined on (a,b) is called *quasi-increasing* if there exists a positive c such that

$$f(x) \le cf(y) \text{ for all } a < x < y < b.$$

The process $\chi^2(\cdot)$ was studied by Iscoe and McDonald (1986, 1989), Schmuland (1988), Csörgő and Lin (1990), Csáki and Csörgő (1992), Csáki, Csörgő and Shao (1991). We note that the real valued process $\chi^2(\cdot)$ is continuous almost surely if and only if the necessary and sufficient conditions of Fernique (1989) for the almost sure continuity of $Y(\cdot) \in \ell^2$ hold true (*cf.* Corollary 5.1 of Csáki and Csörgő (1992)). Csörgő and Lin (1990) investigated the problem of moduli of continuity for $\chi^2(\cdot)$ under the condition

(1.8) $$\sum_{k=1}^{\infty}\frac{\gamma_k^2}{\lambda_k} < \infty$$

and, in this case, they proved the following results: Assume that T_h is non-increasing with $\lim_{h\to 0}\log T_h/\log\frac{1}{h} = \infty$. Then

(1.9) $$\lim_{h\to 0}\sup_{|t|\le T_h}\sup_{0\le s\le h}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8hM)^{\frac{1}{2}}\log(T_h/h)} = 1 \text{ a.s.}$$

where $M = \max_{k\ge 1}\gamma_k^2/\lambda_k$.

Csáki, Csörgő and Shao (1991) obtained the following similar results to (1.9): Assume that $\tilde\sigma(h)/h^\alpha$ and $\tilde\sigma^*(h)/h^\alpha$ are quasi-increasing on (0,1) for some $\alpha > 0$. If

(1.10) $$\tilde\sigma^*(h)\Big(\log\frac{1}{h}\Big)^{\frac{1}{2}} = o\big(\tilde\sigma(h)\big) \text{ as } h\to 0,$$

then

(1.11) $$\limsup_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\frac{|\chi^2(t+s)-\chi^2(t)|}{\tilde\sigma(h)(2\log\frac{1}{h})^{\frac{1}{2}}} \le 1 \text{ a.s.}$$

If

(1.12) $$\tilde\sigma(h) = o\Big(\tilde\sigma^*(h)\big(\log\frac{1}{h}\big)^{\frac{1}{2}}\Big) \text{ as } h\to 0,$$

then

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{|\chi^2(t+s)-\chi^2(t)|}{\tilde{\sigma}^*(h)\log\frac{1}{h}} \le 1 \text{ a.s.} \tag{1.13}$$

At the end of their paper, Csáki, Csörgő and Shao (1991) conjectured that (1.11) remains valid with equality instead of the inequality under condition (1.10) and that there is no function $\theta(h)$ such that

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \theta(h)|\chi^2(t+s)-\chi^2(t)| = 1 \text{ a.s.}$$

if (1.12) is satisfied.

In this paper we give a partial answer to the above conjecture in that we give conditions for exact, though randomized, moduli of continuity for the ℓ^2-norm squared process $\chi^2(\cdot)$. We present our main results in Section 2. Their proofs are given in Section 3.

Throughout this note we will use the following notations: $\log x = \ln\max(x,e)$, where ln is the natural logarithm; $\Phi(x)$ denotes the standard normal distribution function; $[x]$ denotes the integer part of x.

2. **Main results.** Let $\chi^2(\cdot)$ be the ℓ^2-norm squared process defined by (1.2). We put

$$N = \#\{k : \gamma_k > 0\} \tag{2.1}$$

and define the real valued process

$$\Gamma^2(t,h) = 4\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1-e^{-2\lambda_k h})X_k^2(t), \quad t,h\ge 0. \tag{2.2}$$

THEOREM 2.1. *Assume (1.1), $N\ge 2$ and that as $h\longrightarrow 0$*

$$\sigma^2(h) = o\Big(\big(h/\log\frac{1}{h}\big)^{\frac{1}{2}}\Big), \tag{2.3}$$

$$\sum_{\lambda_k\ge\frac{1}{h}} \Big(\frac{\gamma_k}{\lambda_k}\Big)^2 = o\Big(h/\log\frac{1}{h}\Big). \tag{2.4}$$

Then

$$\lim_{h\to 0} \sup_{0\le t\le h^\theta} \sup_{0\le s\le h} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h)(2\log\frac{1}{h})^{\frac{1}{2}}} = (1-\theta)^{\frac{1}{2}} \text{ a.s.} \tag{2.5}$$

for every $0\le\theta<1$.

THEOREM 2.2. *Assume (1.1), $N\ge 1$ and that as $h\longrightarrow 0$*

$$\sigma^2(h) = o\Big(\big(h/\log\log\frac{1}{h}\big)^{\frac{1}{2}}\Big), \tag{2.6}$$

$$\sum_{\lambda_k\ge\frac{1}{h}} \Big(\frac{\gamma_k}{\lambda_k}\Big)^2 = o\Big(h/\log\log\frac{1}{h}\Big). \tag{2.7}$$

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Then

$$\limsup_{h\to 0}\sup_{0\le s\le h}\frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h)(2\log\log\frac{1}{h})^{\frac{1}{2}}}=1 \text{ a.s.} \tag{2.8}$$

$$\limsup_{h\to 0}\frac{\chi^2(t+h)-\chi^2(t)}{\Gamma(t,h)(2\log\log\frac{1}{h})^{\frac{1}{2}}}=1 \text{ a.s.} \tag{2.9}$$

for each t.

The next two corollaries follow from Theorems 2.1 and 2.2 immediately.

COROLLARY 2.1. *Assume that*

$$\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}<\infty, \tag{2.10}$$

$$\sum_{k=1}^{\infty}\gamma_k\log\lambda_k<\infty. \tag{2.11}$$

Then (2.8) and (2.9) hold true if $N\ge 1$ *and so does (2.5) if* $N\ge 2$.

COROLLARY 2.2. *Let* $\lambda_k=k^{\alpha}$, $\gamma_k=k^{\beta}$, $\alpha>\max(0,1+\beta,2+2\beta)$. *Then (2.5), (2.8) and (2.9) hold true.*

Our next example shows that Theorem 2.1 may fail if $N=1$.

EXAMPLE 2.1. Let $\lambda_k\equiv 1$, $\gamma_1=1$, $\gamma_i=0$ for each $i\ge 2$. Then, for all positive functions $f(h)$

$$P\left(\liminf_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\frac{\chi^2(t+s)-\chi^2(t)}{\Gamma(t,h)f(h)}=\infty\right)>0.$$

The proof of this example follows by noting that, on account of an Ornstein-Uhlenbeck process $X(\cdot)$ being nowhere differentiable, with positive probability there exists a random t_0 such that $0\le t_0\le 1$, $X(t_0)=0$ and $\sup_{0\le s\le h}X^2(t_0+s)>0$ for every $0<h<1$. Hence

$$\sup_{0\le s\le h}\frac{X^2(t_0+s)-X^2(t_0)}{|X(t_0)|}=\infty$$

for all $h>0$, from which we conclude Example 2.1.

3. **Proof of theorems.** We start with a preliminary lemma.

LEMMA 3.1. *Let* $\{\xi_n, n\ge 1\}$ *be a sequence of independent standard normal random variables and* $\{a_n, n\ge 1\}$ *be a sequence of non-negative numbers with* $\sum_{n=1}^{\infty}a_n<\infty$. *Then*

$$P\left\{\left|\left(\sum_{i=1}^{\infty}a_i\xi_i^2\right)^{\frac{1}{2}}-E\left(\sum_{i=1}^{\infty}a_i\xi_i^2\right)^{\frac{1}{2}}\right|\ge x\right\}\le 2\exp\left(-\frac{x^2}{2\max_{i\ge 1}a_i}\right) \tag{3.1}$$

for each $x > 0$.

PROOF. It is well-known that one can rewrite

$$\Big(\sum_{i=1}^{\infty} a_i \xi_i^2\Big)^{\frac{1}{2}} = \sup_{\|b\|_{\ell_2}\le 1} \sum_{i=1}^{\infty} b_i a_i^{\frac{1}{2}} \xi_i,$$

where $b = (b_1, b_2, \ldots)$. By Borel's inequality (*cf.* Adler (1990)) we have

$$\begin{aligned} P\Big\{\Big|\Big(\sum_{i=1}^{\infty} a_i\xi_i^2\Big)^{\frac{1}{2}} - E\Big(\sum_{i=1}^{\infty} a_i\xi_i^2\Big)^{\frac{1}{2}}\Big| \ge x\Big\} \\ &= P\Big\{\Big|\sup_{\|b\|_{\ell_2}\le 1}\sum_{i=1}^{\infty} b_i a_i^{\frac{1}{2}}\xi_i - E\sup_{\|b\|_{\ell_2}\le 1}\sum_{i=1}^{\infty} b_i a_i^{\frac{1}{2}}\xi_i\Big| \ge x\Big\} \\ &\le 2\exp\Big(-\frac{x^2}{2\sup_{\|b\|_{\ell_2}\le 1}\sum_{i=1}^{\infty} b_i^2 a_i}\Big) \\ &\le 2\exp\Big(-\frac{x^2}{2\max_{i\ge 1} a_i \sup_{\|b\|_{\ell_2}\le 1}\sum_{i=1}^{\infty} b_i^2}\Big) \\ &= 2\exp\Big(-\frac{x^2}{2\max_{i\ge 1} a_i}\Big), \end{aligned}$$

as desired.

LEMMA 3.2. *Let* $\{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ *be a sequence of independent Ornstein-Uhlenbeck processes defined as in Section 1. Let* $\hat\sigma(h)$ *and* $\hat\sigma^*(h)$ *be non-decreasing functions such that* $\sigma(h) \le \hat\sigma(h)$ *and* $\sigma^*(h) \le \hat\sigma^*(h)$. *Assume that* $\hat\sigma(h)/h^\alpha$ *and* $\hat\sigma^*(h)/h^\alpha$ *are quasi-increasing on (0,1) for some* $\alpha > 0$. *Then*

$$\limsup_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\frac{\sum_{k=1}^{\infty}(X_k(t+s)-X_k(t))^2}{\hat\sigma^{*2}(h)\log\frac{1}{h} + \hat\sigma^2(h)} \le 8 \ a.s. \tag{3.2}$$

PROOF. This is Theorem 4.1 of Csáki, Csörgő and Shao (1991) with minor modifications. It was assumed in their Theorem 4.1 that $\sigma(h)/h^\alpha$ and $\sigma^*(h)/h^\alpha$ are quasi-increasing.

PROOF OF THEOREM 2.1. We first list two facts.

(3.3) $\Gamma^2(t,h)$ is a non-increasing function of h for each t fixed,

(3.4) $\Gamma^2(t,h)/h^{\frac{1}{9}}$ is a non-decreasing function of h on $(0,\delta)$ for each t fixed if

(3.5) $\sum_{\lambda_k\ge\frac{1}{h}}\frac{\gamma_k}{\lambda_k}X_k^2(t) \le \frac{1}{18}\Gamma^2(t,h)$ for $0 < h < \delta$.

Let $\{W_k(t), t\ge 0\}$ be a sequence of independent standard Wiener processes. It is easy to see that

$$\Big\{\Big(\frac{\gamma_k}{\lambda_k}\Big)^{\frac{1}{2}}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}}, -\infty < t < \infty\Big\}_{k=1}^{\infty} \text{ and } \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$$

have the same distribution. Hence, we can redefine

$$X_k(t) = \Big(\frac{\gamma_k}{\lambda_k}\Big)^{\frac{1}{2}} \frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}}, \quad k = 1, 2, \ldots, \tag{3.6}$$

$$\Gamma^2(t,h) = 4\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2 (1 - e^{-2\lambda_k h})\frac{W_k^2(e^{2\lambda_k t})}{e^{2\lambda_k t}}, \quad h \geq 0. \tag{3.7}$$

For t, $s \geq 0$, we have

(3.8)

$$\begin{aligned}\chi^2(t+s) - \chi^2(t) &= 2\sum_{k=1}^{\infty}\big(X_k(t+s) - X_k(t)\big)X_k(t) + \sum_{k=1}^{\infty}\big(X_k(t+s) - X_k(t)\big)^2 \\ &= 2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k(t+s)}) - W_k(e^{2\lambda_k t})}{e^{\lambda_k(t+s)}}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}} \\ &\quad - 2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k^2(e^{2\lambda_k t})}{e^{2\lambda_k t}}(1 - e^{-\lambda_k s}) + \sum_{k=1}^{\infty}\big(X_k(t+s) - X_k(t)\big)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\chi^2(t+s) - \chi^2(t) &\leq 2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k(t+s)}) - W_k(e^{2\lambda_k t})}{e^{\lambda_k(t+s)}}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}} \\ &\quad - 2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k^2(e^{2\lambda_k t})}{e^{2\lambda_k t}}(1 - e^{-\lambda_k s}) \\ &:= \Delta_1(t,s) - \Delta_2(t,s),\end{aligned} \tag{3.9}$$

where

$$\Delta_1(t,s) = 2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k(t+s)}) - W_k(e^{2\lambda_k t})}{e^{\lambda_k(t+s)}}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}}, \tag{3.10}$$

$$\Delta_2(t,s) = 2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k^2(e^{2\lambda_k t})}{e^{2\lambda_k t}}(1 - e^{-\lambda_k s}). \tag{3.11}$$

We show first that

$$\limsup_{h\to 0}\sup_{0\leq t\leq h^\theta}\sup_{0\leq s\leq h}\frac{|\chi^2(t+s) - \chi^2(t)|}{\Gamma(t,h)(2\log\frac{1}{h})^{\frac{1}{2}}} \leq (1-\theta)^{\frac{1}{2}} \text{ a.s.} \tag{3.12}$$

Let $1 < \delta < 2$ and $h_k = \delta^{-k}$, $k = 1, 2, \ldots$. Then, by (3.8) and (3.3)

$$\limsup_{h\to 0} \sup_{0\le t\le h^\theta} \sup_{0\le s\le h} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h)(2\log\frac{1}{h})^{\frac{1}{2}}}$$

$$\le \limsup_{j\to\infty} \sup_{0\le t\le h_j^\theta} \sup_{0\le s\le h_j} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h_{j+1})(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$

$$\le \delta^{\frac{1}{2}} \limsup_{j\to\infty} \sup_{0\le t\le h_j^\theta} \sup_{0\le s\le h_j} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$

$$\le \delta^{\frac{1}{2}} \limsup_{j\to\infty} \sup_{0\le t\le h_j^\theta} \sup_{0\le s\le h_j} \frac{|\Delta_1(t,s)|}{\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$

$$+ 2\limsup_{j\to\infty} \sup_{0\le t\le 1} \frac{\Delta_2(t,h_j)}{\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$

$$+ 2\limsup_{j\to\infty} \sup_{0\le t\le 1} \sup_{0\le s\le h_j} \frac{\sum_{k=1}^\infty (X_k(t+s)-X_k(t))^2}{\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}} \tag{3.13}$$

$$\le \delta^{\frac{1}{2}} \limsup_{j\to\infty} \max_{0\le i\le \frac{h_j^{\theta-1}}{\gamma-1}} \sup_{(\gamma-1)ih_j\le t\le (i+1)(\gamma-1)h_j} \sup_{0\le s\le h_j}$$

$$\left(\frac{2\left|\sum_{k=1}^\infty \frac{\gamma_k}{\lambda_k} \frac{W_k(e^{2\lambda_k(t+s)})-W_k(e^{2\lambda_k t})}{e^{\lambda_k(t+s)}} - \frac{W_k(e^{2\lambda_k ih_j(\gamma-1)})}{e^{\lambda_k ih_j(\gamma-1)}}\right|}{\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}\right)$$

$$+ 32\limsup_{j\to\infty} \sup_{0\le t\le 1} \frac{\Delta_2(t,h_j)}{\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$

$$+ 32\limsup_{j\to\infty} \sup_{0\le t\le 1} \sup_{0\le s\le h_j} \frac{\sum_{k=1}^\infty (X_k(t+s)-X_k(t))^2}{\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$

$$:= \delta^{\frac{1}{2}} I_1 + 32 I_2 + 32 I_3.$$

It follows from (3.3) that

$$\inf_{0\le t\le 1} \Gamma^2(t,h) \ge h \inf_{0\le t\le 1} \Gamma^2(t,1) \text{ for each } 0 < h \le 1. \tag{3.14}$$

Since $N \ge 2$, there are at least two of the $\{\gamma_k\}$ which are positive. Without loss of generality, assume $\gamma_1, \gamma_2 > 0$. Then

$$\inf_{0\le t\le 1} \Gamma^2(t,1) \ge 4 \min_{1\le k\le 2} \left(\frac{\gamma_k}{\lambda_k}\right)^2 (1-e^{-2\lambda_k})e^{-2\lambda_k} \inf_{0\le t\le 1}\left(W_1^2(e^{\lambda_1 t}) + W_2^2(e^{\lambda_2 t})\right)$$
$$> 0 \text{ a.s.} \tag{3.15}$$

by a theorem of Kakutani (1944) (*cf.* Theorem 2.10 of Knight (1981)). Therefore,

$$\inf_{j\ge 1} \frac{\inf_{0\le t\le 1}\Gamma^2(t,h_j)}{h_j} > 0 \text{ a.s.} \tag{3.16}$$

by (3.14) and (3.15). Using Lemma 3.2 and (2.3), we obtain

$$\limsup_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\frac{\sum_{k=1}^{\infty}(X_k(t+s)-X_k(t))^2}{h^{\frac12}(\log\frac1h)^{\frac12}}=0 \text{ a.s.} \tag{3.17}$$

A combination of (3.17) with (3.16) yields

$$I_3 = 0 \text{ a.s.} \tag{3.18}$$

Notice that

$$\begin{aligned} I_2 &\le 2\limsup_{j\to\infty}\frac{\max_{0\le i\le h_j^{-1}}\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k^2(e^{2\lambda_k i h_j})}{e^{2\lambda_k i h_j}}(1-e^{-\lambda_k h_j})}{\inf_{0\le t\le 1}\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac12}} \\ &\quad + 2\limsup_{j\to\infty}\sup_{0\le t\le 1}\sup_{0\le s\le h_j}\frac{\sum_{k=1}^{\infty}(X_k(t+s)-X_k(t))^2}{\inf_{0\le t\le 1}\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac12}} \\ &= 2\limsup_{j\to\infty}\frac{\max_{0\le i\le h_j^{-1}}\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_j})\frac{W_k^2(e^{2\lambda_k i h_j})}{e^{2\lambda_k i h_j}}}{\inf_{0\le t\le 1}\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac12}} \text{ a.s.} \end{aligned} \tag{3.19}$$

by (3.18). It follows from Lemma 3.1 that

$$\begin{aligned} &P\Big\{\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_j})\frac{W_k^2(e^{2\lambda_k i h_j})}{e^{2\lambda_k i h_j}}\ge 16\Big(\sigma^2(h_j)+\sigma^{*2}(h_j)\log\frac{1}{h_j}\Big)\Big\} \\ &\le P\Big\{\Big(\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_j})\frac{W_k^2(e^{2\lambda_k i h_j})}{e^{2\lambda_k i h_j}}\Big)^{\frac12} \\ &\qquad - E\Big(\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_j})\frac{W_k^2(e^{2\lambda_k i h_j})}{e^{2\lambda_k i h_j}}\Big)^{\frac12}\ge 2\sigma^*(h_j)\Big(\log\frac{1}{h_j}\Big)^{\frac12}\Big\} \\ &\le 2\exp\Big(-4\log\frac{1}{h_j}\Big) \\ &= 2h_j^4 \end{aligned}$$

and hence

$$\limsup_{j\to\infty}\frac{\max_{0\le i\le h_j^{-1}}\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_j})\frac{W_k^2(e^{2\lambda_k i h_j})}{e^{2\lambda_k i h_j}}}{\sigma^2(h_j)+\sigma^{*2}(h_j)\log\frac{1}{h_j}}\le 16 \text{ a.s.,} \tag{3.20}$$

which, together with (3.16), (2.3) and (3.19), implies

$$I_2 = 0 \text{ a.s.} \tag{3.21}$$

Note that

$$
\begin{aligned}
&\limsup_{j\to\infty} \sup_{0\le t\le 1} \sup_{0\le s\le h_j} \frac{\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1-e^{-2\lambda_k h_j})(X_k(t+s)-X_k(t))^2}{h_j} \\
&\le \limsup_{j\to\infty} \sup_{0\le t\le 1} \sup_{0\le s\le h_j} \frac{\sigma^{*2}(2h_j)\sum_{k=1}^{\infty}(X_k(t+s)-X_k(t))^2}{h_j} \qquad (3.22)\\
&\le \limsup_{j\to\infty} \sup_{0\le t\le 1} \sup_{0\le s\le h_j} \frac{\sigma^2(2h_j)+\sigma^{*2}(2h_j)\log\frac{1}{h_j}}{h_j\log\frac{1}{h_j}} \sum_{k=1}^{\infty}\big(X_k(t+s)-X_k(t)\big)^2 \\
&= 0 \text{ a.s.}
\end{aligned}
$$

by (3.17) and (2.3). We conclude from (3.22) and (3.16) that

$$
\lim_{j\to\infty} \max_{0\le i\le \frac{1}{\delta-1}h_j^{\theta\ 1}} \left| \frac{\inf_{i(\delta-1)h_j\le t\le (i+1)(\delta-1)h_j} \Gamma(t,h_j)}{\Gamma(i(\delta-1)h_j,h_j)} - 1 \right| = 0 \text{ a.s.}
$$

Thus, we have (putting $h_{j,i} = i(\delta-1)h_j$)

(3.23)

$$
\begin{aligned}
I_1 &\le \limsup_{j\to\infty} \max_{0\le i\le h_j^{\theta-1}/(\delta-1)} \sup_{h_{j,i}\le t\le h_{j,i+1}} \sup_{0\le s\le h_j} \left(\frac{2\left|\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k(t+s)})-W_k(e^{2\lambda_k t})}{e^{\lambda_k(t+s)}}\frac{W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k h_{j,i}}}\right|}{\Gamma(h_{j,i},h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}} \right) \\
&\le \delta \limsup_{j\to\infty} \max_{0\le i\le h_j^{\theta\ 1}/(\delta-1)} \sup_{0\le s\le \delta h_j} \frac{2\left|\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k(s+h_{j,i})})-W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k(s+h_{j,i})}}\frac{W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k h_{j,i}}}\right|}{\Gamma(h_{j,i},\delta h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}} \\
&\quad + \limsup_{j\to\infty} \max_{0\le i\le h_j^{\theta\ 1}/(\delta-1)} \sup_{h_{j,i}\le t\le h_{j,i+1}} \sup_{0\le s\le h_j} \left(\frac{2\left|\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k t})-W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k(t+s)}}\frac{W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k h_{j,i}}}\right|}{\Gamma(h_{j,i},h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}} \right) \\
&:= I_{1,1} + I_{1,2}
\end{aligned}
$$

by (3.3). We write

$$
\begin{aligned}
I_{1,1}(i,j) &= \sup_{0\le s\le \delta h_j} \frac{2\left|\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k(s+h_{j,i})})-W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k(s+h_{j,i})}}\frac{W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k h_{j,i}}}\right|}{\Gamma(h_{j,i},h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}} \\
I_{1,2}(i,j) &= \sup_{h_{j,i}\le t\le h_{j,i+1}} \sup_{0\le s\le h_j} \frac{2\left|\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k t})-W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k(t+s)}}\frac{W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k h_{j,i}}}\right|}{\Gamma(h_{j,i},(\delta-1)h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}.
\end{aligned}
$$

For $0 \le s' \le s \le \delta h_j$, we have

$$E\Big(2\Big(\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\Big(\frac{W_k(e^{2\lambda_k(s+h_{j,i})})-W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k(s+h_{j,i})}}-\frac{W_k(e^{2\lambda_k(s'+h_{j,i})})-W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k(s'+h_{j,i})}}\Big)$$

$$\cdot\frac{W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k h_{j,i}}}\Big)^2\Big|W_k(e^{2\lambda_k h_{j,i}}),\quad k=1,2,\dots\Big)$$

$$=4\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2\frac{W_k^2(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k h_{j,i}}}(1-e^{-2\lambda_k s}+1-e^{-2\lambda_k s'}-2e^{-\lambda_k(s-s')}+2e^{-\lambda_k(s+s')})$$

$$\le 8\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2\frac{W_k^2(e^{2\lambda_k h_{j,i}})}{e^{2\lambda_k h_{j,i}}}(1-e^{-\lambda_k(s-s')})$$

$$=2\Gamma^2\Big(h_{j,i},\frac{1}{2}(s-s')\Big)$$

and

$$E\Big(2\Big|\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k(s+h_{j,i})})-W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k h_{j,i}}}\frac{W_k(e^{2\lambda_k h_{j,i}})}{e^{\lambda_k h_{j,i}}}\Big|^2\Big|W_k(e^{2\lambda_k h_{j,i}}),\quad k=1,2,\dots\Big)$$
$$=\Gamma^2(h_{j,i},s).$$

We prove below that $\Gamma^2(h_{j,i},s)/s^{1/9}$ is a.s. non-decreasing on $(0,\delta h_j)$. It follows from (3.1) and (2.4) that for each $\varepsilon>0$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{h_j^{\theta}\ 1/(\delta-1)}P\Bigg(\sup_{0\le h\le h_j}\frac{\sum_{\lambda_k\ge\frac{1}{h}}(\frac{\gamma_k}{\lambda_k})^2\frac{W_k^2(e^{2\lambda_k h_{j,i}})}{e^{2\lambda_k h_{j,i}}}}{h}\ge\varepsilon\Bigg)$$

$$\le\sum_{j=1}^{\infty}\sum_{i=1}^{\frac{1}{h_j(\delta-1)}}\sum_{\ell=0}^{\infty}P\Big(\sum_{\lambda_k\ge\frac{2^\ell}{h_j}}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2\frac{W_k^2(e^{2\lambda_k h_{j,i}})}{e^{2\lambda_k h_{j,i}}}\ge\varepsilon\frac{2^{\ell-1}}{h_j}\Big)$$

$$\le\sum_{j=1}^{\infty}\sum_{i=1}^{\frac{1}{(\delta-1)h_j}}\sum_{\ell=0}^{\infty}P\Big(\sum_{\lambda_k\ge\frac{2^\ell}{h_j}}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2W_k^2(1)-\sum_{\lambda_k\ge\frac{2^\ell}{h_j}}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2\ge\frac{\varepsilon}{2}\frac{2^{\ell-1}}{h_j}\Big)$$

$$\le\sum_{j=1}^{\infty}\sum_{i=1}^{\frac{1}{(\delta-1)h_j}}\sum_{\ell=0}^{\infty}2\exp\Bigg(-\frac{\varepsilon 2^{\ell-2}h_j^{-1}}{2\max_{\lambda_k\ge\frac{2^\ell}{h_j}}(\frac{\gamma_k}{\lambda_k})^2}\Bigg)$$

$$\le\sum_{j=1}^{\infty}\sum_{i=1}^{\frac{1}{(\delta-1)h_j}}\sum_{\ell=0}^{\infty}2\exp\Big(-4\log\frac{2^\ell}{h_j}\Big)$$

$$<\infty.$$

This proves that $\Gamma^2(h_{j,i},s)/s^{1/9}$ is a.s. non-decreasing on $(0,\delta h_j)$ for all j, i, by (3.4), (3.14) and (3.15). Applying now Lemma 2.2 of Csáki, Csörgő and Shao (1991), we arrive at

(3.24)
$$P\Big(I_{1,1}(i,j) \geq \delta^2(1-\theta)^{\frac{1}{2}}\Big) = EP\Big(I_{1,1}(i,j) \geq \delta^2(1-\theta^{\frac{1}{2}}) \big| W_k(e^{2\lambda_k h_{j,i}}), k=1,2,\ldots\Big)$$
$$\leq K_\delta \exp\Big(-\delta(1-\theta)\log\frac{1}{h_j}\Big)$$
$$\leq K_\delta h_j^{\delta(1-\theta)}$$

where K_δ is a constant depending only on δ. Therefore

$$I_{1,1} \leq \delta^3(1-\theta)^{\frac{1}{2}} \text{ a.s.} \tag{3.25}$$

by (3.24) and the Borel-Cantelli lemma.

Similarly, we can obtain that

$$\limsup_{j\to\infty} \max_{0\leq i\leq h_j^{\theta-1}\frac{1}{\delta-1}} I_{1,2}(i,j) \leq 4(1-\theta)^{\frac{1}{2}} \text{ a.s.} \tag{3.26}$$

Hence

$$I_{1,2} \leq 4(1-\theta)^{\frac{1}{2}}(\delta-1)^{\frac{1}{18}} \text{ a.s.,} \tag{3.27}$$

since $\Gamma(h_{j,i}, s)/s^{\frac{1}{9}}$ is a.s. non-decreasing on $(0, h_j)$.

Putting the above together, we conclude that

$$\limsup_{h\to 0} \sup_{0\leq t\leq h^\theta} \sup_{0\leq s\leq h} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h)(2\log\frac{1}{h})^{\frac{1}{2}}} \leq \delta^4(1-\theta)^{\frac{1}{2}} + 4(1-\theta)^{\frac{1}{2}}(\delta-1)^{\frac{1}{18}} \text{ a.s.} \tag{3.28}$$

This proves (3.12) by the arbitrariness of $\delta > 1$.

We show next that

$$\liminf_{h\to 0} \sup_{0\leq t\leq h^\theta} \sup_{0\leq s\leq h} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h)(2\log\frac{1}{h})^{\frac{1}{2}}} \geq (1-\theta)^{\frac{1}{2}} \text{ a.s.} \tag{3.29}$$

Again, we let $1 < \delta < 2$, $h_j = \delta^{-j}$. Then

(3.30)
$$\liminf_{h\to 0} \sup_{0\leq t\leq h^\theta} \sup_{0\leq s\leq h} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h)(2\log\frac{1}{h})^{\frac{1}{2}}} \geq \liminf_{j\to\infty} \sup_{0\leq t\leq h_j^\theta} \sup_{0\leq s\leq h_j} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h_{j-1})(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$
$$\geq \frac{1}{\sqrt{\delta}} \liminf_{j\to\infty} \sup_{0\leq t\leq h_j^\theta} \sup_{0\leq s\leq h_j} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$
$$\geq \frac{1}{\sqrt{\delta}} \liminf_{j\to\infty} \max_{0\leq i\leq h_j^{\theta-1}} \frac{\chi^2((i+1)h_j)-\chi^2(ih_j)}{\Gamma(ih_j,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$
$$\geq \frac{1}{\sqrt{\delta}} \liminf_{j\to\infty} \max_{0\leq i\leq h_j^{\theta-1}} \frac{\Delta_1(ih_j,h_j)}{\Gamma(ih_j,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$
$$- \frac{1}{\sqrt{\delta}} \limsup_{j\to\infty} \max_{0\leq i\leq h_j^{\theta-1}} \frac{\Delta_2(ih_j,h_j)}{\Gamma(ih_j,h_j)(2\log\frac{1}{h_j})^{\frac{1}{2}}}$$
$$:= J_1 - J_2$$

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by (3.3), (3.9), (3.10), (3.11). From (3.18), we get

$$J_2 = 0 \text{ a.s.} \tag{3.31}$$

Since $\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} \frac{W_k(e^{2\lambda_k(i+1)h_j}) - W_k(e^{2\lambda_k ih_j})}{e^{\lambda_k(i+1)h_j}} \frac{W_k(e^{2\lambda_k ih_j})}{e^{\lambda_k ih_j}}$ is a normal random variable when conditioned on $W_k(e^{2\lambda_k ih_j})$, $k = 1, 2, \ldots$, we have

(3.32)

$$\begin{aligned}
P&\left\{\max_{0\le i\le [h_j^{\theta-1}]} \frac{\Delta_1(ih_j, h_j)}{\Gamma(ih_j, h_j)(2\log\frac{1}{h_j})^{\frac12}} \le \delta^{-3}(1-\theta)^{\frac12}\right\} \\
&= EI\left\{\max_{0\le i<[h_j^{\theta-1}]} \frac{\Delta_1(ih_j, h_j)}{\Gamma(ih_j, h_j)(2\log\frac{1}{h_j})^{\frac12}} \le \delta^{-3}(1-\theta)^{\frac12}\right\} \\
&\quad \cdot P\left(\frac{\Delta_1([h_j^{\theta-1}]h_j, h_j)}{\Gamma([h_j^{\theta-1}]h_j, h_j)(2\log\frac{1}{h_j})^{\frac12}} \le \delta^{-3}(1-\theta)^{\frac12} \,\middle|\, W_k(e^{2\lambda_k[h_j^{\theta-1}]h_j}), k = 1, 2, \ldots,\right) \\
&= \Phi\left(\delta^{-3}(1-\theta)^{\frac12}\left(2\log\frac{1}{h_j}\right)^{\frac12}\right) P\left\{\max_{0\le i<[h_j^{\theta-1}]} \frac{\Delta_1(ih_j, h_j)}{\Gamma(ih_j, h_j)(2\log\frac{1}{h_j})^{\frac12}} \le \delta^{-3}(1-\theta)^{\frac12}\right\} \\
&= \left(\Phi\left(\delta^{-3}(1-\theta)^{\frac12}\left(2\log\frac{1}{h_j}\right)^{\frac12}\right)\right)^{1+[h_j^{\theta-1}]} \\
&\le \left(1 - \frac{\exp(-\delta^{-6}(1-\theta)\log\frac{1}{h_j})}{4(\delta^{-3}(1-\theta)^{\frac12}(2\log\frac{1}{h_j})^{\frac12} + 1)}\right)^{h_j^{\theta-1}} \\
&\le \exp\left(-\frac{h_j^{(\theta-1)(1-\delta^{-6})}}{4(\delta^{-3}(1-\theta)^{\frac12}(2\log\frac{1}{h_j})^{\frac12} + 1)}\right).
\end{aligned}$$

Consequently,

$$J_1 \ge \delta^{-4}(1-\theta)^{\frac12} \text{ a.s.} \tag{3.33}$$

by (3.32) and the Borel-Cantelli lemma.

This proves (3.29) by (3.30), (3.31), (3.33) and the arbitrariness of $\delta > 1$. The proof of Theorem 2.1 is now complete.

PROOF OF THEOREM 2.2. Without loss of generality, assume $t = 0$. Note that

$$\frac{\Gamma^2(0,h)}{h} \ge \Gamma^2(0,1) = 2\sum_{k=1}^{\infty}\left(\frac{\gamma_k}{\lambda_k}\right)^2(1 - e^{-\lambda_k})W_k^2(1) > 0 \text{ a.s.} \tag{3.34}$$

Proceeding again as in the proof of (3.12), we can obtain

$$\limsup_{h\to 0} \sup_{0\le s\le h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\Gamma(0,h)(2\log\log\frac{1}{h})^{\frac12}} \le 1 \text{ a.s.} \tag{3.35}$$

For establishing the lower bound, it suffices to prove only that

$$\limsup_{h\to 0} \frac{\chi^2(h) - \chi^2(0)}{\Gamma(0,h)(2\log\log\frac{1}{h})^{\frac12}} \ge 1 \text{ a.s.} \tag{3.36}$$

Let $1 < \delta < 2$ and define

$$h_k = e^{-k^\delta}, \quad k = 1, 2, \ldots. \tag{3.37}$$

Then

$$\begin{aligned}
\limsup_{h\to 0} \frac{\chi^2(h)-\chi^2(0)}{\Gamma(0,h)(2\log\log\frac{1}{h})^{\frac12}} &\geq \limsup_{j\to\infty} \frac{\chi^2(h_j)-\chi^2(0)}{\Gamma(0,h_j)(2\log\log\frac{1}{h_j})^{\frac12}} \\
&\geq \limsup_{j\to\infty} \frac{2\sum_{k=1}^\infty \frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k h_j})-W_k(1)}{e^{\lambda_k h_j}} W_k(1)}{\Gamma(0,h_j)(2\log\log\frac{1}{h_j})^{\frac12}} \\
&\quad - \limsup_{j\to\infty} \frac{2\sum_{k=1}^\infty \frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_j})W_k^2(1)}{\Gamma(0,h_j)(2\log\log\frac{1}{h_j})^{\frac12}} \\
&\geq \limsup_{j\to\infty} \frac{2\sum_{k=1}^\infty \frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k h_j})-W_k(e^{2\lambda_k h_{j+1}})}{e^{\lambda_k h_j}} W_k(1)}{\Gamma(0,h_j)(2\log\log\frac{1}{h_j})^{\frac12}} \\
&\quad - \limsup_{j\to\infty} \frac{2\left|\sum_{k=1}^\infty \frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k h_{j+1}})-W_k(1)}{e^{\lambda_k h_j}} W_k(1)\right|}{\Gamma(0,h_j)(2\log\log\frac{1}{h_j})^{\frac12}} \\
&\quad - \limsup_{j\to\infty} \frac{2\sum_{k=1}^\infty \frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_j})W_k^2(1)}{\Gamma(0,h_j)(2\log\log\frac{1}{h_j})^{\frac12}} \\
&:= L_1 - L_2 - L_3,
\end{aligned} \tag{3.38}$$

by (3.9).

In terms of Lemma 3.1 and (2.6), similarly to (3.21), we have

$$L_3 = 0 \text{ a.s.} \tag{3.39}$$

Noting that $\sum_{k=1}^\infty \frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k h_{j+1}})-W_k(1)}{e^{\lambda_k h_j}} W_k(1)$ is a normal random variable when conditioned on $W_k(1)$, $k = 1, 2, \ldots$, we have

$$P\left\{ \frac{2\left|\sum_{k=1}^\infty \frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k h_{j+1}})-W_k(1)}{e^{\lambda_k h_j}} W_k(1)\right|}{(4\sum_{k=1}^\infty(\frac{\gamma_k}{\lambda_k})^2 e^{-2\lambda_k h_j}(e^{2\lambda_k h_{j+1}}-1)W_k^2(1))^{\frac12}} \geq x \right\} = \big(2\Phi(x)-1\big)$$

for each $x > 0$, which yields immediately that

$$\limsup_{j\to\infty} \frac{2\left|\sum_{k=1}^\infty \frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k h_{j+1}})-W_k(1)}{e^{\lambda_k h_j}} W_k(1)\right|}{(4\sum_{k=1}^\infty(\frac{\gamma_k}{\lambda_k})^2 e^{-2\lambda_k h_j}(e^{2\lambda_k h_{j+1}}-1)W_k^2(1))^{\frac12}(2\log\log\frac{1}{h_j})^{\frac12}} \leq 1 \text{ a.s.} \tag{3.40}$$

We show next that for each $j \geq 1$

$$\sum_{k=1}^\infty \Big(\frac{\gamma_k}{\lambda_k}\Big)^2 e^{-2\lambda_k h_j}(e^{-2\lambda_k h_{j+1}}-1)W_k^2(1) \leq 3\Big(\frac{h_{j+1}}{h_j}\Big)^{\frac12} \sum_{k=1}^\infty \Big(\frac{\gamma_k}{\lambda_k}\Big)^2 (1-e^{-2\lambda_k h_j})W_k^2(1). \tag{3.41}$$

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We note that we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2 e^{-2\lambda_k h_j}(e^{2\lambda_k h_{j+1}}-1)W_k^2(1)\\
&\le \sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2 e^{-\lambda_k h_j}(1-e^{-2\lambda_k h_{j+1}})W_k^2(1)\\
&=\Big(\sum_{\lambda_k\le(h_jh_{j+1})^{-\frac12}}+\sum_{\lambda_k>(h_jh_{j+1})^{\frac12}}\Big)\Big(\frac{\gamma_k}{\lambda_k}\Big)^2 e^{-\lambda_k h_j}(1-e^{-2\lambda_k h_{j+1}})W_k^2(1)\\
&\le \sum_{\lambda_k\le(h_jh_{j+1})^{-\frac12}}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2 2\lambda_k h_{j+1}W_k^2(1)+\sum_{\lambda_k>(h_jh_{j+1})^{-\frac12}}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2 e^{-(\frac{h_j}{h_{j+1}})^{\frac12}}W_k^2(1)\\
&\le \sum_{\lambda_k\le(h_jh_{j+1})^{-\frac12}}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2\frac{2\lambda_k h_j}{1+2\lambda_k h_j}\frac{h_{j+1}}{h_j}(1+2\lambda_k h_j)W_k^2(1)\\
&\quad+2\sum_{\lambda_k>(h_jh_{j+1})^{-\frac12}}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2 e^{-(\frac{h_j}{h_{j+1}})^{\frac12}}(1-e^{-2\lambda_k h_j})W_k^2(1)\\
&\le \frac{h_{j+1}}{h_j}\Big(1+2\big(\frac{h_j}{h_{j+1}}\big)^{\frac12}\Big)\sum_{\lambda_k\le(h_jh_{j+1})^{-\frac12}}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2(1-e^{-2\lambda_k h_j})W_k^2(1)\\
&\quad+2e^{-(\frac{h_j}{h_{j+1}})^{\frac12}}\sum_{\lambda_k>(h_jh_{j+1})^{-\frac12}}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2(1-e^{-2\lambda_k h_j})W_k^2(1)\\
&\le 3\Big(\frac{h_{j+1}}{h_j}\Big)^{\frac12}\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2(1-e^{-2\lambda_k h_j})W_k^2(1).
\end{aligned}
$$

This proves (3.41). A combination of (3.40) with (3.41) implies

$$L_2 = 0 \text{ a.s.} \tag{3.42}$$

It is easy to see that we have

$$
\begin{aligned}
&P\left\{\frac{2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k h_j})-W_k(e^{2\lambda_k h_{j+1}})}{e^{\lambda_k h_j}}W_k(1)}{\Gamma(0,h_j)(2\log\log\frac{1}{h_j})^{\frac12}}\le\delta^{-4}, j\ge n\right\}\\
&\le P\left\{\frac{2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\frac{W_k(e^{2\lambda_k h_j})-W_k(e^{2\lambda_k h_{j+1}})}{e^{\lambda_k h_j}}W_k(1)}{\Gamma(0,h_j-h_{j+1})(2\log\log\frac{1}{h_j})^{\frac12}}\le\delta^{-3}, j\ge n\right\}\\
&=\prod_{j\ge n}\Phi\Big(\delta^{-3}\big(2\log\log\frac{1}{h_j}\big)^{\frac12}\Big).
\end{aligned}
\tag{3.43}
$$

An elementary calculation yields

$$\prod_{j\ge n}\Phi\Big(\delta^{-3}\big(2\log\log\frac{1}{h_j}\big)^{\frac12}\Big)\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{3.44}$$

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Therefore,

$$L_1 \geq \delta^{-4} \text{ a.s.} \tag{3.45}$$

and (3.36) now follows from (3.38), (3.39), (3.42), (3.45) and the arbitrariness of $\delta > 1$. This completes the proof of Theorem 2.2.

The proofs of Corollaries 2.1 and 2.2 are trivial by Theorems 2.1 and 2.2, and hence omitted.

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