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DIFFERENTIAL OPERATORS WITH ABSTRACT BOUNDARY CONDITIONS

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1. Introduction. Suppose F is a topological vector space. Let $AC_m \equiv AC_m[a, b]$ be the absolutely continuous m-dimensional vector valued functions y on the compact interval $[a, b]$ with essentially bounded components. Consider the boundary value problem

$$ly = A_0y' + Ay = f, \quad Uy = r, \tag{1.1}$$

where A_0, A are respectively $m \times m$ continuously differentiable invertible, and continuous matrices on $[a, b]$, and U: $AC_m[a, b] \to F$ is a continuous linear operator with range in F.

The main purpose of this article is to construct an adjoint for (1.1) as well as higher order generalizations of the problem when ly is viewed as an operator in various L^p spaces, and to show normal solvability. Thus we generalize the results of many recent papers (see the survey article [**14**] for a list) where F is finite dimensional and U is represented by a Stieltjes measure. Having done this, a secondary goal will be to show the remarkable applicability of this concept to questions of current interest in the theory of splines and interpolation.

To outline the paper in greater detail: notation and preliminary facts concerning the representation of U are developed in Section 2. This furnishes the tools by which reasoning valid when dim $F < \infty$ can be generalized. The homogenous case $r = \theta$ is studied in Section 3. We define "minimal" and "maximal" operators, and construct "parametric" adjoints. Fredholm alternatives are stated and the operators are shown to be normally solvable. Furthermore significant and interesting differences between the structure of the adjoint in the finite and infinite dimensional cases arising from a classical and deep lemma of Grothendieck are pointed out. These differences make the material of this section a nontrivial generalization of previous work. Next (§ 4) the nonhomogenous case is considered. Here the system is viewed as an operator with range in $L^p \times F$ and the adjoint is obtained via the application of a generalized Green's formula. This leads to a criterion of solvability of the system (1.1) (Corollary 4.3). Section 5 extends the results of the previous sections to nth order regular differential operators. Several examples including problems with boundary conditions at infinite sets of points are given in this section. Finally (§ 6) to indicate the applicability of our theory in the direction of splines we show that the "variational approach to splines" (i.e., the approach

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defining splines as solutions to constrained variational problems) follows from a generalized Euler-LaGrange equation (Theorem 6.1) for a calculus of variations problem under abstract constraints. Several rederivations of known results are given to illustrate the generality of this approach. As a more challenging exercise we conclude the paper by using a Fredholm Alternative technique to prove a Whitney-Golomb extension theorem (Theorem 6.3) for a problem with infinitely many integral boundary conditions and to obtain an upper bound for the norm of the derivative of a minimal extension in L^p, $1 < p \leqq \infty$.

Let us close this section with the admission that the present paper by no means exhausts the subject of generalized boundary value problems. Notable omissions include interface conditions, differential boundary operators (i.e., systems like (1.1) where the differential expression involves a boundary condition) singular theory, characterizations of self-adjointness, Green's functions, and spectral theory. In the finite dimensional case, information concerning some or all of the above topics may be found in [**4**; **13**; **15**; **16**] and many other papers cited in [**14**]. A full generalization of these results to the present context would be a challenging exercise which we hope to attempt later.

2. Notation and preliminaries. $L_m^p \equiv L_m^p[a, b]$, $1 \leqq p \leqq \infty$, is the space of m-dimensional L^p integrable functions y with norm

$$||y||: = \left[\int_a^b \left\{ \sum_{i=1}^m |y_i|^2 \right\}^{p/2} dt \right]^{1/p}.$$

Similarly, $C_m \equiv C_m[a, b]$ is the space of m dimensional continuous functions on $[a, b]$. X^* is the dual, adjoint, or conjugate transpose of X according to the context. $\mathbf{C}^m$ is complex m-dimensional Euclidian space, and $[\cdot\, , \cdot]$ indicates the natural antisymmetric pairing on $X \times X^*$ over $\mathbf{C}$ (that is, $[x, y] = \overline{[y, x]}$). If $S \subset X$ or X^*, $S^\perp$ and ${}^\perp S$ represent respectively the annihilators of S in X^* or X.

Suppose F is a Banach space. Then since U is bounded, it can be represented by an abstract regular countably additive regular measure on the Borel sets of $[a, b]$ (cf. [**7**, p. 318, 492]). Such an approach would generalize earlier work (e.g., [**3**; **4**]) in which the boundary conditions were represented by systems of ordinary Stieltjes integrals, and has been recently employed by Hönig in a monograph [**11**] discussing the existence and properties of Green's functions for problems like (1.1).

In this paper, however, a different and simpler point of view is possible. Since in all important calculations the boundary operator U appears in the form $[Uy, \phi]$ where ϕ is a variable element in F^*, there exists a $1 \times m$ vector valued measure $d\langle \phi, U\rangle$ of bounded variation on $[a, b]$ such that

$$(2.1) \qquad \int_a^b d\langle \phi, U\rangle y = [Uy, \phi].$$

The representation (2.1) beside avoiding tedious refinements of vector valued measure theory, allows F to be a topological vector space rather than a Banach space, a generalization which will be seen to have significant applications.

The following specialized notation will be convenient throughout the paper. Set

$$\langle \phi, U\rangle (t): = \int_a^t d\langle \phi, U\rangle$$

$$S: = \{\langle \phi, U\rangle^*: \phi \in F^*\}.$$

Clearly $S \subset L_m^q$ for all $1 \leqq q \leqq \infty$.

For some $\psi \in \bar{S}$ (norm closure if $1 \leqq p < \infty$, weak* closure if $p = \infty$) define

$$l^+(z, \psi): = -(A_0^*z + \psi)' + A^*z,$$

$$B[y, z](b^-, a^+]: = y^*(A_0^*z + \psi)|_{a+}^{b-} - \int_a^b y^{*\prime}\psi dt, \tag{2.2}$$

$$W[y, z](b^-, a^+): = y^*(b^-)(A_0^*z(b^-) - d\langle\phi, U\rangle^*[b]) - y^*(a)(A_0^*z(a^+) + d\langle \phi, U\rangle^*[a]),$$

it being understood that y, z are functions at least of bounded variation so that the indicated operations make sense.

2.1 LEMMA. *If $\psi \in S$, $y \in AC_m$ and z is of bounded variation, then*

$$B[y, z](b^-, a^+) = W[y, z](b^-, a^+) + [\phi, Uy].$$

Proof. Let $\psi \in S$, i.e., $\psi = \langle \phi, U\rangle$.

$$\int_a^b y^{*\prime}\psi dt = y^*(b)\langle \phi, U\rangle^*(b^-) - y^*(a)d\langle \phi, U\rangle^*[a] - \int_{a+}^{b-} y^* d\langle \phi, U\rangle^*$$

upon integrating by parts. Hence

$$B[y, z](b^-, a^+) = W[y, z](b^-, a^+) + \int_a^b y^* d\langle\phi, U\rangle^*.$$

The lemma follows from (2.1).

3. L and its adjoint. We now construct "maximal" and "minimal" operators L, L_0 on L_m^p determined by (1.1) and study the adjoints L^*, L_0^* of these operators.

3.1. *Definition.* Let

$$D_p': = \{y \in AC_m: ly \in L_m^p\},$$

$$D_{0p}': = \{y \in D_p': y(a) = y(b) = 0\},$$

for a fixed p, $1 \leqq p \leqq \infty$.

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Then L is the restriction of ly to

$$D: = \{y \in D_p' : Uy = 0\},$$

and L_0 is the restriction of L to

$$D_0: = \{y \in D \cap D_{0p}'\}.$$

3.2. *Definition.* If $1/p + 1/q = 1$, let

$$\tilde{D}^+: = \{z: z + \psi \in AC_m, l^+(z, \psi) \text{ exists for some } \psi \in \bar{S}\},$$

$$D^+: = \{z \in \tilde{D}^+: \psi \in S\},$$

$$\tilde{D}_0^+: = \{z \in D^+: B[y, z](b^-, a^+) = 0 \text{ for all } y \in D\},$$

$$D_0^+: = \{z \in D^+: W[y, z](b^-, a^+) = 0 \text{ for all } y \in D\},$$

$$D_0^{+\prime}: = \{z \in D^+: z(a^+) = -\langle \phi, U\rangle^*[a]; z(b^-) = \langle \phi, U\rangle^*[b]\}.$$

Finally, let $\tilde{L}^+$, L^+, $\tilde{L}_0^+$, L_0^+, $L_0^{+\prime}$ be the restrictions of $l^+(z, \psi)$ to $\tilde{D}^+$, D^+, $\tilde{D}_0^+$, D_0^+ and $D_0^{+\prime}$.

3.3. *Remarks.* (1) $\tilde{L}^+$, L^+, $\tilde{L}^+$, L_0^+, $L_0^{+\prime}$ in general are linear relations in $L_m^q \times L_m^q$ rather than operators since $l^+(z, \psi)$ depends on both z and ψ.

(2) The following inclusions are clear: $L_0 \subset L$, $L_0^{+\prime} \subset L_0^+ \subset L^+ \subset \tilde{L}^+$. Also from Lemma 2.1 and the definitions of D_0^+, $\tilde{D}_0^+$, $L_0^+ \subset \tilde{L}_0^+$.

If F is finite dimensional the structure of L^*, L_0^* is well understood when the boundary condition is given by a Stieltjes integral. The following theorem, whose proof may be found in [**3**], summarizes the situation.

3.4. THEOREM. *If* $\dim F < \infty$, *then for* $1 < p \leqq \infty$

$$L_0^{+\prime *} = L \quad \textit{and} \quad L^{+*} = L_0.$$

For $1 \leqq p < \infty$,

$$L^* = L_0^+ = L_0^{+\prime} \quad \textit{and} \quad L_0^* = L^+.$$

3.5. *Remark.* If F is finite dimensional, $S = \bar{S}$ so that $\tilde{L}^+ = L^+$ and $\tilde{L}_0^+ = L_0^+$. Thus $L_0^{+\prime}$, L_0^+, $\tilde{L}_0^+$ and L^+, $\tilde{L}^+$ collapse into a pair of operators respectively adjoint to L and L_0.

In the infinite dimensional case, however, $L_0^{+\prime}$, L_0^+, $\tilde{L}_0^+$, L^+ and $\tilde{L}^+$ play more complicated roles. We now state the appropriate generalization of Theorem 3.4.

3.6. THEOREM. *If F is not finite dimensional, then for* $1 < p \leqq \infty$

$$L_0^{+\prime *} = L_0^{+*} = \tilde{L}_0^{+*} = L \quad \textit{and} \quad L^{+*} = \tilde{L}^{+*} = L.$$

For $1 \leqq p < \infty$,

$$L_0^* = \tilde{L}^+ \quad \textit{and} \quad L^* = \tilde{L}_0^+.$$

The proof of Theorem 3.5 will be clear from the next seven lemmas.

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3.7. LEMMA. *Let T_0: $L_m^p \to L_m^p$ be defined by y' on D_{0p}'. Then for $1 \leqq p < \infty$, $R(T_0)^*$ is isometrically isomorphic to L^q/K where K denotes the space of constant vectors.*

Proof. For $m = 1$ it is known (cf. [**8**, ch. V]) that $T_0^* = T$, where T: $L^q \to L^q$ is the maximal operator determined $lz = z'$ on D_q. Also T_0 is normally solvable. The case $m — 1$ offers no complications. Since $N(T) = K$, $R(T_0)^\perp = K$. The lemma now follows from standard facts about duality (cf. [**19**, p. 91]).

3.8. LEMMA (Green's Relations). *If $y \in D_p'$ and $z \in \tilde{D}^+$, then*

$$(3.1) \qquad [ly, z] - [y, l^+(z, \psi)] = B[y, z](b^-, a^+).$$

If $z \in D^+$, then

$$(3.2) \qquad [ly, z] - [y, l^+(z, \psi)] = W[y, z](b^-, a^+) + [\phi, Uy]$$

Proof. Suppose $z \in \tilde{D}^+$ and $y \in W_m^{1,p}$. Then

$$\int_a^b l^+(z, \psi)^* y dt = \int_a^b \{-[A_0^* z + \psi]' + Az^*\}^* y dt$$

$$= -[A_0^* z + \psi]^* y|_{a+}^{b-} + \int_a^b [A_0^* z + \psi]^* y' dt + \int_a^b (A^* z)^* y dt.$$

$$= -\{A_0^* z + \psi\}^* y|_{a+}^{b-} + \int_a^b z^* ly dt + \int_a^b \psi^* y' dt.$$

(3.2) now is an immediate consequence of (3.1) and Lemma 2.1.

3.9. *Remark.* From Lemma 3.8 and Lemma 2.1, it is now clear that

$$(3.3) \qquad L^+ \subset \tilde{L}^+ \subset L_0^*, \quad L_0^{+\prime} \subset L_0^+ \subset \tilde{L}_0^+ \subset L_0^*.$$

3.10. LEMMA. *Let Φ be a fundamental matrix of ly. Then*

(1) *$f \in R(L)$ if and only if there exists a vector $C \in \mathbf{C}^m$ such that*

$$U\left[\Phi(x)\left(\int_a^x \Phi^{-1} A_0^{-1} f dt + C\right)\right] = 0;$$

(2) *$f \in R(L_0)$ if and only if the additional conditions*

$$C = 0, \quad \int_a^b \Phi^{-1} A_0^{-1} f dt = 0$$

hold;

(3) $N(L^+) = \left\{A_0^{*-1} \Phi^{*-1}\left[\int_a^t \Phi^* d\langle\phi, U\rangle^* + C\right); \phi \in F^*, C \in \mathbf{C}^m\right\}$;

(4) *$z \in N(L_0^+) \subset N(L^+)$ if and only if $C = 0$ and ϕ satisfies*

$$(3.4) \qquad y^*(b) \Phi^{*-1}(b) \int_a^b \Phi^* d\langle\phi, U\rangle^* = 0$$

for all y in D.

Proof. By variation of parameters, functions in $D(L)$ can be written

$$y = \Phi(x)\left(\int_a^x \Phi^{-1}f(x)dt + C\right).$$

We derive (1) by requiring the boundary conditions to be satisfied. The additional conditions for (2) follow by requiring y to vanish at the end-points. Proceeding to (3): $f \in R(L^+)$ if and only if there exists z in D^+ such that $l^+(z, \psi) = f$; i.e.,

$$(3.5) \qquad -(A_0{}^*z + (\phi, U)^*)' + A^*z.$$

Set $Z: = \Phi^* A_0{}^*z$ and substitute $z = A_0{}^{*-1}\, \Phi^{*-1}Z$ into (3.4). We obtain

$$[\Phi^{*-1}\, \Phi^{*\prime} + A^*\, A_0{}^{*-1}]\, \Phi^{*-1}Z - \Phi^{*-1}Z' = f + \langle \phi, U\rangle^{*\prime}.$$

Since the first term vanishes

$$Z' = -\, \Phi^*(\langle \phi, U\rangle^{*\prime} + f).$$

From the definition of D^+ and Z

$$Z + \Phi^*\langle \phi, U\rangle^* \in AC_m.$$

Therefore,

$$Z + \Phi^*\langle \phi, U\rangle^* = -\int_a^t \Phi^*(\langle \phi, U\rangle^{*\prime} + f)dt + \int_a^t (\Phi^*\langle \phi, U\rangle^{*\prime}dt + C$$
$$= \int_a^t (\Phi^{*\prime}(\langle \phi, U\rangle^* + \Phi^* f)dt + C$$

where C is a constant. Integrating $\Phi^{*\prime}\langle \phi, U\rangle^*$ by parts and taking $f = 0$, we finally obtain

$$Z = -\int_a^t \Phi^* d\langle \phi, U\rangle^* + C.$$

Equivalently

$$z = A_0{}^{*-1}\Phi^{*-1}\left(\int_a^t \Phi^* d\langle \phi, U\rangle^* + C\right).$$

That C ranges over all of $\mathbf{C}^m$ follows from the fact (which is easy to verify) that $N(L^+)$ contains all functions of the form $A_0{}^{*-1}\, \Phi^{*-1}C$, where C is arbitrary.

Finally, the representation (4) for $N(L_0{}^+)$ is obtained by forcing $z \in N(L^+)$ to belong to $D_0{}^+$, i.e., by making z satisfy $W[y, z][b^-, a^+) = 0$. The details are omitted.

3.11. *Remark.* If F is finite dimensional, (4) is equivalent to the condition

$$(3.6) \qquad \int_a^b \Phi^* d\langle \phi, U\rangle^* = 0$$

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on the parameter ϕ. To see this, note that 3 defines a functional η on D_p' whose kernel contains "D". Suppose F has finite dimension m. Then

$$D = \left\{ y \in D_p' : \bigcap_{i=1}^{m} \pi_i(Uy) = 0 \right\},$$

where π_i denotes the projection functional of $\mathbf{C}^m$ onto its ith coordinate. By the *linear dependence principle* (cf. [**19**, p. 62]), η is a linear combination of the functionals $\pi_i \cdot U_j$, i.e.,

$$\eta(y) = [Uy, C] \tag{3.7}$$

for some $C \in \mathbf{C}^m$ and *all* y in D_p'. Since $y(b)$ is *now* arbitrary, (3.6) follows at once from (3.4) and (3.7).

3.12. Lemma. *For* $1 < p \leqq \infty$, $R(L_0) = N(L^+)^{\perp}$. *If* $p = 1$, *then* $R(L_0) = {}^{\perp}N(L^+)$.

Proof. By (1) and (2) of Lemma 3.10, $f \in R(L_0)$ if and only if

$$U\left[\Phi(x) \int_a^x \Phi^{-1} A_0^{-1} f dt \right] = 0 \tag{3.8}$$

with

$$\int_a^b \Phi^{-1} A_0^{-1} f dt = 0. \tag{3.9}$$

Now (3.8) holds if and only if

$$\left\langle \phi, U\left[\Phi(x) \left(\int_a^x \Phi^{-1} A_0^{-1} f dt \right) \right] \right\rangle = 0$$

for all $\phi \in F^*$. This in turn becomes

$$\int_a^b d\langle \phi, U[a, x] \rangle \left(\Phi(x) \int_a^b \Phi^{-1} A_0^{-1} f dt \right) = 0. \tag{3.10}$$

Applying Fubini's Theorem and taking congugate transposes, (3.10) is equivalent to

$$\int_a^b f^* A_0^{*-1} \Phi^{*-1} \int_a^t \Phi^* d\langle \phi, U \rangle^* = 0. \tag{3.11}$$

Also (3.9) holds if and only if

$$C^* \int_a^b \Phi^{-1} A_0^{-1} f dt = 0$$

where C is an arbitrary m vector, which is equivalent to

$$\int_a^b f^* A_0^{*-1} \Phi^{*-1} C = 0. \tag{3.12}$$

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Combining (3.11) and (3.12), $f \in R(L_0)$ if and only if it is orthogonal to every function in $N(L^+)$ as given in (3) of Lemma 3.10. The proof is complete.

3.13. LEMMA. *For* $1 \leqq p < \infty$, $R(L_0)^\perp = N(\tilde{L}^+)$. *Thus* $N(\tilde{L}^+) = N(L^+)$.

Proof. Note first that $R(L_0)^\perp$ is the weak* closure of $N(L^+)$ for $1 \leqq p < \infty$. The weak * closure of $N(L^+)$ furthermore equals its norm closure if $1 < p < \infty$. (As before—cf. (2.2)—we will use the notation $\overline{N(L^+)}$ for either closure depending on the context.)

If $\langle z_n \rangle$ is a sequence in $N(L^+)$, it is not difficult to see that

$$(3.13) \quad \psi_n = -A_0{}^*z_n + \int_a^t A^*z_n + c_n$$

where $c_n \in K$. Now suppose that z is in the weak * closure of $N(L^+)$. Set

$$(3.13a) \quad \psi := -A_0{}^*z + \int_a^t A^*z.$$

Let $a \in L_m{}'$. Integrating (3.13) by parts gives

$$(3.13b) \quad [a, \psi_n] = \left[-\left(A_0 a + A \int_a^t a ds \right), z_n \right] + [a, d_n],$$

where $d_n \in K$. Obviously this identity holds with respect to ψ, z and a constant d.

It follows from the definition of the weak * topology that z belongs to the weak * closure of $N(L^+)$ if and only if for every finite set $A \subset L_m{}^p$ there exists a sequence $\langle z_n \rangle$ in $N(L^+)$ such that $[a, z_n] \to [a, z]$ for all $a \in A$. Similarly $\psi \in \bar{S}$ if and only if there exists a sequence $\langle \psi_n \rangle$ in S such that $[a, \psi_n] \to [a, \psi]$ for all $a \in A$. Now given $A \subset L_m{}^p$ define

$$\tilde{A} := \left\{ -\left(A_0 a + A \int_a^t a ds \right) : a \in A \right\}.$$

Clearly $\tilde{A}$ is finite. Hence there exists a sequence $\langle z_n \rangle$ in $N(L^+)$ such that $[\tilde{a}, z_n] \to [\tilde{a}, z]$ for all $\tilde{a}$ in $\tilde{A}$. Choosing ψ_n and ψ according to (3.13) and (3.13a) it follows from (3.13b) that

$$(3.13c) \quad [a, \psi_n - d_n - d] \to [a, \psi].$$

Now (3.13c) implies that ψ lies in the weak * closure of $S + K$. Because K is finite dimensional $\psi = \psi_1 + k$ where $\psi_1 \in \bar{S}$ and $k \in K$. Differentiation of (3.13a) shows that $z \in N(\tilde{L}^+)$. Hence $\overline{N(L^+)} \subset N(\tilde{L}^+)$. By Lemma 3.8 $N(\tilde{L}^+) \subset R(L_0)^\perp$. Since $R(L_0)^\perp = \overline{N(L^+)}$, it follows that $N(\tilde{L}^+) \subset \overline{N(L^+)}$ and that $R(L_0)^+ = N(\tilde{L}^+)$. The proof is complete.

3.14. LEMMA. *For* $1 \leqq p < \infty$, $L_0{}^* = \tilde{L}^+$ *and* $L^* = \tilde{L}_0{}^+$.

Proof. Suppose $(\alpha, \beta) \in G(L_0{}^*)$. Note that

$$L^+ \subset \tilde{L}^+ \subset L_0{}^*.$$

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(cf. Remark 3.9). Also L^+ is onto since it contains the ordinary maximal operator $l^+(z, \theta)$ defined on $D_q' \subset L_m^q$. Hence

$$(3.14) \quad l^+(z, \theta) = \beta$$

for some z in $D_q' \subset \tilde{D}^+$. Therefore $\alpha - z \in N(L_0^*)$. If L_0^* is an operator it is true that $R(L_0)^\perp = N(L_0^*)$. The same, however, is true for mutually adjoint linear relations (cf. [**1**, Proposition 3.31]). We conclude that $\alpha - z \in R(L_0)^\perp$. By Lemma 3.13, $R(L_0)^\perp = N(\tilde{L}^+)$. Further, since $N(\tilde{L}^+) \subset \tilde{D}^+$, it follows that $\alpha \in \tilde{D}^+$. Hence there exists $\psi \in \bar{S}$ such that

$$l^+(\alpha - z, \psi) = 0.$$

That is,

$$(3.15) \quad -(A_0^*(\alpha - z) + \psi)' + A^*(\alpha - z) = 0.$$

Since A_0^*z and $A_0^*(\alpha - z) + \psi$ are absolutely continuous, so is $A_0^*\alpha + \psi$. Adding (3.14) and (3.15), we find that

$$-[A_0^*\alpha + \psi]' + A^*\alpha = \beta = l^+(\alpha, \psi).$$

This shows that $(\alpha, \beta) \in G(\tilde{L}^+)$ or equivalently that $G(\tilde{L}^+) \subset G(L_0^*)$. Since the reverse inequality is true, we conclude that the relations are equal.

Next, since $L_0 \subset L$, $L^* \subset L_0^*$. From the previous part of the lemma it follows that $G(L^*) \subset G(\tilde{L}^+)$. It is now a direct consequence of the Green's identity (3.2) that $G(L^*)$ consists of exactly those $(\alpha, \beta) \in G(\tilde{L}^+)$ for which $B[y, z](b^-, a^+) = 0$ for all y in D. Hence $L^* = \tilde{L}_0^+$. The proof is complete.

3.15. Lemma. *For* $1 < p \leqq \infty$, $L^{+*} = \tilde{L}^{+*} = L_0$ *and* $L_0^{+\prime *} = L_0^{+*} = \tilde{L}_0^{+*} = L$.

Proof. Since L^+ contains an ordinary maximal operator defined by l^+z on $[a, b]$, it follows that L^{+*}, is a restriction of the minimal operator defined by ly on $[a, b]$. Hence if $(y, ly) \in G(L^{+*})$ the Green's formula (3.1) gives $\langle \phi, Uy \rangle = 0$. Since ϕ is arbitrary, $Uy = 0$. Thus $L^{+*} \subset L_0$. On the other hand, (Remark 3.9) $L_0 \subset \tilde{L}^{+*} \subset L^{+*}$. These inclusions yield the first statement of the lemma. In the same way the second statement of the lemma will follow from Remark 3.9 if we can show that $L = L_0^{+\prime *}$. We first note that $L_0^{+\prime *}$ (the reasoning is the same as for L^{+*}) is a restriction of the maximal operator determined by ly on $[a, b]$. Let $(y, ly) \in G(L_0^{+\prime *})$. Just as for L^{+*}, Green's formula implies that $\langle \phi, Uy \rangle = 0$.

For any ϕ, there is an absolutely continuous function ξ in the domain of the maximal operator ly on $[a, b]$ satisfying the endpoint conditions

$$\xi(a^+) = 0; \quad \xi(b^-) = \langle \phi, U[a, b] \rangle^*.$$

Since $\xi(t) - \langle \phi, U[a, t] \rangle^* \in D_0^{+\prime}$, this shows that ϕ is arbitrary over D_0^+ (just as it is over D^+). Hence $Uy = 0$; i.e., $y \in D$ and $L_0^{+\prime *} \subset L$. Since the

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reverse inclusion $L \subset L_0^{+\prime *}$ is immediate from Green's formula, we conclude that $L_0^{+\prime *} = L$.

Lemma 3.15 completes the proof of Theorem 3.6. The next two corollaries follow from the ordinary theory of adjoints applied to Theorem 3.6.

3.16. COROLLARY (Fredholm Alternatives). *For* $1 < p < \infty$,
(1) $R(L)^{\perp} = N(\tilde{L}_0^+)$.
(2) $R(L_0)^{\perp} = N(\tilde{L}^+)$.
(3) $R(L) = N(\tilde{L}_0^+)^{\perp} = N(L_0^+)^{\perp} = N(L_0^{+\prime})^{\perp}$.
(4) $R(L_0) = N(\tilde{L}^+)^{\perp} = N(L^+)^{\perp}$.
For $p = 1$, (1) *and* (2) *hold. But* $N(\tilde{L}^+)$ *and* $N(\tilde{L}_0^+)$ *are weak* closed, and*
(5) $R(L) = {}^{\perp}N(\tilde{L}_0^+) = {}^{\perp}N(L_0^+) = {}^{\perp}N(L_0^{+\prime})$.
(6) $R(L_0) = {}^{\perp}N(\tilde{L}^+) = {}^{\perp}N(L^+)$.
For $p = \infty$, (3) *and* (4) *hold. But* $R(L)$ *and* $R(L_0)$ *are weak* closed, and*
(7) ${}^{\perp}R(L) = N(\tilde{L}_0^+)$
(8) ${}^{\perp}R(L_0) = N(\tilde{L}^+)$.

3.17. COROLLARY. *For* $1 \leqq p \leqq \infty$ L, L_0, $\tilde{L}^+$, $\tilde{L}_0^+$ *are normally solvable. Further,* $\tilde{L}^+ = \bar{L}^+$ *and* $\tilde{L}_0^+ = \bar{L}_0^+ = \bar{L}_0^{+\prime}$.

We have already noted that if F is finite dimensional, then $\tilde{L}^+ = L^+$ and $\tilde{L}_0^+ = L_0$. We close this section with a converse result.

3.18. THEOREM. *Suppose* F *is locally convex and* $R(U) = F$. *Then if* $\tilde{L}^+ = L^+$, F *is finite dimensional.*

Proof. Since $\tilde{L}^+ = L^+, S = \bar{S}$. By a classical and deep lemma of Grothendieck (cf. [**19**, p. 111]), S is finite dimensional.

Next consider the adjoint boundary operator $U^* : F^* \to C_m^*$. Then by (2.1) and the definition of U^*,

$$[Uy, \phi] = \int_a^b d\langle \phi, U\rangle y = [y, U^*\phi].$$

If we restrict y to D_{0p}' and integrate the last two expressions by parts, we obtain

$$\left[y', \int_a^t U^*\phi ds\right] = [y', \langle \phi, U\rangle]. \tag{3.16}$$

By Lemma 3.7 it follows that

$$S = \left\{\int_a^t U^*\phi ds \colon \phi \in F^*\right\} \tag{3.17}$$

modulo the constant vectors. The finite dimensionality of S and (3.17) now imply that $R(U^*)$ is finite dimensional. Because, U is onto, $N(U^*) = \{0\}$. Hence $R(U^*) \cong F^*$ and F^* is finite dimensional. Therefore $F^* \cong F^{**}$. The local convexity of F implies (by the Hahn-Banach Theorem) that F^* separates points. Thus the natural mapping ϕ: $F \to F^{**}$ defined by $\phi(f) = [f, f^*]$ for f^* in F^* is $1 - 1$. Therefore F is finite dimensional.

3.19 Corollary. *Suppose $R(U)$ is a Banach space. Then the following are equivalent;*

(1) dim $R(U) < \infty$.

(2) $L^+ = \tilde{L}^+$.

(3) *The operator $V: L^p \to F$ given by $U \circ \int_a^t(\cdot)$ has a closed range.*

Proof. We have already noted that (1) $\Rightarrow$ (2). Since a Banach space is locally convex, (2) $\Rightarrow$ (1) by Theorem 3.18.

If $y \in D_{0p}'$, $Vy' = Uy$. Hence

$$[Uy, \phi] = [y', V^*\phi].$$

By (3.16) of Theorem 3.18, $R(V^*) = S$ modulo the constant vectors. Since (2) $\Rightarrow S$ is closed, (2) $\Rightarrow R(V^*)$ is closed. However, the closure of $R(V)$ is equivalent to the closure of $R(V^*)$ by the closed range theorem, which proves (2) $\Leftrightarrow$ (3).

4. The non-homogeneous case. We now consider the adjoint theory of (1.1) when $r \neq \theta$. In this case just as before the boundary value problem determines a pair of "maximal" and "minimal" operators with range in $L_m^p \times F$.

4.1. *Definition.* Let $\hat{L}$: $AC_m[a, b] \to L^p[a, b] \times F$ and $\hat{L}^+$: $D^+ \times F_m^* \to L_m^q[a, b]$ be given respectively by $\hat{L}y: = (ly, Uy)^t$ and $\hat{L}^+(z, \phi): = l^+(z, \psi)$ where $\psi = \langle \phi, U\rangle^*$ (i.e., $\psi \in S$ and ϕ corresponds to z in D^+).

Definitions of $\hat{L}_0$, $\hat{L}_0^+$, are obvious and will be omitted. Note that $\hat{L}^+$, L_0^+ are automatically operators and that $\hat{L}^+$ is onto L_m^q (since its image coincides with that of the operator L^+ which was previously shown to be onto).

For $\alpha = (x, y) \in L^p \times F$, $\beta = (x', y') \in L^q \times F_m^*$, define $[\alpha, \beta] = [x, x'] + [y, y']$. Using the inner product $[\cdot, \cdot]$, and the operators L, L_0, L^+, L_0^+ Green's formula (Lemma 3.8) can be written in the alternative forms

$$(4.1) \qquad \begin{aligned} [\hat{L}y, Z] &= [y, \hat{L}_0^+Z] \\ [\hat{L}_0y, Z] &= [y, \hat{L}^+Z], \end{aligned}$$

where $Z = (z, \phi)$ and $z \in D_0^+$ or D^+.

The main result of this section is the following:

4.2. Theorem. *For $1 < p < \infty$,*

$$(4.2) \qquad \hat{L}_0^{+*} = \hat{L}, \quad \hat{L}^{+'*} = \hat{L}_0$$

$$(4.3) \qquad \hat{L}_0^* = \hat{L}^{+'}, \quad \hat{L}^* = \hat{L}_0^+.$$

If $p = \infty$, (4.2) is true, and if $p = 1$, (4.3) is true.

Proof. We need only show $\hat{L}_0^* \subset \hat{L}^{+'}$, $\hat{L}^{+'*} \subset \hat{L}_0$, etc., since the reverse inclusions are obvious from (4.1). Suppose $(z, \phi) \in D(\hat{L}_0^*)$. By Green's relation,

$$(4.4) \qquad [ly, z] + [\phi, Uy] = [y, \hat{L}_0^*(z, \phi)]$$

for $y \in D_{0p}'$. Since $\hat{L}_0^* \hat{L}^{+\prime}$, and $\hat{L}^{+\prime}$ restricted to $D_q' \times \{\theta\}$ is onto $L_m{}^q$, we can find $z' \in D_q'$ such that $l^+(z', \theta) = \hat{L}_0^*(z, \phi)$. Further

(4.5) $\quad [ly, z'] = [y, l^+(z', \psi]$.

Hence, subtracting (4.5) from (4.4) gives

(4.6) $\quad [ly, z - z'] + [\phi, Uy] = 0.$

Upon integrating $(z - z')^*A$ and $[\phi, Uy]$ by parts, (4.6) becomes

$$\left((z - z')^*A_0 - \langle \phi, U \rangle - \int_0^t (z - z')^*A\right)y'dt = 0$$

for all y in D_p'. By Lemma 3.7 the term in parentheses is constant. Taking conjugate transposes and differentiating once, it follows that $z - z'$ (and therefore z) belongs to D^+. Differentiating again gives

$$l^+(z - z', \psi) = l^+(z, \psi) - l^+(z, \theta) = 0,$$

thus $l^+(z, \phi) = \hat{L}_0^*(z, \phi)$, proving $\hat{L}_0^* \subset \hat{L}^{+\prime}$ and that the two operators are equal. To show that $\hat{L}^{+\prime *} = \hat{L}_0$, $1 < p \leqq \infty$, we prove that $\hat{L}_0$ is closed. Let $y_n \to y$ and $\hat{L}_0 y_n \to (z, \alpha)$ where $y_n \in D'$ and $(z, \alpha) \in L_m{}^p \times F$. Since ly (i.e., "T_0" on D_{0p}' is closed), we have at once that $y \in D_0'$ and $ly = z$. Since $U: D' \to F$ is bounded, $Uy = \alpha$.

The remaining adjoint relations $\hat{L}^* = \hat{L}_0^+$, $\hat{L}_0^{+*} = \hat{L}$ are demonstrated in a similar way. We omit the details.

4.3. COROLLARY. *$\hat{L}$, $\hat{L}_0$, $\hat{L}^+$, $\hat{L}_0^+$ are normally solvable. For $1 < p < \infty$*
(1) $R(\hat{L})^\perp = N(\hat{L}_0^+)$,
(2) $R(\hat{L}) = N(\hat{L}_0^+)^\perp$,
(3) $R(\hat{L}_0)^\perp = N(\hat{L}^+)$,
(4) $R(\hat{L}_0) = N(\hat{L}^+)^\perp$.
*For $p = 1$, (1) and (3) hold. But $N(\hat{L}^+)$ and $N(\hat{L}_0^+)$ are weak * closed, and*
(5) $R(\hat{L}) = N(\hat{L}_0^+)$,
(6) $R(\hat{L}_0) = {}^\perp N(\hat{L}^+)$.
*For $p = \infty$, (2) and (4) hold. But $R(\hat{L})$ and $R(\hat{L}_0)$ are weak * closed, and*
(7) ${}^\perp R(\hat{L}) = N(\hat{L}_0^+)$,
(8) ${}^\perp R(\hat{L}_0) = N(\hat{L}_0^+)$.

4.4. *Remarks.* (1) A short computation based on Corollary 4.3 and the definition of the inner product gives: $(f, r)^t \in R(\hat{L})$ if and only if

(4.7) $$\int_a^b \int_a^t (d\langle \phi, U\rangle \Phi^{-1} A_0 f dt) = \phi^* r$$

for all $\phi \in F^*$; $(f, r)^t \in R(\hat{L}_0)$ if and only if in addition to (4.7),

$$\int_a^b \Phi_0^{-1} A_0^{-1} f dt = 0.$$

Solvability conditions similar to (4.7) have been given in the case of functional differential equations (see [**10**; **21**]). Note that the adjoint theory for the non-homogeneous case is much simpler than in § 3. In particular $\hat{L}^+$, $\hat{L}_0^+$ are always closed operators and analogues of $\tilde{L}^+$, $\tilde{L}_0^+$ play no role.

5. Extensions to higher order operators. In this section we show how the construction of adjoint relations for higher order operators under abstract boundary conditions can be generalized from the first order case.

Suppose

$$(5.1) \qquad ly = \sum_{i=1}^{n} a_i y^{n-i}$$

in an nth order regular scalar differential operator (i.e., $a_0 = 1, a_i \in C^{n-i}[a, b]$) on $[a, b]$. Assume

$$Uy = \sum_{i=1}^{n} V_i(y^{n-i}),$$

where V_i: $C[a, b] \to F$ are bounded operators.

Define D', D, L as in Definition 3.1. Let

$$D_{0p}' : = \{y \in D' : \hat{y}(a) = \hat{y}(b) = 0\},$$

where

$$\hat{y}(a) : = (y(a), Y'(a), \ldots, y^{n-1}(a))^t$$
$$\hat{y}(b) : = (y(b), y'(b), \ldots, y^{n-1}(b))^t.$$

Also let L_0 be the restriction of L to D_0: $= D \cap D_{0p}'$. Set

$$S : = \left\{ \sum_{i=1}^{n} (-1)^{i-1} I^{(i-1)} \langle \phi, V_i \rangle^* : \phi \in F^* \right\},$$

where the notation $I^{(j)}(\cdot)$ stands for the j fold repeated integral.

$$\int_a^t \ldots \int_a^{t_{j-1}} (\cdot) dt_{j-1} \ldots dt_1 dt.$$

Clearly S is a subspace of $m \times 1$ vector valued functions of bounded variation. (In particular, everything in S is L^q integrable). Likewise introduce the sequence of partial adjoints

$$\begin{aligned} l_0^+(z, \psi) &= a_0^* z + \psi, \\ l_1^+(z, \psi) &= - l_0^{+\prime}(z, \psi) + a_1^* z, \\ (5.2) \qquad l_2^+(z, \psi) &= - l_1^{+\prime}(z, \psi) + a_2^* z, \\ &\cdot \quad \cdot \quad \cdot \\ l_n^+(z, \psi) &= - l_{n-1}^+(z, \psi) + a_n^* z, \end{aligned}$$

where $\psi \in \bar{S}$, $i = 1, \ldots, n$, and set

$$W[y, z](b^-, a^+) = \sum_{j=1}^{n} y^{(n-j)*}(b^-)(l_{j-1}{}^+(z, \theta)(b^-) - d\langle \phi, V_j\rangle^*[b])$$

$$(5.3) \qquad - y^{(n-j)*}(a^+)(l_{j-1}{}^+(z, \theta)(a^+) + d\langle \phi, V_j\rangle^*[a]).$$

$$B[y, z](b^-, a^+) = \sum_{j=1}^{n} y^{(n-j)*} l_{j-1}{}^+(z, \psi)|_{a+}^{b-} + \int_a^b y^{(n)*}\psi dt.$$

The next result generalizes Lemma 2.1. The proof is a formal calculation done by repeatedly integrating the integral term in $B[y, z][b^-, a^+)$ by parts.

5.1. Lemma. *Suppose* $l_j{}^+(z, \psi)$, $j = 0, \ldots, n - 1$, *exist for* $\psi \in S$ *and* $y \in D_p{}'$ *Then*

$$B[y, z](b^-, a^+) = W[y, z](b^-, a^+) + [\phi, Uy].$$

5.2. *Definition.*

$\tilde{D}^+ := \{z: l_j{}^+(z, \psi) \in AC_k; l_n{}^+(z, \psi) \in L_k{}^q$ for some $\psi \in \bar{S}\}$.
$\tilde{D}_0{}^+ := \{z \in \tilde{D}^+; B[y, z](b^-, a^+) = 0, y \in D\}$.
$\tilde{L}^+ := l_n{}^+(z, \psi)$ on $\tilde{D}^+$.
$\tilde{L}_0{}^+ := \tilde{L}^+$ on $\tilde{D}_0{}^+$.

We construct L^+, $L_0{}^+$, $L^{+\prime}$, $L_0{}^{+\prime}$ (as in the first order case) by restricting ψ to S and substituting $W[y, z](b^-, a^+)$ for $B[y, z](b^-, a^+)$.

With these new definitions Green's relations Lemma 3.8 hold as stated for the higher order case. The proofs which we omit amount to repeated integration by parts. Similarly the inclusions

$$L^+ \subset \tilde{L}^+ \subset L_0{}^* \quad \text{and} \quad L_0{}^{+\prime} \subset L_0{}^+ \subset \tilde{L}_0{}^+ \subset L_0{}^*$$

hold as in the first order case (cf. Remark 3.9).

We now show that Theorem 3.6 and its corollaries also extend to the higher order case. The only nontrivial step in the proof will be to generalize Lemmas 3.12 and 3.13. For, once it is known that $R(L_0)^\perp = N(\tilde{L}^+)$, $1 \leqq p < \infty$, we can retrace the proofs of the remaining lemmas almost verbatim.

To this end associate with L_0 the first order minimal operator $\mathscr{L}_0$ with domain and range in $L_n{}^p[a, b]$ determined (as in Definition 3.1) by

$$\tilde{l}y = y' + Ay, \quad \mathscr{V}y = 0,$$

where A is the $n \times n$ matrix

$$\begin{bmatrix} 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & -1 \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_1 \end{bmatrix}$$

and $\mathscr{V}: C_n[a, b] \to F$ is given by

$$\mathscr{V}y := [V_n, \ldots, V_1]y.$$

By Lemma 3.12 (old version)

$$(5.4) \quad R(\mathscr{L}_0) = N(\mathscr{L}^+)^{\perp},$$

where $N(\mathscr{L}^+)$ consists of all $z \in BV_n[a, b]$ such that

$$(5.5) \quad -(z + \langle \phi, \mathscr{V} \rangle^*)' + A^*z = 0$$

$$(5.6) \quad z + \langle \phi, \mathscr{V} \rangle^* \in AC_n[a, b], \psi \in S.$$

Now

$$A^* = \begin{bmatrix} 0 & 0 & \ldots & 0 & a_n \\ -1 & 0 & \ldots & 0 & a_{n-1} \\ 0 & -1 & \ldots & 0 & a_{n-2} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & & -1 & a_1 \end{bmatrix}.$$

Further

$$\langle \phi, \mathscr{V} \rangle^* = (\langle \phi, V_n \rangle, \ldots, \langle \phi, V_1 \rangle)^*.$$

Now suppose $f \in R(L_0)$. This is the case if and only if $(0, \ldots, 0, f)^t \in R(\mathscr{L}_0)$. By (5.4) we conclude that $f \in R(L_0)$ if and only if f is orthogonal to $\pi_n N(\mathscr{L}^+)$ where π_n is the projection operator of $N(\mathscr{L}^+)$ onto its nth component. More precisely: for $1 < p \leqq \infty$, $R(L_0) = \pi_n N(\mathscr{L}^+)^{\perp}$; for $p = 1$, $R(L_0) = {}^{\perp}\pi_n$-$(N(\mathscr{L}^+)$. We now wish to characterize $\pi_n(N(\mathscr{L}^+))$. Set $z = (z_1, \ldots, z_n)^t$. Then (5.5) is equivalent to the system of equations

$$(5.7) \quad \begin{aligned} -(z_{n-j} + \langle \phi, V_{j+1} \rangle^*)' - z_{n-(j+1)} + a_{j+1}{}^*z_n = 0, \quad j = 0, \ldots, n-2, \\ -(z_1 + \langle \phi, V_n \rangle^*)' + a_n{}^*z_n = 0, \quad j = n-1. \end{aligned}$$

Furthermore (5.7) becomes

$$(5.8) \quad z_{n-j} + \langle \phi, V_{j+1} \rangle^* \in AC, \quad j = 1, \ldots, n-1.$$

If we introduce the notation

$$(5.9) \quad \begin{aligned} K_0 z &:= z^*z \\ K_1 z &:= [K_0 z + \langle \phi, V_1 \rangle^*]' + a_1{}^*z \\ K_2 z &:= [K_1{}^*z + \langle \phi, V_2 \rangle^*]' + a_2{}^*z \\ &\cdot \quad \cdot \quad \cdot \\ K_n z &:= -[K_{n-1} z + \langle \phi, V_n \rangle^*]' + a_n{}^*z, \end{aligned}$$

it is apparent from (5.7), (5.9) and the structure of the partial adjoints (5.2) that

$$z_{n-1} = -K_1 z_n = -l_1^+(z, \psi) + \sum_2^n (-1)^{i-1} I^{i-2}\langle \phi, V_i\rangle^*$$

(5.10) $$z_{n-2} = -K_2 z_n = -l_2^+(z, \psi) + \sum_3^n (-1)^{i-2} I^{i-3}\langle \phi, V_i\rangle^*$$

$$\ldots$$

$$z_1 = -K_{n-1} z_n = -l_{n-1}^+(z, \psi) - \langle \phi, V_n\rangle^*.$$

Also (setting $j = n - 1$), $K_n z_n = l_n^+(z, \psi) = 0$. Moreover, from (5.8) and (5.10),

(5.11) $$-l_j^+(z_n, \psi) - \langle \phi, V_{j+1}\rangle + \sum_{j+1}^n (-1)^{i-j} I^{i-j+1}\langle \phi, V_i\rangle^* \in AC.$$

This implies that $l_j^+(z_n, \psi)$, $j = 0, \ldots, n - 1$, are absolutely continuous. From the definition of L^+, it follows that $z_n \in N(L^+)$. On the other hand, if $z \in N(L^+)$, a similar computation shows that $(K_0 z, \ldots, K_{n-1} z)$ exists and is in $N(\mathscr{L}^+)$. Thus $N(L^+) = \pi_n N(\mathscr{L}^+)$ and we have shown the following.

5.3. Lemma. *For* $1 < p \leqq \infty$, $R(L_0) = N(L^+)^\perp$. *For* $p = 1$, $R(L_0) = {}^\perp N(L^+)$.

We now generalize Lemma 3.13.

5.4. Lemma. *For* $1 \leqq p < \infty$, $R(L_0)^\perp = N(\tilde{L}^+)$.

The proof of Lemma 5.4 is exactly the same as that of Lemma 3.13 except that

$$\psi_k = \sum_{i=0}^n (-1)^{i+1} I^{(i)}[A_i^* z_k] + \pi_{n-1}, \quad \text{etc.}$$

where $\pi_{n-1} \in K^{(n-1)}$.

As mentioned at the beginning of this section the generalization of Theorem 3.6 and its corollaries is routine. First repeating the proof of Lemma 3.14 *verbatim* yields the adjoint relations

$$L_0^* = \tilde{L}^+; \quad L^* = \tilde{L}_0^+ \quad 1 \leqq p < \infty.$$

The proof of Lemma 3.15 is also valid in the higher order case, provided we can show that ϕ is arbitrary over $D_0^{+\prime}$. By standard theory there exists $\xi \in D_q'$ satisfying the two point boundary conditions

$$\xi(a^+) = 0, \quad \xi^{(j)}(b^-) = \sum_{i=j+1}^n (-1)^{i-(j+1)} I^{(j+1)}\langle \phi_i, V_i\rangle^*(b)$$

for any $\phi \in F^*$. One easily checks that $\xi - \psi \in D_0^{+\prime}$ where ψ corresponds to ϕ. Thus ϕ is arbitrary.

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Corollaries 3.16, 3.17, and 3.18 generalize without difficulty to the higher order case. This task is left to the reader.

5.5. *Examples.*

1. *Operators with Stieltjes boundary conditions.* Let

$$ly = \sum_{i=0}^{n} a_j y^{(n-j)}, \quad a_i \in C^{n-i}[0, 1]$$

$$Uy = \sum_{i=1}^{n} \int_0^1 dV_j y(n-j),$$

where the dV_i are $1 \times m$ vector valued real measures with components of bounded variation. Here $F \cong R^m = F^*$. Since F is finite dimensional, $\tilde{D}_0^+ = D_0^+ = D_0^{+\prime}$ and $L^* = L_0^{+\prime}$. We now show that the characterization of L^* read off from Definition 5.2 is equivalent to one obtained in an earlier paper [**3**] dealing with Stieltjes boundary value problems. We can assume for convenience that the measures $dV_j, j = 1, \ldots, n-1$, are singular with respect to Lebesque measure. For if dV_j has an absolutely continuous part V_{jc}, it satisfies

$$\int_0^1 dV_{jc} y^{(n-j)} = V_{jc} y^{(n-j-1)} \Big|_0^1 - \int_0^1 dV_{jc}' y^{(n-j-1)}.$$

By assuming that the Radon-Nikodym derivative is also of bounded variation, etc., repeated integration by parts results in singular measures on (0, 1) plus point mass measures at the endpoints. This means, in particular, that

(5.12) $\quad l_j^+(z, \psi) = l_j^+(z, \theta), \quad j = 0, \ldots, n-1.$

Moreover the requirement that $l_j^+(z, \psi)$ be absolutely conditinous (5.11) and (5.12) lead to the condition

$$-l_j^+(z, 0) + \langle \phi, V_{j+1} \rangle \in AC, \quad j = 0, \ldots, n-1.$$

The endpoint conditions may be read off from (5.3). Up to a change of sign therefore the domain of the adjoint is the same as given in Theorem 5.1 of [**3**].

2. *An operator with boundary conditions at infinitely many points.* Let $T \subset (0, 1)$ be the finite union of disjoint countable sets $T_j = \{t_{ij}\}, j = 1, \ldots, m$ such that $t_{ij} < t_{lk}, i < l, j \leqq k$. We characterize the adjoint L^* associated with the system

(5.13) $\quad ly = D^n y$

$$y(t_{ij}) = r_{ij}, \quad t_{ij} \in T$$

To use the machinery of this paper it is necessary to construct a t.v.s. F such that

$$Uy: = (\langle y(t_{i1}) \rangle, \ldots, \langle y(t_{im}) \rangle)$$

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is bounded. Associate with each set T_j a copy F_j of $\mathbf{R}^\omega$ and set

$$F: = \bigtimes_{j=1}^{m} F_j.$$

For $1 \leqq j \leqq m$, let $\langle c_{ij} \rangle$ be an element of c_{00}, the subspace of $\mathbf{R}^\omega$ having finitely many nonzero components. Let Γ be the m-fold direct sum of c_{00}. Identify Γ with a total family of functionals on Γ via the pairing

$$[r, c] = \sum_{i,j} r_{ij} c_{ij}.$$

Endow F with the weak topology relative to Γ, and Γ with the weak * topology relative to F. Then F and Γ are locally convex topological vector spaces. Furthermore, $F^* = \Gamma$ and $\Gamma^* = F$, and U is bounded.

In this framework we define L and L_0 as in § 5.

Setting

$$K(t_{ij}): = \sum \phi_{lk}, \quad 1 \leqq l \leqq i, 1 \leqq k \leqq j, \phi \in F^*$$

$$\Delta_{ij}: = [t_{ij}, t_{i+1j}],$$

we can write

(5.14) $$\langle \phi, U \rangle(t) = \sum_{i,j} K(t_{ij}) \lambda(t) \Delta_{ij}$$

Hence

(5.15) $$S = \left\{ \int_0^t \frac{(t-s)^{n-2}}{(n-2)!} \sum_{i,j} K(t_{ij}) \lambda(s) \Delta_{ij} ds, \quad \phi \in F^* \right\}$$

(Note that all sums are finite because of the nature of F^*).

Now let $\tilde{K}(t)$ be any step function with jumps on the points of T. From (5.14) and (5.15), it is readily seen that we can approximate $I^{n-1}[\tilde{K}]$ as closely as we please in the L^∞ norm by a sequence of elements in S. On the other hand, because of the local finite dimensionality of S, $\bar{S}_\infty = \bar{S}_p$, $1 \leqq p < \infty$. Therefore if $\psi \in \bar{S}$, ψ can be uniformly approximated by a sequence ψ_i in S. On every subinterval $(t_{ij}, t_{i+1j})\psi_i$ is a polynomial of degree n. Hence on this interval its uniform limit is also. Similarly, by considering the restriction of ψ_i to (t_{ij}, t_{i+2j}), we find that ψ inherits the smoothness of ψ_i at t_{i+1j}. The above reasoning shows that

$$\bar{S} = \{I^{n-1}[K]: K \text{ is a step function with jumps on } T\}.$$

Direct inspection of Definition 5.2 shows that $l_n^+(z, \psi) = (-1)^n z^{(n)}$. Now, $D_0^{+\prime} = D_0^+$ and consists of those functions satisfying

(1) $z^{(j)}(0) = z^{(j)}(1) = 0, \quad j = 0, \ldots, n-1.$

(2) $z^{(n-2)} \in AC$; $z^{(n-1)}$ has jump discontinuities at some finite subset of T.

(3) $z^{(n)} \in L^q$.

$\tilde{D}_0^+$, besides satisfying (3), can have jump discontinuities in the $n-1$st derivative at every point of $\bar{T}$. To determine the endpoint conditions on $\tilde{D}_0^+$, note that the condition $B[y, z](1^-, 0^+) = 0$ for all y in D yields

$$\sum_{j=1}^{n} (-1)^{j-1} z^{(j-1)} y^{(n-j)} \Big|_{0^+}^{1^-} = 0.$$

Since y is arbitrary at $z^{(j)}(0) = z^{(j)}(1) = 0, j = 0, \ldots, n-1$. Thus $\tilde{D}_0^+$ also satisfies (1).

To summarize, the adjoint L^* of L consists of the operator $(-1)^n z^{(n)}$ such that $z^{(n-2)}$ is absolutely continuous, $z^{(n-1)}$ has an arbitrary discontinuity at each point in T, and z along with its first $n-1$ derivatives vanishes at a and b. Further $L_0^{+\prime} = L_0^+$ and is a *proper* restriction of L^*. Note also that the null space of L^* consists of polynomial splines of order n and knot of set T.

3. Let ly be T, F, F^* be the same as in the previous example. We consider the integral boundary conditions

$$(5.16) \quad \int_{\Delta_{ij}} y dt = r_{ij}, \quad t_{ij} \in T.$$

The boundary operator U corresponding to (5.16) is easily seen to be continuous. Further, if $\phi \in F^*$

$$\begin{aligned} \langle \phi, U \rangle (t) &= \int_0^t \sum_{ij} \phi_{ij} \lambda(s) \Delta_{ij} ds \\ (5.17) \qquad &= \sum_{ij} \phi_{ij} (t_{i+1j} - t_{ij}) \lambda(t)_{[t_{i+1j},1]} \\ &\quad + \sum_{ij} (t - t_{ij}) \phi_{ij} \lambda(t)_{[t_{ij}, t_{i+1j}]}. \end{aligned}$$

The reader may verify the following assertions directly from (5.17) and the definitions of $l^+(z, \psi)$, S, $\bar{S}$, $\tilde{D}_0^+$ etc. The reasoning is similar to the previous example:

$$\bar{S}: = \left\{ (-1)^{n-1} \int_0^t \frac{(t-s)^{n-1}}{(n-1)} K(s) ds \right\}, \quad 1 \leqq q \leqq \infty,$$

where K is an arbitrary piecewise constant function L^q with jumps on T. S is a subspace of $\bar{S}$ such that K has support on finitely many intervals Δ_{ij}.

$$\begin{aligned} (5.18) \quad l^+(z, \psi) &= (-1)^n z^{(n)} + (-1)^{n-1} K; \quad D_0^{+\prime} = D_0^+ = \tilde{D}_0^+ = D_{0,q}{}' \\ &= \{z: z^{(n-1)} \in AC; z^{(n)} \in L^q; z^{(j)}(0) = z^{(j)}(1) = 0, j = 1, \ldots, n-1\}. \end{aligned}$$

Thus L^* is a linear relation with graph $(z, l^+(z, \psi))$ where z ranges over $D_{0_q}{}'$ which is the same domain as the ordinary minimal operator of degree n on $[0, 1]$. Setting (5.18) equal to zero, $N(L^*)$ is evidently the set of splines of order $n + 1$ and knot of set T.

We can also view the system as determining an operator $\hat{L}: D_p{}' \to L^p \times F$.

In this case, by Theorem 4.2 we see that the adjoint is given by $(-1)^n z^{(n)}$ on

(5.19) $\{(z, \phi): z \in D_0^{+\prime}\} \subset L^q \times F^*.$

By Corollaries 4.3 and 4.4 it follows that $(f, r) \in R(\hat{L})$ if and only if

(5.20) $$\int_0^1 fzdt = [\phi, r]$$

for all (z, ϕ) in the set (5.19). In the next section it will be seen how this fact leads to a nontrivial Whitney extension theorem for histosplines.

6. Some applications to splines. In this section we show how theory of adjoints developed in the previous sections can be applied to problems in interpolation. First the variational approach to splines will be reinterpreted from the point of view of generalized boundary value problems. Secondly, we will prove a new result concerning the existence of quadratic histospline extensions of minimum L^p norm.

We begin by discussing a generalized calculus of variations problem of interest in its own right. Let

$$J(y) := \int_a^b g(y, ly, t)dt$$

$$D_r := \{y : y^{n-1} \in AC; y^n \in L^p; Uy = r\},$$

where $r \in F$ and U is bounded. We seek an extremal in D_r maximizing or minimizing J. Our solution to this problem yields a striking application of the adjoint $\tilde{L}_0^+$ constructed in the previous sections.

Assume that g is continuous in y, ly, and t, and has continuous partial derivatives with respect to y and ly. Denote these derivatives evaluated at a fixed $\bar{y}$ by $g_{\bar{y}}$ and $g_{l\bar{y}}$.

We have the following result.

6.1. THEOREM. *$\bar{y}$ is an extremal for J only if $(g_{\bar{y}}, -g_{l\bar{y}}) \in G(\tilde{L}_0^+)$, where L_0^+ is the L^q adjoint of the operator L; $L^p \to L^p$ determined by l_y and $Hy = \theta$. In other words, $g_y \in \hat{D}_0^+$ and there exists $\psi \in \bar{S}$ such that $l^+(g_{\bar{y}}, \psi) = g_{l\bar{y}}$.*

Proof. If $\bar{y}$ is an extremal, the Gateaux differential $\delta J(\bar{y}; h)$ must vanish for all h in D (cf. [**17**, Theorem 1, p. 178]). Together with our assumptions on g, this implies

$$\begin{aligned}\delta J(y; h) &:= \frac{d}{d\alpha}\int_a^b g(\bar{y}(t) + \alpha h(t), l(\bar{y}(t) + \alpha h(t)), t)dt \\ &= \int_a^b (g_{\bar{y}}h + g_{l\bar{y}}lh)dt \\ &= [h, g_{\bar{y}}] + [lh, g_{l\bar{y}}] \\ &= 0.\end{aligned}$$

Because the last relation is true for all $h \in D$,

$$(g_{\bar{y}}, -g_{l_{\bar{y}}}) \in G(L^*).$$

Since $G(L^*) = G(\tilde{L}_0{}^+)$, the theorem is immediate.

To illustrate the preceeding theorem we consider the problem of minimizing $||ly||_2$ over D_r. In this case $g(t, y, ly) = (ly)^2$. Theorem 6.1 immediately gives the following necessary conditions on the minimizing solution or "spline" $\bar{y} \in D_r$

$$(6.1) \qquad \begin{aligned} & l\bar{y} \in D_0{}^+ \\ & \tilde{L}_0{}^+(l_{\bar{y}}) = 0. \end{aligned}$$

Just how (6.1) yields a concrete characterization of both known and unknown spline functions will be shown shortly by example.

The significance of Theorem 6.1 lies in the demonstration that highly peculiar variational problems can still be handled by "ordinary" calculus of variations. Even for "splines" the Euler equation exists! There is however a more powerful functional analytic approach which we now develop. What follows owes much to the analysis of DeBoor [**6**]. Our generalized boundary value problem approach, however, gives greater generality.

First let us give a simple derivation of (6.1) using Hilbert space methods: by definition $l\bar{y}$ is an element of minimum norm in the flat

$$R(L_r) := \{ly, y \in D_r\}.$$

Since $R(L)$ is closed, $l\bar{y}$ exists and is orthogonal to $R(L)$. Hence

$$l\bar{y} \in N(L^*) \equiv N(\tilde{L}_0{}^+)$$

which is equivalent to (6.1). (Note that this argument gives sufficiency as well as necessity.)

A similar idea may be used for the more difficult problem of minimizing $||R(L_r)||_p$ for $1 \leqq p \leqq \infty$. By a standard lemma in approximation theory (cf. [**17**, pp. 119-121])

$$(6.2) \qquad \inf_{y \in D_r} ||ly||_p = \inf_{(s \text{ fixed in } D_r)} ||ls + R(L)||_p = \sup_{||z||_q = 1 \;\; z \in N(\tilde{L}_0{}^+)} [ls, z]$$

(Note that (6.2) is a special case of the isometric isomorphism $X^*/M^\perp \cong M^*$ with $N(\tilde{L}_0{}^+) \equiv M$.)

If $p > 1$, a solution $\bar{y}$ exists since $L^p = (L^q)^*$. Moreover, if the supremum on the right is attained by an actual z^* (certainly the case if $\dim F < \infty$), $l\bar{y}$ must be "aligned" with z^*. In other words equality in Holder's equality holds yielding

$$(6.3) \qquad l\bar{y} = K|z^*|^{q-1} \operatorname{sgn} z^*(t), \quad 1 < p < \infty,$$

$$(6.4) \qquad l\bar{y} = K \operatorname{sgn} z^*(t), \quad p = \infty,$$

where $K: = \sup [ls, z]$. Note that (6.4) characterizes $\bar{y}$ completely only if $z^*(t)$ does not vanish on a set of positive measure. Also, if $p = 2$, (6.3) easily implies (6.1). If $p = 1$, $\bar{y}$ may not exist since L^1 is not the dual of any normed space. In fact it is easily seen from (6.4) that $\bar{y}$ *cannot* exist if $N(L^*)$ consists of continuous functions vanishing at the endpoints (cf. Examples 2, 3 § 5) except in the degenerate case $K = 0$.

6.2 *Examples.*

1. *Polynomial splines with infinitely many knots.* Let $ly = y^{(n)}$, $p = 2$. Let the constraints be given by

$$y(t_{ij}) = r_{ij}, \quad t_{ij} \in T.$$

In this case L is as in Example 2, § 5, and (6.1) implies that $\bar{y}$ is a natural polynomial spline of degree $2n - 1$ (order $2n$) with knot set T.

2. *Polynomial Lg-splines.* We modify the previous example only by introducing the constraints

$$\sum_{ij} \alpha_{ij}^{k} y(t_{ij}) = r_{ij}^{k}, \quad t_{ij} \in T, k = 1, \ldots, m.$$

The homogenous operator L' in this case is obviously an extension of L. Hence $L'^* \subset L^*$. By the Fredholm Alternative we conclude that $\bar{y}$ is also a natural polynomial spline of degree $2n - 1$.

3. *Histosplines.* Keeping ly, etc., as before we introduce the integral constraints

$$\int_{t_{ij}}^{t_{i+1j}} y dt = r_{ij}, \quad t_{ij} \in T.$$

Then L is the operator constructed in Example 3, § 5. Since $N(L^*)$ consists of polynomial splines of order $n + 1$ (degree n) and knot set T such that the first $n - 1$ derivatives vanish at 0 and 1, (6.2) implies that $\bar{y}$ is a polynomial spline of degree $2n$. Thus if $n = 2$, the solution is a quartic spline, or "histospline" a fact demonstrated by I. J. Schoenberg [**20**, p. 117] in his discussion of the smoothing of histograms by splines.

The reader may readily supply further examples of his own. Obvious ones would include the *plg/lg* splines of Jerome and Schumaker [**12**], (their structural properties follow at once from Example 1, § 5), "perfect splines", minimization under inequality constraints, etc. However since what has already been done demonstrates that a general class of constrained minimization problems yielding spline solutions can be handled in a unified way, we will not pursue these matters further. Instead we close the paper with a hopefully deeper application of our theory to splines. (An interesting anticipation of the method developed in this paper may be found in Reid [**18**]. Reid studied minimum norm solutions of boundary value problems under *two* point conditions, using functional analytic ideas similar to (6.2)-(6.4).)

In what follows we shall prove a Whitney or "$H^{n,p}$" extension theorem appropriate to an infinite set of integral interpolation conditions. Our results are analogous to a recent theorem of DeBoor [**5**] giving necessary and sufficient conditions for the existence of such extensions for point evaluation data. More interesting, perhaps, than the result, however, is the technique which shows the close relation of this and similar problems to the Fredholm Alternatives for nonhomogenous generalized boundary value problems developed in Section 4.

To simplify notation we will assume $T \subset (0, 1)$ to be a set with at most *one* limit point $\notin T$ at sup $T < 1$. Otherwise the formalism of Examples 2, 3, § 5 is retained. As will be seen, no generality is lost by this assumption.

6.3. THEOREM. *A sufficient condition that an absolutely continuous function y exists on* $[0, 1]$ *such that*

$$y' \in L^p, \quad 1 \leqq p < \infty,$$

and

$$\int_{t_i}^{t_{i+1}} y dt = r_i, \quad t_i \in T, \tag{6.5}$$

is that

$$\langle \Delta_i^{1/p} [r_i \Delta_i^{-1}, r_{i+1} \Delta_{i+1}^{-1}] \rangle \in l^p, \tag{6.6}$$

where Δ_i *denotes the length of* (t_i, t_{i+1}) *and* $[\,\cdot\,,\ \cdot\,]$ *is the divided difference*

$$(r_{i+1} \Delta_{i+1}^{-1} - r_i \Delta_i^{-1}) \Delta_i^{-1}.$$

Proof. Consider both the homogenous and nonhomogenous operators L and $\hat{L}$ determined by y' and the boundary conditions (6.5). Recall from Example 3, § 4 that if $n = 1$, $N(L_0^{+\prime})$ is the set of piecewise linear functions with corners on T vanishing on $[0, t_k]$ $[t_m, 1]$ for integers k, m. Furthermore,

$$z(t) = (t_{i+1} - t) \phi_i + z(t_{i+1}).$$

Define ψ: $N(L_0^{+\prime}) \to F$ by

$$\psi(z) = \langle z(t_{i+1}) \Delta_i^{1/q} \rangle.$$

Since $\psi(z)$ has finitely many non zero terms, ψ maps into l^q. We now investigate the continuity of ψ. Let z_{Δ_i} be the restriction of $z \in N(L_0^{+\prime})$ to Δ_i. Consider the following sequence of maps

$$z \xrightarrow{\varphi_i} z_{\Delta_i} \xrightarrow{\psi_i} z(t_{i+1}) \Delta_i^{1/q}.$$

Clearly ϕ_i considered as map from L^q to $L^q(\Delta_{ij})$ has norm 1. Further

$$||\psi_i||_q^q = \sup \frac{z^q(t_{i+1}) \Delta_i}{||z_{\Delta_i}||^q}, \quad z \in N(L_0^{+\prime}).$$

We estimate $||\psi_i||_q$ first in the case $q = 1$ (corresponding to $p = \infty$) and then

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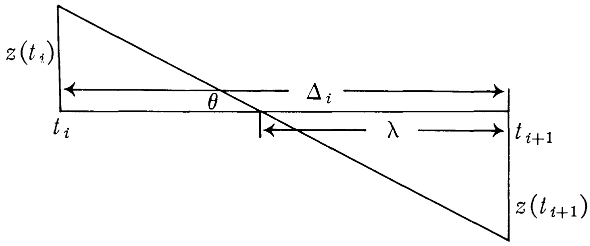
use Holder's inequality to consider the case $q > 1$. Finally we use this bound, which is independent of i, to bound ψ.

Case 1. $z(t_i)\, z(t_{i+1}) > 0$. In this case it is clear that $\|\psi_i\|_1 = 2$. The proof is left to the reader.

Case 2. $z(t_i)\, z(t_{i+1}) \leqq 0$. Assume that $z(t_i)\, z(t_{i+1}) < 0$. Let λ, θ be as in the figure below.



Then

$$(6.7) \quad \frac{|z(t_{i+1})|\Delta_i}{\|z_{\Delta_i}\|} = \frac{2\lambda\Delta_i}{\lambda^2 + (\Delta_i - \lambda)^2} = \frac{2\Delta_i}{\lambda + (\Delta_i - \lambda)^2/\lambda}.$$

By the derivative test this function has a maximum on $[0, \Delta_i]$ at $\lambda = \Delta_i/2$, and we find

$$\|\psi_i\|_1 = 1/(\sqrt{2} - 1) = 2.414.$$

Since the limit of (6.7) is 0 if $\lambda \to 0$ and 2 if $\lambda \to \Delta_i$, we can dispose of the case $z(t_i)\, z(t_{i+1}) = 0$. Suppose now $\infty > q > 1$. By Holder's inequality,

$$\|z\|_1 \leqq \|z\|_q\, \Delta_i^{1-1/q}.$$

Hence

$$\frac{|z(t_{i+1})|^q \Delta_i}{\|z_{\Delta_i}\|_q^q} \leqq \left(\frac{|z(t_{i+1})|\Delta_i}{\|z_{\Delta_i}\|_1}\right)^q,$$

and

$$\|\psi_i\|_q \leqq \left(\frac{1}{\sqrt{2} - 1}\right)^q.$$

Since

$$\begin{aligned} \|\psi(z)\|_q^q &= \sum |z(t_i)|^q \Delta_i \leqq \sum \|\psi_i\|_q^q \|z_{\Delta_i}\|_q^q \\ &\leqq \left(\frac{1}{\sqrt{2} - 1}\right)^q \sum \|z_{\Delta_i}\|_q^q \\ &= \left(\frac{1}{\sqrt{2} - 1}\right)^q \|z\|_q^q, \end{aligned}$$

we obtain

$$||\psi||_q \leqq 1/(\sqrt{2} - 1).$$

Thus for any $1 \leqq q < \infty$, we conclude that ψ is bounded independently of Δ_i and q. Now suppose the condition (6.5) holds. Set

$$\hat{r}: = \langle \Delta_i^{1/q}[r_i \Delta_i^{-1}, \quad r_{i+1} \Delta_{i+1}^{-1}] \rangle.$$

Then $\hat{r}$ induces a continuous functional on $R(\psi)$ and thus on $N(L_0^{+\prime})$ via the natural pairing on $l^p \times l^q$. Further, the norm of $\hat{r}$ on $N(L_0^{+\prime})$ is less than or equal to $||\hat{r}||_p/(\sqrt{2} - 1)$. By the Hahn-Banach theorem, this functional has a norm preserving extension to all of L^q. Let $f \in L^p = (L^q)^*$ be the Riesz representor of this functional. Thus restricted to $N(L_0^{+\prime})$

$$\text{(6.8)} \qquad \int_0^1 zfdt = \hat{r} \circ \psi(z).$$

But

$$\begin{aligned} \hat{r} \circ \psi(z) &= \sum_{i=1} \Delta_i^{1/p}[r_i\Delta_i^{-1}, r_{i+1}\Delta_{i+1}^{-1}]z(t_{i+1})\Delta_i^{1/q} \\ &= \sum_{i=1}^{\infty} (r_i\Delta_i^{-1} - r_{i+1}\Delta_{i+1}^{-1})z(t_{i+1}) \\ &= \sum_{i=1}^{\infty} r_{i+1}\Delta_{i+1}^{-1}(z(t_{i+2}) - z(t_{i+1})) + r_1\Delta_1^{-1}z(t_2) \\ &= \sum_{i=1} r_i\phi_i. \end{aligned}$$

So

$$\text{(6.9)} \qquad \int_0^1 zfdt = [r, \phi], \quad z \in N(L_0^{+\prime}).$$

From what has been shown in § 4, (6.9) says that f is orthogonal to $N(\hat{L}^*)$. Hence by the Fredholm Alternative (Corollary 4.3), $(f, r)^t \in R(\hat{L})$. In other words, there exists $y \in AC$ such that (6.5) is satisfied and $y' \in L^p$. The proof of the theorem is complete.

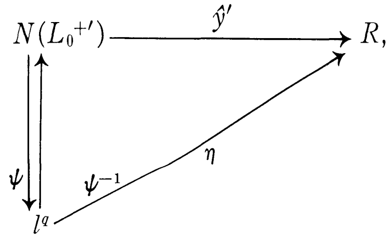
6.4. Corollary. *Condition* (6.6) *is necessary if*

$$\text{(6.10)} \qquad \sup \Delta_{i-1}^{-1} \Delta_i = N < \infty.$$

Proof. We begin by reversing the reasoning of Theorem 6.3. If $(f, r)^t \in R(\hat{L})$ then (6.9) holds and y' induces a continuous functional y' on $N(L_0^{+\prime})$. Moreover

$$\begin{aligned} [r, \phi] &= \sum y(t_i)\phi_{i+1} \\ &= \sum y(t_i)\Delta_i^{-1}(z(t_{i+1}) - z(t_i)) \\ \text{(6.11)} \qquad &= \sum (y(t_i)\Delta_i^{-1} - y(t_{i+1})\Delta_{i+1}^{-1})z(t_{i+1}) \\ &= \sum \Delta_i^{1/p}[y(t_i)\Delta_i^{-1}, y(t_{i+1})\Delta_{i+1}^{-1})]z(t_{i+1})\Delta_i^{1/q}. \end{aligned}$$

It is clear that ψ^{-1} exists. Also (6.9) and (6.11) imply the commuting diagram



where η represents the action of

$$\langle \Delta_i^{1/p} [y(t_i)\ \Delta_i^{-1}, y(t_{i+1})\ \Delta_{i+1}^{-1}] \rangle$$

on $R(\psi) \subset l^q$. Since $\eta \circ \psi = \hat{y}'$, we can conclude that η is continuous, i.e., (6.6) holds provided ψ^{-1} is continuous. Suppose, therefore, that $\bar{c} \in c_{00}$ and $z: = \psi^{-1}\,\bar{c}$. Then

$$z(t_i): = c_i\,\Delta_{i-1}^{-1/q},$$

and

$$||\psi^{-1}||_q{}^q = \sup \frac{||z||_q{}^q}{||\bar{c}||_q{}^q} \leqq 1/2 \sum \frac{(|c_{i-1}|^q \Delta_{i-1}{}^{-1} + |c_i|^q \Delta_i{}^{-1})\Delta_i}{||\bar{c}||_q^q} \leqq \frac{M+1}{2}.$$

6.4. *Discussion.* 1. As previously mentioned the assumption that T contains only one limit point at 1 was made only to simplify notation. Our results extend without difficulty to a general set T of the first species considered in § 5. This is done by repeating the proofs verbatim on each T_j and taking direct sums where necessary. Thus (6.6) should be replaced by

$$\sum_{\oplus j} \Delta_{ij}{}^{1/p}[r_{ij}\Delta_{ij}{}^{-1}, r_{i+1j}\Delta_{i+1j}{}^{-1}] \in \sum_{\oplus j} l_p.$$

2. The proof of Theorem 6.3 gives both an extension of minimum norm and an upper bound on this norm. To see this, suppose (6.6) holds so that an extension exists. We know (cf. 6.2)

$$\text{(6.12)} \quad \inf ||y' + R(L)||_p = \sup\,[y', z] \quad ||z|| = 1, z \in N(\tilde{L}_0{}^+)$$

and the minimum is attained (because $L^p = (L^q)^*$, $1 \leqq q < \infty$). However since $N(L_0{}^{+\prime}) = N(\tilde{L}_0{}^+)$ it follows from (6.12) that the supremum on the right is just

$$||y'||_p: = ||f||_p \leqq ||\hat{r}||\ ||\psi||_q \leqq \frac{||\hat{r}||}{2-1}.$$

(If $p = 2$, one can obtain the sharper bound $||\psi||_2 \leqq 2$ by a direct argument similar to the one for $p = 1$.) In any event these bounds yield the important practical result that one obtains information about the norm of the derivative of a minimizing histospline in terms of the histogram without having to actually compute the histospline.

3. For point evaluation, the condition analogous to (6.6) for $n \geqq 1$ is

$$\langle \Delta_i^{1/p} [t_i, \ldots, t_{i+n}] \rangle \in l^p.$$

In this case necessity is easy (cf. [**20**, p. 59]). However, until [**5**] (e.g. [**9**]) a mesh ratio requirement similar to (6.10) was required to prove sufficiency. In this paper, on the other hand, the order of difficulty is reversed! Whether or not (6.10) is required for necessity and our technique can be extended to the case $n > 1$ and/or more general functionals are interesting questions we hope to consider elsewhere.

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