Appendix A

Green–Kubo formula for transport coefficients

Transport coefficients of a gauge theory plasma, such as the shear viscosity η, can be extracted from correlation functions of the gauge theory via a relation known as the Green–Kubo formula. Here, we derive this relation, used in both Chapters 3 and 6, for the case of the shear viscosity. Let us consider a system in equilibrium and let us work in the fluid rest frame, meaning that $u^\mu = (1, \mathbf{0})$. Deviations from equilibrium are studied by introducing a small external source of the type

$$S = S_0 + \frac{1}{2} \int d^4x \, T^{\mu\nu} h_{\mu\nu} \, , \tag{A.1}$$

where S_0 (S) is the action of the theory in the absence (presence) of the perturbation $h_{\mu\nu}$. To leading order in the perturbation, the expectation value of the stress tensor is

$$\langle T^{\mu\nu}(x) \rangle = \langle T^{\mu\nu}(x) \rangle_0 - \frac{1}{2} \int d^4y \, G_R^{\mu\nu,\alpha\beta}(x-y) h_{\alpha\beta}(y), \tag{A.2}$$

where the subscript 0 indicates the unperturbed expectation value and the retarded correlator is given by

$$i G_R^{\mu\nu,\alpha\beta}(x-y) \equiv \theta\left(x^0 - y^0\right) \left\langle \left[T^{\mu\nu}(x), T^{\alpha\beta}(y) \right] \right\rangle \, . \tag{A.3}$$

To extract the shear viscosity, we concentrate on an external perturbation of the form

$$h_{xy}(t, z) \, . \tag{A.4}$$

Upon Fourier transforming, this is equivalent to using rotational invariance to choose the wave vector of the perturbation, $\mathbf{k}$, along the $\hat{z}$ direction. The off-diagonal components of the stress tensor are then given by

$$\left\langle T^{xy}(\omega, k) \right\rangle = -G_R^{xy,xy}(\omega, k) h_{xy}(\omega, k) \, , \tag{A.5}$$

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to linear order in the perturbation. In the long-wavelength limit, in which the typical variation of the perturbed metric is large compared with any correlation length, we obtain

$$\left\langle T^{xy}\right\rangle(t,z) = -\int \frac{d\omega}{2\pi} e^{-i\omega t} G_R^{xy,xy}(\omega, k=0) h_{xy}(\omega, z) \,. \tag{A.6}$$

This long-wavelength expression may be compared to the hydrodynamic approximation by studying the reaction of the system to the source within the effective theory. The source $h_{\mu\nu}$ can be interpreted as a modification on the metric,

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \,. \tag{A.7}$$

To leading order in the perturbation, the shear tensor defined in Eq. (2.16) is given by

$$\sigma_{xy} = 2\,\Gamma^0_{xy} = \partial_0 h_{xy} \,, \tag{A.8}$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols. The hydrodynamic approximation is valid in the long time limit, when all microscopic processes have relaxed. In this limit, we can compare the linear response expression (A.5) to the expression obtained upon making the hydrodynamic approximation, namely (2.14). We conclude that

$$\eta = -\lim_{\omega\to 0} \frac{1}{\omega} \lim_{k\to 0} \mathrm{Im} G_R^{xy,xy}(\omega, \mathbf{k}) \,. \tag{A.9}$$

This result is known as the Green–Kubo formula for the shear viscosity.

The above discussion for the stress tensor can also be generalized to other conserved currents. In general, the low frequency limit of $G_R(\omega, \vec{k})$ for a conserved current operator O defines a transport coefficient χ

$$\chi = -\lim_{\omega\to 0} \lim_{\vec{k}\to 0} \frac{1}{\omega} \mathrm{Im} G_R(\omega, \mathbf{k}) \,, \tag{A.10}$$

where the retarded correlator is defined analogously to Eq. (A.3)

$$i G_R(x-y) \equiv \theta\left(x^0 - y^0\right) \left\langle\left[O(x), O(y)\right]\right\rangle \,. \tag{A.11}$$

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