Glasgow Math. J. **56** (2014) 551–568.
doi:10.1017/S0017089514000032.

THE HOMOGENISED ENVELOPING ALGEBRA OF THE LIE ALGEBRA $s\ell(2, \mathbb{C})$

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(Received 9 November 2012; revised 29 May 2013; accepted 20 June 2013)

Abstract. In this paper, we study the homogenised algebra B of the enveloping algebra U of the Lie algebra $s\ell(2, \mathbb{C})$. We look first to connections between the category of graded left B-modules and the category of U-modules, then we prove B is Koszul and Artin–Schelter regular of global dimension four, hence its Yoneda algebra $B^!$ is self-injective of radical five zeros, and the structure of $B^!$ is given. We describe next the category of homogenised Verma modules, which correspond to the lifting to B of the usual Verma modules over U, and prove that such modules are Koszul of projective dimension two. It was proved in Martínez-Villa and Zacharia (Approximations with modules having linear resolutions, *J. Algebra* **266**(2) (2003), 671–697)] that all graded stable components of a self-injective Koszul algebra are of type ZA_∞. Here, we characterise the graded $B^!$-modules corresponding to the Koszul duality to homogenised Verma modules, and prove that these are located at the mouth of a regular component. In this way we obtain a family of components over a wild algebra indexed by $\mathbb{C}$.

2000 *Mathematics Subject Classification.* Primary 16S30, 17B35; Secondary 17B10

1. Introduction. In the papers [**9**, **10**], we considered the homogenised Weyl algebra B_n of the Weyl algebra A_n [**3**]. We proved that such an algebra is Artin–Schelter regular and Koszul, and explored the relation between the category of graded B_n-modules and the category of A_n-modules. Since the connection between these two categories is well understood, we can study the finitely generated graded B_n-modules instead of finitely generated A_n-modules. In the paper we applied the Koszul duality, and related the finitely graded B_n-modules with the finitely generated graded modules over the Yoneda algebra $B_n^!$. In this way we provided dictionaries relating modules over the Weyl algebras A_n with graded modules over the finite dimensional self-injective Koszul algebra $B_n^!$.

In this paper, we will concentrate on the study of the enveloping algebra U of the Lie algebra $s\ell(2, \mathbb{C})$ and its homogenised algebra B. In this case B is a Koszul Artin–Schelter regular algebra of global dimension 4 and we can extend the methods used for the Weyl algebras to study the relations among U-modules, graded B-modules and graded $B^!$-modules.

It is generally agreed that the classification of all modules over a wild finite dimensional algebra is impossible. It makes sense to look to families of indecomposable modules over a particular finite dimensional wild algebra, this is what we do in this paper for the self-injective wild algebra $B^!$. By the standard methods of the representation theory we do not know much about the $B^!$-modules, we only know by the result of [**11**] that the graded stable Auslander–Reiten components are all of type ZA_∞. However, the algebra B is the only Lie algebra for which all irreducible modules are known, in particular a family of modules $\{M(\lambda) \mid \lambda \in \mathbb{C}\}$, called the Verma modules, is fully described. The so-called weight modules and the related category $\mathcal{O}$ have been fully studied for $s\ell(2, \mathbb{C})$ [**12**]. We will here consider the corresponding category of homogenised Verma modules $V(\lambda)$ with $\lambda \in \mathbb{C}$. These modules are Koszul of projective dimension 2, and we can describe the modules $W(\lambda)$ over the Yoneda algebra $B^!$corresponding to $V(\lambda)$ under Koszul duality. We prove that such modules $W(\lambda)$ belong to the mouth of a regular graded component, obtaining in this way an infinite number of components parametrised by $\lambda \in \mathbb{C}$.

2. The homogenised algebra B of the enveloping algebra U of $s\ell(2, \mathbb{C})$. In this section, we study the basic properties of the homogenised algebra B of the enveloping algebra U of $s\ell(2, \mathbb{C})$ and its Yoneda algebra $B^!$. We concentrate on the structure of these algebras, and our main tool is the resolution given by Anick in [**1**]. The situation is very similar to the case of the Weyl algebras considered in [**9**].

Throughout the paper, $\mathbb{C}$ will denote the complex numbers, and $s\ell(2, \mathbb{C})$ denotes the $\mathbb{C}$-vector subspace of the space of two by two matrices $M_{2\times 2}(\mathbb{C})$ consisting of the matrices with zero trace. $s\ell(2, \mathbb{C})$ is the Lie algebra with a bracket product $[X, Y] = XY - YX$. The enveloping algebra U of $s\ell(2, \mathbb{C})$ is given by generators and relations by $U = \mathbb{C} < e, f, h > / L$, where $\mathbb{C} < e, f, h >$ is the free algebra with three generators: e, f, h, and L is the ideal generated by the relations: $[e, f]$–h, $[h, e]$–$2e$, $[h, f] + 2f$. It is well known that U has a Poincare–Birkoff basis [**5**, **12**], which means that every element of $u \in U$ can be written in a unique way as a combination $u = \sum_\ell \sum_{i+j+k=\ell} c_{i,j,k} e^i f^j h^k$ and $c_{i,j,k} \in \mathbb{C}$.

We will denote by B the algebra defined by generators and relations as: $B = \mathbb{C} < e, f, h, z > / I$, where $\mathbb{C} < e, f, h, z >$ is the free algebra in four generators and I is the ideal generated by the relations: $[e, f]$–hz, $[h, e]$–$2ez$, $[h, f]$+$2fz$, $[e, z]$, $[f, z]$, $[h, z]$. We will call B the homogenised enveloping algebra of $s\ell(2, \mathbb{C})$ [**8**, **16**].

We prove that B has a Poincare–Birkoff basis, it is Koszul and is an Artin–Schelter regular.

In the next proposition, we describe relations between the homogenised enveloping algebra B and the usual enveloping algebra of $s\ell(2, \mathbb{C})$. We leave the details of the proof for the reader.

PROPOSITION 1. *There is an isomorphism of* $\mathbb{C}$*-algebras* $B/(z-1)B \cong U$.

THEOREM 1. *The algebra B has a Poincare–Birkoff basis.*

Proof. Since the enveloping algebra U has Poincare–Birkoff basis $\{\mathrm{e}^j\mathrm{f}^k\mathrm{h}^\ell\}$, it follows easily from the previous proposition that B has Poincare–Birkoff basis $\{\mathrm{z}^i\mathrm{e}^j\mathrm{f}^k\mathrm{h}^\ell\}$. □

We will freely use the definitions and theorems from [**1**].

We give to the generators e, f, h, z the following order $z < f < e < h$ and to the words of $\mathbb{C} < e, f, h, z >$ the order grade lexicographic. There is a natural surjection φ: $\mathbb{C} < e, f, h, z > \to B$.

Let W be a monoid consisting of all words of $\mathbb{C} < e, f, h, z >$. The monoid W is well ordered.

Given $u, v \in W$, following [**1**], we will say that v is a submonomial of u, $v < u$ if $v = 1$ or $v = x_{i_m}x_{i_{m-1}}\dots x_{i_s}$ and $u = x_{i_1}x_{i_2}\dots x_{i_t}$, with $1 \leq m \leq s \leq t$ and $x_{i_j} \in W$.

Submonomial is a partial order on W.

A subset $M \subset W$ is an order ideal of monomials (o.i.m.) if and only if $u \in M$ and $v < u$ implies $v \in M$.

Let M is defined as $M = \{x \in W \mid \varphi(x) \notin span\varphi(y), y < x\}$.

We have from [**1**] the following.

LEMMA 1. *The set M is an o.i.m. and the elements $\varphi(x)$ for $x \in M$ form a basis of B as $\mathbb{C}$-module.*

We compute in our case these basis.

The following diagrams illustrate the order of the monomials involved in the relations defining the ideal I.

$$\begin{array}{c} h \nearrow\searrow z \\ \cdot \xrightarrow{e} \cdot \xrightarrow{f} \cdot \\ f \searrow\nearrow e \end{array}, \text{ where } fe < ef < hz,$$

$$\begin{array}{c} h \nearrow\searrow e \\ \cdot \xrightarrow{e} \cdot \xrightarrow{h} \cdot \\ e \searrow\nearrow z \end{array}, \text{ where } ez < eh < he,$$

$$\begin{array}{c} h \nearrow\searrow f \\ \cdot \xrightarrow{f} \cdot \xrightarrow{h} \cdot \\ f \searrow\nearrow z \end{array}, \text{ where } fz < fh < hf,$$

$$\begin{array}{c} e \nearrow\searrow z \\ z \searrow\nearrow e \end{array}, \text{ where } ze < ez,$$

$$\begin{array}{c} f \nearrow\searrow z \\ z \searrow\nearrow f \end{array}, \text{ where } zf < fz,$$

$$\begin{array}{c} h \nearrow\searrow z \\ z \searrow\nearrow h \end{array}, \text{ where } zh < hz.$$

It follows from the defining relations and the order we gave to the monomials that the paths: hz, he, hf, ez, fz do not belong to M.

Moreover, $hz = ef - fe$ and $hz = zh$, hence $ef - fe - zh = 0$ and $ef > fe, fe > zh$. Therefore: $ef \notin M$.

M does not contain the paths $\{hz, he, hf, ez, fz, ef\}$.

Since B has Poincare–Birkoff basis $\{\mathrm{z}^i\mathrm{e}^j\mathrm{f}^k\mathrm{h}^\ell\}$, it is clear that a word is in M if and only if it is of the form $z^i f^j e^k h^\ell$.

Using the Poincare–Birkhoff basis, we define a good filtration on B, as follows:

Let b be an element of B. Then b is written as $b = \sum_{i,j,k,\ell\geq 0} c_{i,j,k,\ell} z^i f^j e^k h^\ell$, with $c_{i,j,k,\ell} \in \mathbb{C}$ such that all but a finite number of them are zero.

The degree of b is the largest integer n such that some $c_{i,j,k,\ell} \in \mathbb{C}$ with $j + k + \ell = n$ is non-zero.

Let $\mathcal{F}_n$ be defined by $\mathcal{F}_n = \{\, b \in B \mid \text{degree(b)} \leq n\}$.

Then $\mathcal{F}_{-1} = 0, \mathcal{F}_0 = \mathbb{C}[z]$, $\mathcal{F}_i \subset \mathcal{F}_{i+1}$, for all $i \geq 0$, $B = \cup_{n\geq 0}\mathcal{F}_n$, $\mathcal{F}_n\mathcal{F}_m \subset \mathcal{F}_{n+m}$ and B is generated as algebra by $\mathcal{F}_1$.

The associated graded algebra $\oplus_{i\geq 0}\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to the polynomial ring $\mathbb{C}[z, e, f, h]$.

https://doi.org/10.1017/S0017089514000032 Published online by Cambridge University Press

Proceeding as in [**9**], we can check the properties of the above filtration. We leave details for the reader.

A filtration $\mathcal{F}$ such that the associated graded ring $\mathrm{Gr}(\mathcal{F})$ is noetherian, and $\mathrm{Gr}_1(\mathcal{F})$ generates $\mathrm{Gr}(\mathcal{F})$ as a $\mathcal{F}_0$ algebra, is called a 'good' filtration.

It was proved in [**13**] that algebras with a 'good' filtration are noetherian; as a consequence we obtain the following.

THEOREM 2. *The algebra B is noetherian.*

Let V_M be the obstruction set $V_M = \{v \in W \mid v \notin M, u \lneqq v \text{ implies } u \in M\}$.

It is clear that $V_M = \{hz, he, hf, ez, fz, ef\}$.

According to [**1**], 0-chains are precisely the arrows $V_0 = \{e, f, h, z\}$, 1-chains are $V_1 = V_M$, 2-chains are the 2-overlaps V_2 ={efz, hfz, hez, hef} and 3-chains are the 3-overlaps $V_3 = $ {hefz}.

There are no 4-chains.

According to Anick [**1**], there is a projective resolution of the only graded simple $S : *)$

$$0 \to V_3 \otimes_{\mathbb{C}} B \to V_2 \otimes_{\mathbb{C}} B \to V_1 \otimes_{\mathbb{C}} B \to V_0 \otimes_{\mathbb{C}} B \to B \to S \to 0.$$

This is a minimal linear resolution, hence B is the Koszul algebra of global dimension 4.

We will compute the Yoneda algebra $B^{!}$ of B.

We know by the general theory of Koszul algebras [**6**, **7**] that $B^!$ is defined by quiver and relations as $B^! = \mathbb{C} < e, f, h, z > / \ I^{\perp}$, where $\mathbb{C} < e, f, h, z >$ is the free algebra in four generators and $I^{\perp}$ is the ideal generated by the relations orthogonal to the relations defining I with respect to the bilinear form defined in the paths of length two by $< \alpha\beta, \alpha'\beta' >= \begin{cases} 1 & \text{if } \alpha = \alpha', \beta = \beta' \\ 0 & \text{otherwise} \end{cases}$. We will use the following notation: $(a, b) = ab + ba$.

We prove next that $I^{\perp}$ is generated by S=$\{e^2, f^2, h^2, z^2, (e, f), (e, h), (f, h), (h, z) + ef, (e, z) - 2eh, (f, z) - 2hf\}$.

Let's compute $I^{\perp}$.

(1) $e^2, f^2, h^2, z^2 \in I^{\perp}$.

By definition of the bilinear form:

$< e^2, ef - fe - hz >=< e^2, he - eh - 2ez >$=0

$< e^2, hf - fh + 2fz >=< e^2, fz - zf >$=0

$< e^2, ez - ze >=< e^2, hz - zh >$=0

Then $e^2 \in I^{\perp}$. Similarly, $f^2, h^2, z^2 \in I^{\perp}$.

(2) $ef + fe, he + eh, hf + fh \in I^{\perp}$.

$< ef + fe, ef - fe - hz >=< ef + fe, he - eh - 2ez >=< ef + fe, hf - fh + 2fz >=$

$< ef + fe, fz - zf >=< ef + fe, ez - ze >=< ef + fe, hz - zh >= 0$.

It follows that $ef + fe \in I^{\perp}$.

We leave for the reader to check the inclusions: $he + eh, hf + fh \in I^{\perp}$.

(3) $hz + zh + ef, ez + ze - 2eh, fz + zf - 2hf \in I^{\perp}$.

$< hz + zh + ef, ef - fe - hz >=< hz + zh + ef \ , he - eh - 2ez >=$

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$< hz + zh + ef\ , hf - fh + 2fz >=< hz + zh + ef\ , fz - zf >=$
$< hz + zh + ef\ , ez - ze >=< hz + zh + ef, hz - zh >= 0.$

We have proved $hz + zh + ef \in I^{\perp}$.

The proof of the inclusions: $ez + ze - 2eh, fz + zf - 2hf \in I^{\perp}$ is similar.

We have the inclusion $S \subset I^{\perp}$. We check that S generates $I^{\perp}$.

Let b be an element of degree two in $I^{\perp}$. Then b has the following form:

$b = a_1e^2 + a_2 f^2 + a_3h^2 + a_4 z^2 + a_{12}fh + a_{21}hf + a_{13}fe + a_{31}ef + a_{14}fz + a_{41}zf + a_{23}he + a_{32}eh + a_{24}hz + a_{42}zh + a_{34}ez + a_{43}ze$,

which can be rewritten as:

$b = a_1e^2 + a_2 f^2 + a_3h^2 + a_4 z^2 + a_{12}(fh + hf) + a_{13}(fe + ef) + a_{14}(fz + zf - 2hf) + a_{23}(he + eh) + a_{24}(hz + zh + ef) + a_{34}(ez + ze - 2eh) + b'$,

where $b' = (a_{21} - a_{12} + 2a_{14})hf + (a_{31} - a_{13} - a_{24})ef + (a_{32} - a_{23} + 2a_{34})eh + (a_{41} - a_{14})zf + (a_{42} - a_{24})zh + (a_{43} - a_{34})ze$.

It follows $b' \in I^{\perp}$.

Then $< b', fz - zf >= a_{14} - a_{41} = 0$, $< b', hz - zh >= a_{24} - a_{42} = 0$, $< b', ez - ze >= a_{34} - a_{43} = 0$.

$< b', hf - fh - 2fz >= 0 = a_{21} - a_{12} + 2a_{14}$, $< b',\ ef - fe - hz >= 0 = a_{31} - a_{13} - a_{24}$, $< b', he - eh - 2ez >= -((a_{32} - a_{23} + 2a_{34}) = 0$.

We have proved $b' = 0$ and $I^{\perp}$ is generated by S.

The algebra $B^!$ is graded and $B^! = L_0 \oplus L_1 \oplus L_2 \oplus L_3 \oplus L_4$, with $L_0 = \mathbb{C}1$, $L_1 = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}z$, L_2 generated by $\{fe, he, fh, fz, hz, ez\}$, L_3 generated by $\{fhe, fez, fhz, hez\}$ and L_4 generated by $\{fhz\}$.

We must prove $dim_{\mathbb{C}} L_2 = 6$, $\dim_{\mathbb{C}} L_3 = 4$, $\dim_{\mathbb{C}} L_4 = 1$.

Using the resolution for the simply graded B-module S *) and the Koszul property [**6**], we get what we claimed, $dim_{\mathbb{C}} L_2 = 6$, $dim_{\mathbb{C}} L_3 = 4$, $\dim_{\mathbb{C}} L_4 = 1$.

It follows that the algebra $B^!$ has simple socle, hence it is self-injective [**17**].

By the Koszul theory [**16**], the algebra B is Artin–Schelter regular.

It will be clear from the relations that the algebra $B^!$ has a structure related with the exterior algebra.

Let $C^!$ be the exterior algebra: $C^! = \mathbb{C}<e, f, h>/<e^2, f^2, h^2, (e, f), (e, h), (f, h)>$. Then $C^!$ is a subalgebra of $B^!$ and $B^!$ decomposes $B^! = C^! \oplus C^! z$ as a left and $B^! == C^! \oplus zC^!$ as a right $C^!$-module .

It is clear that $B/zB \cong C$, where C is the polynomial algebra $\mathbb{C}[e.f, h]$.

Observe that as in [**9**], C appears as a quotient of B and $C^!$ as a subalgebra of $B^!$.

We next give another characterization of U.

We will follow very closely the ideas and results from [**9**]; although the next propositions were given in [**9**], we include them here for completeness.

By construction, B is a $\mathbb{C}[z]$-algebra.

Consider the multiplicative subset $S = \{1, z, z^2, ..\}$ of $\mathbb{C}[z]$ and denote by $\mathbb{C}[z]_S$ the graded localisation. It was proved in [**9**] that there is a $\mathbb{C}$-algebras isomorphism $\mathbb{C}[z]_S \cong \mathbb{C}[z, z^{-1}]$, where $\mathbb{C}[z, z^{-1}]$ denotes the Laurent polynomials.

The Algebra $B_z = B \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}]$ is $\mathbb{Z}$-graded with homogeneous elements b/z^k, where b is homogeneous and the degree of b/z^k is $\deg(b/z^k) = \deg(\mathrm{b}) - \mathrm{k}$.

There exists a homomorphism of graded algebras $\varphi : B \to B_z$ given by $\varphi(b) = b/1 = b \otimes 1$. Consider the composition of rings homomorphisms: $B \xrightarrow{\varphi} B_z \xrightarrow{\pi} B_z/(z-1)B_z$. Clearly, $(z - 1)B \subset \ker \pi\varphi$.

Let b be an element of B such that $b/1 \in (z - 1)B_z$. This is: $b/1 = (z - 1)b'/z^k$.

Therefore: $z^{\ell} b = (z - 1)z^t b' = (z - 1)p(z)b + b$ and $p(z)$ is a polynomial in z. Hence, $b = (z - 1)b'' \in (z - 1)B$.

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We have a factorization of $\pi\varphi$:

$$\begin{array}{ccccc} B & \xrightarrow{\varphi} & B_z & \xrightarrow{\pi} & B_z/(z-1)B_z \\ & \searrow & & \nearrow & \\ q & & B/(z-1)B & & \psi \end{array}$$

with ψ being an injective homomorphism of $\mathbb{C}$-algebras.

Let $b/z^k + (z-1)B_z$ be an element of $B_z/(z-1)B_z$. Element b/z^k can be written as: $b/z^k = b_\ell/z^\ell + b_{\ell\text{-}1}/z^{\ell-1} + \dots b_0 + b_1 z + \dots b_m z^m$, where each b_i is a polynomial in e, f, h.

For each $i, b_i/1 = z^i b_i/\ \ z^i = (z-1)p(z)b_i/z^i + b_i/z^i$. Then $b_i/z^i + (z-1)B_z = b_i/1 + (z-1)B_z$.

It follows, $b/z^k + (z-1)B_z = (b_\ell + b_{\ell\text{-}1} + \dots b_0 + b_1 + \dots b_m)/1 + (z-1)B_z$.

Element $u = b_\ell + b_{\ell\text{-}1} + \dots b_0 + b_1 + \dots b_m$ is a polynomial in e, f, h , therefore it is an element of $U = B/(z-1)B$ and $\psi(u) = b/z^k + (z-1)B_z$.

We have proved ψ is an isomorphism with inverse ψ^{-1}. The composition $\tau = \psi^{-1}\pi$ makes the diagram

$$\begin{array}{ccccc} B & & \xrightarrow{\varphi} & & B_z \\ & \searrow & & \swarrow & \\ q & & B/(z-1)B & & \tau \end{array}$$

commute.

The algebra B_z is $\mathbb{Z}$-graded and has in degree zero a subalgebra $(B_z)_0$. There is an inclusion $(B_z)_0 \to B_z$ and a projection $B_z \to B_z/(z-1)B_z$. Denote by $\theta : (B_z)_0 \to B_z/(z-1)B_z$ the composition.

Proposition 2. *The morphism $\theta : (B_z)_0 \to B_z/(z-1)B_z$ is an isomorphism.*

Proof. Let $u = \sum_{i=0}^m g_i(f, h, e)z^{-n_i}$ be an element of $(B_z)_0$; this is: each $g_i(f, h, e)$ is a polynomial in e, f, h of degree n_i and $n_0 > n_1 > \dots > n_m$.

We can rewrite u as $u = z^{-n_0-1}\sum_{i=0}^m g_i(f, h, e)z^{n_0+1-n_i} =$
$z^{-n_0-1}(\sum_{i=0}^m (g_i(f, h, e) + (z-1)p_i(z)g_i(f, h, e))$.

We write $g(f, h, e) = \sum_{i=0}^m (g_i(f, h, e)$.

Hence, $\theta(u) = u + (z-1)B_z = z^{-n_0-1}g(f, h, e) + (z-1)B_z$.

Assume $\theta(u) = 0$. Then $z^{-n_0-1}g(f, h, e) = (z-1)b' z^{-k}$.

It follows that there exist integers $s, t \geq 0$ with
$z^t g(f, h, e) = g(f, h, e) + (z-1)p(z)g(f, h, e) = (z-1)b'' z^s$.

Therefore: $g(f, h, e) = (z-1)\overline{b}$ with
$\overline{b} = b_0(f, h, e) + b_1(f, h, e)z + b_2(f, h, e)z^2 + \dots b_k(f, h, e)z^k$.

As above,
$g(f, h, e) = -b_0(f, h, e) + (b_0(f, h, e) - b_1(f, h, e))z + (b_1(f, h, e) - b_2(f, h, e))z^2 + \dots (b_{k-1}(f, h, e) - b_k(f, h, e))z^k + b_k(f, h, e)z^{k+1}$.

It follows $g(f, h, e) = -b_0(f, h, e)$ and $b_0(f, h, e) = b_1(f, h, e)) = b_2(f, h, e) = \dots = b_k(f, h, e) = 0$.

We have proved $\sum_{i=0}^m (g_i(f, h, e) = 0$ with each $g_i(f, h, e)$ homogeneous of degree n_i.

It follows $g_i(f, h, e) = 0$ for all i and $u = 0$.

We next prove that θ is surjective.

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Let $b/z^k + (z-1)B_z$ be an element of $B_z/(z-1)B_z$, $b = \sum_{i=0}^m b_i$ and b_i are homogeneous components of degree n_i with $n_0 < n_1 < \cdots < n_m$.

As before, $z^{n_m - n_i} b_i = (z-1)p_i(z)b_i + b_i$, the element of B, $b'_i = z^{n_m - n_i} b_i$ is homogeneous of degree n_m and $b' = \sum_{i=0}^m b'_i$ is homogeneous of degree $n_m = \ell$.

Then $b/z^k + (z-1)B_z = b'/z^k + (z-1)B_z$.

If $\ell > k$, then $b'/z^k + (z-1)B_z = z^{\ell-k} b'/z^\ell + (z-1)B_z = b'/z^\ell + (z-1)B_z$ with degree $b'/z^\ell = 0$.

If $\ell < k$, then $b''/z^k = z^{k-\ell} b'/z^k = b'/z^k + (z-1)p(z)b'/z^k$, where $b'' = z^{k-\ell} b$ has degree k.

$b''/z^k + (z-1)B_z = b'/z^k + (z-1)B_z = b/z^k + (z-1)B_z$.

In the first case $\theta(b'/z^\ell) = b/z^k + (z-1)B_z$.

In the second case $\theta(b''/z^k) = b/z^k + (z-1)B_z$.

We have proved θ is an isomorphism. □

With the identification $U \cong B/(z-1)B \cong B_z/(z-1)B_z$ we have proved B_z has a U in degree zero.

THEOREM 3. *There exists an isomorphism of graded rings:*
$U[z, z^{-1}] = U \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \cong B \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}]$.

Proof. Identifying U with $(B_z)_0$, we will prove that the morphism:
$\mu : (B_z)_0 \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \to B_z$ given by $\mu(bz^{-\ell} \otimes z^k) = bz^{k-\ell}$ is a ring isomorphism.
Let b/z^ℓ be an homogeneous element of B_z with degree $b = \ell + k$.
Then $b/z^\ell = z^k b/z^{\ell+k}$ and $\mu(b/z^{\ell+k} \otimes z^k) = b/z^\ell$ and μ is surjective.
Let b/z^k be an homogeneous element of degree zero.
Then $b = \sum_{i=0}^k g_i(f, h, e) z^{k-i}$ with each $g_i(f, h, e)$ a polynomial in f, e, h, which is either zero or degree $(g_i(f, h, e)) = i$.
$\mu(b/z^k \otimes z^\ell) = z^{\ell-k} \sum_{i=0}^k g_i(f, h, e) z^{k-i} = 0$ in B_z.
There exists $t \geq 0$ such that $z^t \sum_{i=0}^k g_i(f, h, e) z^{k-i} = 0$ in B_z.
Then $b/z^k \otimes z^\ell = z^t b/z^{k+t} \otimes z^\ell = 0$.
We have proved that μ is an isomorphism. □

Ring $U[z, z^{-1}]$ is strongly $\mathbb{Z}$-graded. The ring homomorphism $U \to U[z, z^{-1}]$ induces functors:
$U[z, z^{-1}] \otimes_U - : Mod_U \to Gr_{U[z,z^{-1}]}$ and $res_U : Gr_{U[z,z^{-1}]} \to Mod_U$.
By Dade's theorem [**4**] we have the following.

THEOREM 4. *The functors $U[z, z^{-1}] \otimes_U -$ and res_U are inverse exact equivalences.*

COROLLARY 1. *Equivalences $U[z, z^{-1}] \otimes_U -$ and res_U preserve projective and irreducible modules and send left ideals to left ideals giving an order preserving bijection.*

In view of previous statements, the study of U-modules reduces to the study of graded B_z-modules. We next consider the relation between the graded B-modules and the graded modules over the localisation B_z.

We will make use of the following:

DEFINITION 1. Given a B-module M, we define the z-torsion of M as $t_z(M) = \{m \in M \mid \text{there exists } n \geq 0 \text{ with } z^n m = 0\}$. The module M is of z-torsion if $t_z(M) = M$ and z-torsion free if $t_z(M) = 0$.

$t_z(M)$ is a submodule of M and a map $f : M \to N$ restricts to a map $f_{|t_z(M)} : t_z(M) \to t_z(N)$, in this way $t_z(-)$ is a subfunctor of the identity with $t_z(t_z(M)) = t_z(M)$,

For any B-module M there is an exact sequence:

$$0 \to t_z(M) \to M \to M/t_z(M) \to 0$$

with $t_z(M)$ of z-torsion and $M/t_z(M)$ z-torsion free.

The kernel of the natural morphism $M \to M_z = B_z \otimes_B M$ is $t_z(M)$.

In the next proposition we describe as a particular case known facts concerning any localisation [**14**].

Proposition 3. *The following statements hold:*

(*i*) *Given a graded map of B-modules $f : M \to N$ the map induced in the localisation: $f_z : M_z \to N_z$ is zero if and only if f factors through a z-torsion module.*

(*ii*) *Assume that the localised module M_z is finitely generated, and let $\varphi : M_z \to N_z$ be a morphism of graded B_z-modules. Then there exists an integer $k \geq 0$ and a morphism of B-modules $f : z^kM \to N$ such that the composition $M_z \xrightarrow{\sigma} (z^kM)_z \xrightarrow{f_z} N_z, f_z\sigma = \varphi$ and σ is an isomorphism of graded B_z-modules.*

(*iii*) *Let M be a finitely generated B_z-module. Then there exists a finitely generated graded B-submodule $\overline{M}$ of M such that $\overline{M}_z \cong M_z$.*

Proof. (i) Let $f : M \to N$ be a morphism of graded B-modules with $f_z = 0$ and $m \in M$. Then $f(m)/1 = 0$ implies that there exists an integer $k \geq 0$ with $z^kf(m) = 0$, hence $f(M)$ is of z-torsion and f factors $M \to f(M) \to N$.

Conversely, if $f = hg$ with $g : M \to L$ and $h : L \to N$ maps with L of z-torsion, then $L_z = 0$ and $f_z = h_zg_z = 0$.

(ii) Let $\varphi : M_z \to N_z$ be a morphism of graded B_z-module with M_z finitely generated. Assume $m_1, m_2, \ldots, m_k$ generate M_z and degree$(m_i) = d_i$ and let d be $d = \max\{d_i\}$

$\varphi(m_j) = \sum n_{ij} \otimes z^{k_{ij}}$ with degree $n_{ij} + k_{ij} = d_i$.

If $k_{ij} > 0$, then $n_{ij} \otimes z^{k_{ij}} = n_{ij}z^{k_{ij}} \otimes 1$ and we may assume that in the expression $\varphi(m_j) = \sum n_{ij} \otimes z^{k_{ij}}$ all $k_{ij} \leq 0$. Let k be $k = \max\{-k_{ij}\}$. Then $k + k_{ij} \geq 0$. Hence, $\varphi(m_j) = \sum n_{ij}z^{k+k_{ij}} \otimes z^{-k} = n_j \otimes z^{-k}$ and degree $n_j = k + d_j$.

Let $j : M \to M_z = M \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}]$ is the localisation map $j(m) = m/1 = m \otimes 1$ and $\varphi : M_z = M \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}] \to N_z = N \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}]$ is the given map.

Let $f' : z^kM \to N \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}]$ be the map $f(z^km_j) = z^kn_j \otimes z^{-k} = n_j \otimes 1$ and $f'(z^kM) \subset N \otimes 1 \cong N$ restricting the image and identifying $N \otimes 1 = N$ we obtain a map $f : z^kM \to N$ given by $f(z^km_j) = n_j$.

There is an exact sequence $0 \to z^kM \xrightarrow{i} M \to M/z^kM \to 0$, where M/z^kM is of z-torsion. Localising, we obtain an exact sequence: $0 \to (z^kM)_z \xrightarrow{i_z} M_z \to (M/z^kM)_z \to 0$ with $(M/z^kM)_z = 0$, and the map $\sigma = i_z$ is an isomorphism.

The morphism f_z is defined as $f_z(m/z^\ell) = f_z(z^k(m/z^{k+\ell}) = f(z^km)/z^{k+\ell} = \varphi(z^km)/z^{k+\ell} = z^k\varphi(m)/z^{k+\ell} = \varphi(m)/z^\ell = \varphi(m/z^\ell)$.

(iii) Let M be a finitely generated B_z-module with homogeneous generators $m_1, m_2, \ldots, m_k$ of degree$(m_i) = d_i$.

By restriction, M is a B-module. Let $\overline{M}$ be the B-submodule of M generated by $m_1, m_2, \ldots, m_k$.

We need to check that under localisation $\overline{M}_z$ is isomorphic to M as B_z-modules.

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Localising, $\overline{M}_z \cong B_z \otimes_B \overline{M} \cong \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}[z]} B \otimes_B \overline{M} \cong \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}[z]} \overline{M}$. The homogeneous elements of $\overline{M}_z$ are of the form $z^{-k} \otimes m$, let $\mu : \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}[z]} \overline{M} \to M$ be the multiplication map $\mu(z^{-k} \otimes m) = z^{-k}m$, hence $\mu(z^{-k} \otimes m) = 0 = z^{-k}m$ implies $z^k z^{-k} m = m = 0$, and μ is injective.

Let m be an homogeneous element of M of degree k. Then $m = \sum b_i/z^{n_i} m_i$ and degree $b_i + d_i - n_i = k$. Let n be $n = \max\{n_i\}$.

Therefore: $z^n m = \sum z^{n-n_i} b_i/m = \overline{m}$ and $\overline{m} \in \overline{M}$ with degree$(\overline{m}) = k + n$.

It follows, $\mu(z^{-n} \otimes \overline{m}) = m$. □

Corollary 2. *Let M and N be graded B-modules with M finitely generated. A morphism $\varphi : M_z \to N_z$ is an isomorphism if and only if a map $f : z^k M \to N$ such that $f_z = \varphi$ has kernel and co-kernel of z-torsion.*

3. The homogenised Verma modules. In the previous section we saw that there is an exact equivalence between the category of graded B_z-module, Gr_{B_z}, and the category of U-modules, Mod_U, where U is the enveloping algebra of $s\ell(2,\mathbb{C})$, B is its homogenization and B_z is the localisation of B in z. We also sketched relations between the category of graded B-modules and the category of graded B_z-modules. We remarked that the algebra B is Koszul and studied the structure of both B and its Yoneda algebra $B^!$. In this setting, families of modules over the enveloping algebras are known, and it was proved in **[11]** that $B^!$ is a self-injective Koszul wild algebra whose graded stable Auslander–Reiten components are of type ZA_∞. The aim of the section is to lift known categories of U-modules to the categories of B-modules. What we mean is that to describe the categories of B-modules such that when we apply the localisation functor and restrict them to the degree zero part of U of B_z, we obtain the given categories of U-modules. In some cases we will be able to use Koszul duality to obtain families of $B^!$-modules and see how these are distributed among the Auslander–Reiten components. Of special interest are the Verma modules; here we study the homogenised versions of Verma B-modules **[12]**.

3.1. The Verma B-modules. For each $\lambda \in \mathbb{C}$ we define the homogenised left ideal I_λ of B by $I_\lambda = Be + B(h - \lambda z)$ and the homogenised Verma module $V(\lambda) = B/I_\lambda$.

An homogeneous element $\overline{b}$ of $V(\lambda)$ is of the form:

$\overline{b} = b + I_\lambda$ with $b = \sum_{i+j+m+\ell=k} c_{i,j,m,\ell} f^i h^j e^m z^\ell$, which can be written as:

$$b = \sum_{\substack{i+j+m+\ell=k \\ m\geq 1}} c_{i,j,m,\ell} z^\ell f^i h^j e^m + \sum_{i+j+\ell=k} c'_{i,j,\ell} z^\ell f^i h^j$$

and

$$\overline{b} = \sum_{i+j+\ell=k} c'_{i,j,\ell} z^\ell f^i (h - \lambda z) + \lambda z)^j + I_\lambda =$$

$$\sum_{t=0}^{\ell} \sum_{i+j+\ell=k} c'_{i,j,\ell} z^\ell f^i (h - \lambda z)^{j-t} (\lambda z)^t + I_\lambda = \sum_{i+j+\ell=k} c'_{i,j,\ell} \lambda^j z^{\ell+j} f^i + I_\lambda.$$

It follows that the monomials $\{f^i z^m\}$ generate $V(\lambda)$ as $\mathbb{C}$-vector space. We prove they form a basis.

We will prove by induction on k that $\sum_{i+j=k} a_{ij} f^i z^j \in I_\lambda$ implies $a_{ij} = 0$.

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For $k = 1$, $a_{01}z + a_{10}f = b_1 e + b_2(h - \lambda z)$ implies $b_1 = 0$ and $b_2 = 0$, therefore: $a_{01} = a_{10} = 0$.

Assume $k > 1$.

If $\sum_{i+j=k} a_{ij} f^i z^j \in I_\lambda$, then

$$\sum_{i+j=k} a_{ij} f^i z^j = \sum_{i+j+m+\ell=k-1} c_{i,j,m,\ell} z^\ell f^i h^j e^{m+1} + \sum_{i+j+m+\ell=k-1} c'_{i,j,\ell} z^\ell f^i h^j e^m (h - \lambda z).$$

Since $eh = he - 2ez$, it follows by induction that $e^m h = he^m - 2me^m z$ and $e^m(h - \lambda z) = he^m - (2m + \lambda)e^m z$.

Then,

$$\sum_{i+j=k} a_{ij} f^i z^j = \sum_{i+j+m+\ell=k-1} c''_{i,j,m,\ell} z^\ell f^i h^j e^{m+1} + \sum_{i+j+\ell=k-1} d_{i,j,\ell} z^\ell f^i h^j (h - \lambda z).$$

This implies $c''_{i,j,m,\ell} = 0$.

Hence, we have equalities:

$$\begin{aligned}\sum_{i+j=k} a_{ij} f^i z^j &= z\left(\sum_{i+j+\ell=k-1} d'_{i,j,\ell} z^{\ell-1} f^i h^j (h - \lambda z)\right) + \sum_{i+j=k-1} d''_{i,j} f^i h^j (h - \lambda z)\\ &= z\left(\left(\sum_{i+j+\ell=k-1} d'_{i,j,\ell} z^{\ell-1} f^i h^j (h - \lambda z)\right) - \lambda \sum_{i+j=k-1} d''_{i,j} f^i h^j\right) + \sum_{i+j=k-1} d''_{i,j} f^i h^{j+1}.\end{aligned}$$

Therefore: $d''_{i,j} = 0$, which implies $a_{k,0} = 0$

In the equality: $\sum_{i+j=k} a_{ij} f^i z^j = z(\sum_{i+j+\ell=k-1} d'_{i,j,\ell} z^{\ell-1} f^i h^j (h - \lambda z))$ we can cancel z on both sides to have:

$$\sum_{i+j=k} a_{ij} f^i z^{j-1} = \sum_{i+j+\ell=k-1} d'_{i,j,\ell} z^{\ell-1} f^i h^j (h - \lambda z).$$

It follows by induction $a_{ij} = 0$ for all i, j.

We proved the following.

Proposition 4. *For each $\lambda \in \mathbb{C}$ the monomials $\{f^i z^m\}$ form a $\mathbb{C}$-basis of the homogenised Verma module $V(\lambda) = B/I_\lambda$, where $I_\lambda = Be + B(h - \lambda z)$.*

Corollary 3. *The homogenised Verma module $V(\lambda) = B/I_\lambda$ is z-torsion-free.*

The exact sequence: $0 \to I_\lambda \to B \to B/I_\lambda \to 0$ induces by localisation an exact sequence of B_z-modules $0 \to (I_\lambda)_z \to B_z \to (B/I_\lambda)_z \to 0$, restricting to the degree zero part, we get an exact sequence:

$$0 \to ((I_\lambda)_z)_0 \to (B_z)_0 \to ((B/I_\lambda)_z)_0 \to 0.$$

We proved the isomorphisms: $U \cong B/(z-1)B \cong B_z/(z-1)B_z \cong (B_z)_0$.

On the other hand, $(I_\lambda)_z = B_z \otimes_B I_\lambda = B_z \otimes_B Be + B_z \otimes_B B(h - \lambda z) = B_z e + B_z(h - \lambda z) = B_z e/z + B_z(h/z - \lambda)$.

Hence, $((I_\lambda)_z)_0 = (B_z)_0 e/z + (B_z)_0(h/z - \lambda) \cong (B_z/(z-1)B_z)e/z + (B_z/(z-1)B_z)(h/z - \lambda)$.

As above, $e+(z-1)B_z = e/z+(z-1)B_z$ and $h-\lambda z+(z-1)B_z = h-\lambda+(z-1)B_z$.

It follows $((I_\lambda)_z)_0 = (B_z/(z-1)B_z)e + (B_z/(z-1)B_z)(h-\lambda) \cong Ue + U(h-\lambda)$.

Using these isomorphisms, we obtain an exact sequence:

$$0 \to Ue + U(h-\lambda) \to U \to ((B/I_\lambda)_z)_0 \to 0.$$

Therefore: $(V(\lambda)_z)_0$ is the usual Verma U-module, which we will denote by $M(\lambda)$.

It is well known [**12**], which Verma U-modules $M(\lambda)$ are irreducible.

Proposition 5. *A Verma U-module $M(\lambda)$ is irreducible if and only if $\lambda \notin \mathbb{N}\cup\{0\}$. If $n \in \mathbb{N}\cup\{0\}$, then the Verma U-module $M(n)$ is indecomposable. Furthermore, the module $M(-n-2)$ is the unique simple submodule of $M(n)$, and $M(n)/M(-n-2) = V^{(n+1)}$ is the unique finite dimensional simple module of dimension $n+1$.*

As a consequence of this proposition and of the exact equivalences:

$res_U : Gr_{B_z} = Gr_{U[z,z^{-1}]} \to Mod_U$ and $U[z,z^{-1}]\otimes_U - : Mod_U \to Gr_{B_z}$ it follows:

Proposition 6. *The localisation of the homogenised Verma B-module $V(\lambda)_z$ is irreducible if and only if $\lambda \notin \mathbb{N}\cup\{0\}$.*

From this proposition we obtain the following properties of homogenised Verma modules.

Proposition 7. (i) *Given a non-zero submodule X of the homogenised Verma B-module $V(\lambda)$ with $\lambda \notin \mathbb{N}\cup\{0\}$, module $V(\lambda)/X$ is z-torsion.*

(ii) *Module $V(\lambda)$ is indecomposable for any $\lambda \in C$.*

Proof. (i) is a consequence of the previous proposition, and (ii) follows from the fact that $V(\lambda)$ has an indecomposable graded projective cover. □

Since $((I_\lambda)_z)_0 \cong (B_z)_0 1/ze + (B_z)_0 1/z(h-\lambda z)$, then $B_z \otimes_{(B_z)_0} ((I_\lambda)_z)_0 \cong B_z \otimes_{(B_z)_0} ((B_z)_0 1/ze + (B_z)_0 1/z(h-\lambda z)) \cong B_z 1/ze + B_z 1/z(h-\lambda z) \cong B_z e + B_z(h-\lambda z) = I_\lambda$.

Applying the functor $B_z \otimes_{(B_z)_0}$–to the exact sequence

$$0 \to ((I_\lambda)_z)_0 \to (B_z)_0 \to ((B/I_\lambda)_z)_0 \to 0$$

we obtain an exact sequence

$$0 \to (I_\lambda)_z \to B_z \to V(\lambda)_z \to 0.$$

Lemma 2. (i) *For $\lambda \in \Bbbk$, let v_k be the following elements of homogenised Verma module $V(\lambda) : v_0 = 1 + Be + B(h-\lambda z)$ and $v_k = f^k v_0$. Then, $ev_k = k(\lambda-(k-1))z^2 v_{k-1}$ and $hv_k = (\lambda-2k)zv_k$.*

(ii) *Let n be a positive integer. Then there is a monomorphism: $0 \to V(-n-2) \to V(n)$.*

Proof. (i) Using that ef = fe + hz and hf = fh – 2fz, it follows by induction that $ef^k = f^k e + kf^{k-1}hz - k(k-1)f^{k-1}z^2$.

From the equality hf = fh – 2fz, it follows $hf^k = fh^k - 2kf^k z$.

By the above observations, it follows that for $\lambda \in \Bbbk$, the elements v_k of $V(\lambda)$: $v_0 = 1 + Be + B(h-\lambda z)$ and $v_k = f^k v_0$ satisfy the equalities:

$ev_k = ef^k v_0 = (f^k e + kf^{k-1}z(h-\lambda z) - k((k-1)-\lambda)f^{k-1}z^2)v_0 = k(\lambda-(k-1))f^{k-1}z^2)v_0$.

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We also have: $hf^k=f^k(h-\lambda z)+(\lambda-2k)f^kz$, hence $hv_k=(\lambda-2k)zv_k$.

(ii) If $\lambda=$ n, then $ev_{n+1}=(n+1)(n-(n+1-1))z^2v_n=0$ and $hv_{n+1}=(n-2(n+1))zv_{n+1}=(-n-2)zv_{n+1}$ or $(h-(-n-2)z)v_{n+1}=0$.

The map $\overline{\varphi}{:}B\to V(n)$, given by $\overline{\varphi}(1)=v_{n+1}$, contains $Be+B(h-(-n-2)z)$ in the kernel, hence it induces a non-zero homomorphism of left B-modules $\varphi{:}V(-n-2)\to V(n)$. By restriction, we obtain a map $(\varphi_z)_0{:}(V(-n-2)_z)_0\to(V(n)_z)_0$. This map is non-zero and can be identified with a monomorphism of left U-modules $0\to M(-n-2)\to M(n)$.

It follows that $\mathrm{Ker}\varphi$ is of z-torsion and $V(-n-2)$ z-torsion free implies $\mathrm{Ker}\varphi=0$. □

As a consequence of Proposition 5, we have the following.

Proposition 8. *For $n\in\mathbb{N}\cup\{0\}$ the homogenised Verma module $V(n)$ induces exact sequences:*

$0\to V(-n-2)\to V(n)\to V(n)/V(-n-2)\to 0$ and $0\to V(-n-2)_z\to V(n)_z\to(V(n)/V(-n-2))_z\to 0$ with $V(-n-2)_z$ the unique simple submodule of $V(n)_z$ and $(V(n)/V(-n-2))_z$ has in degree zero the simple U-module of dimension $n+1$, $V^{(n+1)}$.

In the next theorem, we prove that the homogenised Verma modules are Koszul of projective dimension two.

Theorem 5. *Let $V(\lambda)$ be a homogenised Verma B-module. Then $V(\lambda)$ has a minimal projective resolution: $0\to B[-2]\xrightarrow{d_2}B\oplus B[-1]\xrightarrow{d_1}B\to V(\lambda)\to 0$ with $d_1(a,b)=ae+b(h-\lambda z)$ and $d_2(b)=b((\lambda+2)z-h,e)$. In particular, $V(\lambda)$ is a Koszul B-module.*

Proof. We look to the map d_1 in degree zero: $(d_1)_0(a,b)=ae+b(h-\lambda z)=0$ with $a,b\in\mathbb{C}$ implies $a=0$ and $b=0$, hence d_1 is a monomorphism in degree zero.

Let $a,b\in B$ be homogeneous elements with degree$(a)=$ degree$(b)=m>0$.

$d_1(a,b)=ae+b(h-\lambda z)=0$ implies $ae=b(\lambda z-h)$.

The elements a,b can be written as:

$$a=\sum_{i+j+k+\ell=m}a_{i,j,k,\ell}f^ih^je^kz^\ell,\ b=\sum_{i+j+k+\ell=m}b_{i,j,k,\ell}f^ih^je^kz^\ell.$$

Then,

$$ae=\sum_{i+j+k+\ell=m}a_{i,j,k,\ell}f^ih^je^{k+1}z^\ell=\sum_{i+j+k+\ell=m}b_{i,j,k,\ell}f^ih^je^kz^\ell(\lambda z-h).$$

By induction we prove, $e^k(\lambda z-h)=(2k+\lambda)e^kz-he^k$.

Then,

$$\begin{aligned}b(\lambda z-h)&=\sum_{i+j+k+\ell=m}b_{i,j,k,\ell}f^ih^j((2k+\lambda)e^kz-he^k)z^\ell\\&=\sum_{i+j+k+\ell=m}(2k+\lambda)b_{i,j,k,\ell}f^ih^je^kz^{\ell+1}-\sum_{i+j+k+\ell=m}b_{i,j,k,\ell}f^ih^{j+1}e^kz^\ell.\end{aligned}$$

Comparing both sides of equality $ae=b(\lambda z-h)$, we obtain $b_{i,j,0,\ell}=0$ for i,j,ℓ with $i+j+\ell=m$.

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We can cancel e on both sides of the equation to get:

$$a = \sum_{i+j+k+\ell=m} a_{i,j,k,\ell} f^i h^j e^k z^\ell = \sum_{i+j+k+\ell=m} (2k+\lambda) b_{i,j,k,\ell} f^i h^j e^{k-1} z^{\ell+1} - \sum_{i+j+k+\ell=m} b_{i,j,k,\ell} f^i h^{j+1} e^{k-1} z^\ell.$$

We prove $he^{k-1} = e^{k-1}h + 2(k-1)e^{k-1}z$.

Substituting in the above equality we have:

$$a = \sum_{i+j+k+\ell=m} (2k+\lambda)\, b_{i,j,k,\ell} f^i h^j e^{k-1} z^{\ell+1} - \sum_{i+j+k+\ell=m} b_{i,j,k,\ell} f^i h^j e^{k-1} z^\ell h$$
$$- \sum_{i+j+k+\ell=m} 2(k-1)\, b_{i,j,k,\ell} f^i h^j e^{k-1} z^{\ell+1} = z \left(\sum_{i+j+k+\ell=m} (2+\lambda)\, b_{i,j,k,\ell} f^i h^j e^{k-1} z^\ell \right)$$
$$- \left(\sum_{i+j+k+\ell=m} b_{i,j,k,\ell} f^i h^j e^{k-1} z^\ell \right) h.$$

Let b' be equal to

$$\sum_{i+j+k+\ell=m} b_{i,j,k,\ell} f^i h^j e^{k-1} z^\ell = \sum_{i+j+k+\ell=m-1} b_{i,j,k,\ell} f^i h^j e^k z^\ell.$$

With this notation, we have proved $a = b'(2+\lambda)z - b'h = b'((\lambda+2)z - h)$ and $b'e = b$, this is $\ker(d_1)_m = b'((\lambda+2)z - h, e)$.

Define $d_2(b') = (b'((\lambda+2)z - h), b'e)$.

Then the sequence: $0 \to B[-2] \xrightarrow{d_2} B \oplus B[-1] \xrightarrow{d_1} B \to V(\lambda) \to 0$ is exact. □

COROLLARY 4. *Let $M(\lambda)$ be a Verma module over the enveloping algebra U of $sl(2,\mathbb{C})$. Then $M(\lambda)$ has projective dimension of either one or two.*

Proof. Localising the exact sequence $0 \to B[-2] \xrightarrow{d_2} B \oplus B[-1] \xrightarrow{d_1} B \xrightarrow{\pi} V(\lambda) \to 0$ we obtain a projective resolution of $V(\lambda)_z$:

$0 \to B_z[-2] \xrightarrow{d_{2z}} B_z \oplus B_z[-1] \xrightarrow{d_{1z}} B_z \xrightarrow{\pi z} V(\lambda)_z \to 0$. Hence, $V(\lambda)_z$ has projective dimension less or equal to two. If $V(\lambda)_z$ is projective , then π_z is an isomorphism and $I_\lambda = \mathrm{Ker}\pi$ is of Z-torsion, a contradiction.

It follows that $V(\lambda)_z$ has a projective dimension of either one or two, and by Corollary 1, $(V(\lambda)_z)_0 = M(\lambda)$ has a projective dimension of either one or two □

Since $V(\lambda)$ is a Koszul module, we can apply Koszul duality to obtain a Koszul $B^!$-module $W(\lambda) = \oplus_{k=0}^{2} Ext_B^k(V(\lambda), B_0)$. $W(\lambda) = W(\lambda)_0 \oplus W(\lambda)_1 \oplus W(\lambda)_2$, where $W(\lambda)_0 = Hom_{\mathbb{C}}(B/JB, \mathbb{C})$, $W(\lambda)_1 = (Hom_{\mathbb{C}}(B/JB \oplus B/JB, \mathbb{C})) = Hom_{\mathbb{C}}(\mathbb{C}e \oplus \mathbb{C}(h - \lambda z), \mathbb{C})$ $W(\lambda)_2 = Hom_{\mathbb{C}}(B/JB, \mathbb{C})$, with $B/JB \cong \mathbb{C}$.

We want to find the structure of $W(\lambda)$ as $B^!$-module.

In $B^!$ we identify the arrows e, f, h, z with the dual basis $\delta_e, \delta_f, \delta_h, \delta_z$. We start computing the product $e.1$ in $W(\lambda)_0$.

We identify e with the extension $\hat{e} : 0 \to \mathbb{C} \to E \to \mathbb{C} \to 0$.

We have the following commutative diagram:

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$$\begin{array}{cccccccc}
0\to & Be+B(h-\lambda z) & \to & B & \to & V(\lambda) & \to 0 \\
 & j\downarrow & & 1\downarrow & & \downarrow & \\
0\to & JB & \to & B & \to & \mathbb{C} & \to 0 \\
 & p\downarrow & & & & & \\
 & \mathbb{C}e\oplus\mathbb{C}f\oplus\mathbb{C}h\oplus\mathbb{C}z & & \downarrow & & 1\downarrow & \\
 & \delta_e\downarrow & & & & & \\
0\to & \mathbb{C} & \longrightarrow & E & \to & \mathbb{C} & \to 0
\end{array}$$

We have a factorization:

$$\begin{tikzcd}
Be+B(h-\lambda z) \arrow[r, "j"] \arrow[dr] & JB \arrow[r, "p"] & \mathbb{C}e\oplus\mathbb{C}f\oplus\mathbb{C}h\oplus\mathbb{C}z \arrow[dr, "\delta_e"] & \\
 & \mathbb{C}e\oplus\mathbb{C}(h-\lambda z) \arrow[ur, "i"] & & \mathbb{C}
\end{tikzcd}.$$

We identify the product $\hat{e}1$ with the map $\delta_e i : \mathbb{C}e\oplus\mathbb{C}(h-\lambda z)\to\mathbb{C}$, where i is the natural inclusion.

The vector space $\mathbb{C}e\oplus\mathbb{C}(h-\lambda z)$ has dual basis $\delta_e, \delta_{h-\lambda z}$ and we write $\delta_e i = a\delta_e + b\delta_{h-\lambda z}$, $a, b \in \mathbb{C}$.

$\delta_e(h-\lambda z) = a\delta_e(h-\lambda z) + b\delta_{h-\lambda z}(h-\lambda z) = b = 0$ and $\delta_e(e) = a\delta_e(e) + b\delta_{h-\lambda z}(e) = 1 = a$.

We have proved $\hat{e}1 = \delta_e$.

Similarly, $\hat{f}1 = \delta_f i = a\delta_e + b\delta_{h-\lambda z}$. It follows, $\delta_f(e) = 0 = a$, $\delta_f(h-\lambda z) = 0 = b$.

Therefore: $\hat{f}1 = 0$.

We compute $\hat{h}1 = \delta_h = a\delta_e + b\delta_{h-\lambda z}$ and $\delta_h(e) = a\delta_e(e) = a = 0$, $\delta_h(h-\lambda z) = 1 = b\delta_{h-\lambda z}(h-\lambda z) = b$.

It follows, $\hat{h}1 = \delta_{h-\lambda z}$.

It only remains to compute $\hat{z}1$. As above, $\hat{z}1 = \delta_z = a\delta_e + b\delta_{h-\lambda z}$ and $\delta_z(e) = a\delta_e(e) + b\delta_{h-\lambda z}(e) = a = 0$ and $\delta_z(h-\lambda z)) = b\delta_{h-\lambda z}(h-\lambda z) = b = -\lambda$.

$\hat{z}1 = -\lambda\delta_{h-\lambda z}$.

It follows, $(z+\lambda h)1 = -\lambda\delta_{h-\lambda z} + \lambda\delta_{h-\lambda z} = 0$.

Therefore, we have the following exact commutative triangle:

$$\begin{tikzcd}
B^! \arrow[rr, "\pi"] \arrow[dr, "q"'] & & W(\lambda) \arrow[r] & 0 \\
 & B^!/B^!f + B^!(z+\lambda h) \arrow[ur, "\varphi"'] & &
\end{tikzcd},$$

the map $\varphi : B^!/B^!f + B^!(z+\lambda h)\to W(\lambda)$ is an epimorphism. Using the dimension argument, we will prove it is an isomorphism.

Since $B^!_0 = \mathbb{C}$, $(B^!/B^!f + B^!(z+\lambda h))_0 = \mathbb{C}$.

The vector space $B^!_1$ has basis $e.\ f, h, z$ hence, $e, f, h, (z+\lambda h)$. It follows that $(B^!/B^!f + B^!(z+\lambda h))_1$ has basis $\{\overline{e}, \overline{h}\}$.

The vector space $B^!_2$ $\{fe, he, fh, fz, hz, ez\}$,where $fe = -fe$ and $fh = -hf$, hence $fe, fh \in (B^!f + B^!(z+\lambda h))_2$.

We have $h(z+\lambda h) = hz + \lambda h^2 = hz$ implies $hz \in (B^!f + B^!(z+\lambda h))_2$.

From the equality $fz + zf = 2hf$, it follows that $fz = (2h - z)f \in (B^!f + B^!(z+\lambda h))_2$.

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Since $e(z+\lambda h) \in (B^! f + B^!(z+\lambda h))_2$, it follows that $\overline{ez} = -\lambda\overline{ze}$ in $B^!/B^! f + B^!(z+\lambda h)$.

We have proved $(B^!/B^! f + B^!(z+\lambda h))_2$ is generated by $\overline{he}$.

Now $B_3^!$ is generated by $\{fhe, fez, fhz, hez\}$. It is easy to see that $fhe, fez, fhz \in B^! f$ and $hez = -ezh$, $-ez(z+\lambda h) = -ez^2 - \lambda ezh = -\lambda ezh$ implies $\overline{hez} = 0$ in $B^!/B^! f + B^!(z+\lambda h)$.

We have proved $(B^!/B^! f + B^!(z+\lambda h))_3 = 0$.

By dimensions, φ is an isomorphism.

We proved the following.

THEOREM 6. *For any $\lambda \in \mathbb{C}$ let $W(\lambda)$ be the $B^!$-module corresponding to the homogenised B-module $V(\lambda)$ under Koszul duality. Then there is an isomorphism $W(\lambda) \cong B^!/B^! f + B^!(z+\lambda h)$.*

We now have a family $\{W(\lambda)\}_{\lambda\in\mathbb{C}}$ of non-isomorphic indecomposable Koszul modules. We know that the graded stable Auslander–Reiten components are of type ZA_∞ [**11**, **15**]. We will prove below that each $W(\lambda)$ is at the mouth of a regular component, but first we need to prove the following.

PROPOSITION 9. *The $B^!$-module $rB^!/socB^!$ is indecomposable.*

Proof. We proved above $B^! = \mathbb{C} < e,f,h,z > /I^\perp$ and $I^\perp = < e^2, f^2, h^2, z^2, (e,f), (h,e), (h,f), (h,z) + ef, (e,z) - 2eh, /f, z) - 2hf >$, the algebra $B^!$ is graded $B^! = L_0 \oplus L_1 \oplus L_2 \oplus L_3 \oplus L_4$, with $L_0 = \mathbb{C}1$, $L_1 = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}z$, $L_2 = \mathbb{C}\ fe \oplus \mathbb{C}\ he \oplus \mathbb{C}fh \oplus \mathbb{C}fz \oplus \mathbb{C}hz \oplus \mathbb{C}ez$, $L_3 = \mathbb{C}fhef \oplus \mathbb{C}ez \oplus \mathbb{C}fhz \oplus \mathbb{C}hez$ and $L_4 = \mathbb{C}fhz$.

To prove $rB^!/socB^! = L_1 \oplus L_2 \oplus L_3$ is indecomposable, we prove that the graded endomorphism ring $Hom_{GrB^!}(rB^!/socB^!, rB^!/socB^!)$ has dimension one as $\mathbb{C}$-vector space.

Let $E : rB^!/socB^! \to rB^!/socB^!$ be a graded B^1-homomorphism.

$E(e) = a_{11}e + a_{12}f + a_{13}h + a_{14}z$,

$E(f) = a_{21}e + a_{22}f + a_{23}h + a_{24}z$,

$E(h) = a_{31}e + a_{32}f + a_{33}h + a_{34}z$,

$E(z) = a_{41}e + a_{42}f + a_{43}h + a_{44}z$.

Then $E(e^2) = 0 = eE(e) = a_{12}ef + a_{13}eh + a_{14}ez = -a_{12}fe - a_{13}he + a_{14}z$.

Hence, $a_{12} = a_{13} = a_{14} = 0$.

Similarly, $E(f^2) = 0 = fE(f) = a_{21}fe + a_{23}fh + a_{24}fz$ implies $0 = a_{21} = a_{23} = a_{24}$.

$E(h^2) = 0 = hE(h) = a_{31}he + a_{32}hf + a_{34}hz = a_{31}he - a_{32}fh + a_{34}hz$ implies that $a_{31} = a_{32} = a_{34} = 0$.

$E(z^2) = 0 = zE(z) = a_{41}ze + a_{42}zf + a_{43}zh = -a_{41}ez - a_{42}fz - a_{43}hz + 2a_{41}eh + 2a_{42}hf - a_{43}ef = -a_{41}ez - a_{42}fz - a_{43}hz - 2a_{41}he - 2a_{42}fh + a_{43}fe$.

Implies $a_{41} = a_{42} = a_{43} = 0$.

On the other hand, we have: $E(ef) = -E(fe) = -a_{11}fe$ and $E(ef) = eE(f) = -a_{22}fe$, hence $a_{11} = a_{22}$.

$E(hf) = a_{22}hf$, $E(fh) = a_{33}fh$, $E(hf) = -E(fh)$ implies that $a_{22} = a_{33}$.

$E(zh) = zE(h) = a_{33}zh$ and $E(hz) = hE(z) = a_{44}hz$,

$E(zh) = -E(hz) - E(ef) = -a_{44}hz - a_{22}ef = a_{33}zh = a_{33}(-hz - ef)$.

Implies $a_{33} = a_{44}$.

We have proved $a_{11} = a_{22} = a_{33} = a_{44} = a$.

We have now $E(fhe) = fhE(e) = afhe$.

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$E(fez) = feE(z) = afez$, $E(fhz) = fhf(z) = afhz$, $E(ehz) = ehE(z) = aehz$. Therefore: $E = a1$. □

By [**2**] there is an almost split sequence: $0 \to rB^! \to B^! \oplus rB^!/socB^! \to B^!/socB^! \to 0$ and $rB^!$ is at the mouth of a stable component of type ZA_∞.

THEOREM 7. *For each $\lambda \in \mathbb{C}$ the $B^!$-module $W(\lambda)$ is at the mouth of a regular component.*

Proof. We first prove that there are no modules of type $W(\lambda)$ in the pre-projective component.

The component of the only indecomposable projective is of the form:

$$\begin{array}{cccccccccccccccc}
 & & \circ & & & & rP & \to & P & \to & P/S & & & & \circ & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
\circ & & & & \circ & & & & rP/S & & & & \circ & & \mathbf{D} & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\mathbf{E} & & \circ & & \mathbf{A} & & \circ & & & & \circ & & \mathbf{B} & & \circ & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
\circ & & & & \circ & & & & \circ & & & & \circ & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
 & & \circ & & & & \circ & & & & \circ & & & & \circ & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \mathbf{C} & \nearrow & & \searrow & & \nearrow & & \searrow \\
\circ & & & & \circ & & & & \circ & & & & \circ & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
 & & \circ & & & & \circ & & & & \circ & & & & \circ &
\end{array}$$

We distinguish five regions in this picture.

Region **A** consists of all the modules on the two going up diagonals ending either in rP or in rP/S and the irreducible maps corresponding to the arrows. Along these diagonals the arrows represent epimorphisms and all the maps parallel to $rP \to P/S$ are also epimorphisms.

Region **B** consists of all modules on the going down diagonals containing either P/S or rP/S. All the irreducible maps on these diagonals are monomorphism and all maps parallel to $rP/S \to P/S$ are monomorphisms.

Region **C** consists of all modules below the going down the diagonal containing rP and below the going up diagonal containing P/S. In this region the irreducible maps on the going up diagonals are epimorphisms and the irreducible maps on the going down diagonals are monomorphisms.

Region **D** consists of all modules above the going down the diagonal containing P/S, and Region **E** consists of all the modules above the going up diagonal containing rP.

In both regions **D** and **E**, all the irreducible maps on the modules along the going up diagonals are epimorphism and all the irreducible maps of modules along the going down diagonals are monomorphisms.

We will see that a module of the form $W(\lambda)$ cannot appear in any of these five regions.

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The module $W(\lambda)$ is of the form:

$$\begin{tikzcd} & \mathbb{C} & \\ \mathbb{C} \arrow[ur, "1"] & & \mathbb{C} \arrow[ul, "1"'] \\ & \mathbb{C} \arrow[ul, "1"] \arrow[ur, "1"'] & \end{tikzcd}$$

and it has as submodules:

$$L = \begin{tikzcd} & 0 & \\ \mathbb{C} \arrow[ur] & & \mathbb{C} \arrow[ul] \\ & \mathbb{C} \arrow[ul, "1"] \arrow[ur, "1"'] & \end{tikzcd}, \quad L_1 = \begin{tikzcd} & 0 & \\ \mathbb{C} \arrow[ur] & & 0 \arrow[ul] \\ & \mathbb{C} \arrow[ul, "1"] \arrow[ur] & \end{tikzcd}, \quad L_2 = \begin{tikzcd} & 0 & \\ 0 \arrow[ur] & & \mathbb{C} \arrow[ul] \\ & \mathbb{C} \arrow[ul] \arrow[ur, "1"'] & \end{tikzcd}, \text{ and } L_3 = \begin{tikzcd} & 0 & \\ 0 \arrow[ur] & & 0 \arrow[ul] \\ & \mathbb{C} \arrow[ul] \arrow[ur] & \end{tikzcd}.$$

We have composition series: $W(\lambda) \supset L \supset L_1 \supset L_3$ and $W(\lambda) \supset L \supset L_2 \supset L_3$.

If $W(\lambda)$ appears in region **D**, then it appears at most at three steps of the mouth, hence $W(\lambda)$ is on a going down diagonal. If it is not at the mouth, then either one of L_3, L_1, L_2 or L is at the mouth, but none of the exact sequences: $0 \to L_3 \to L_2 \to S \to 0$, $0 \to L_3 \to L_1 \to S \to 0$, $0 \to L \to W(\lambda) \to S \to 0$, with S simple is almost split. The reason is that, according to [**2**], for an almost split sequence $0 \to \tau(A) \to E \to A \to 0$, $socA \cong top\Omega(A)$ and all modules L_3, L_2, L_1, L have simple socle but $top\Omega S = rB^!/r^2B^! = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}z$.

Then if $W(\lambda)$ appears in region **D**, it appears at the mouth.

By the same kind of argument if $W(\lambda)$ appears in region **E**, it appears at the mouth.

But we will see that it is impossible for $W(\lambda)$ to appear at the mouth of a pre-projective component.

If it appears in region **D**, then for some integer k, $\Omega^{2k}\mathcal{N}^k\ (W(\lambda))$ is isomorphic to P/S, where $\mathcal{N}$ denotes the Nakayama equivalence, but this is a contradiction because both Nakayama equivalence and syzygy preserve Koszulity and P/S are not Koszul.

If $W(\lambda)$ appears in region **E**, then for some integer k there is an isomorphism $\Omega^{2k}\mathcal{N}^k\ (rP) \cong W(\lambda)$, but this is also impossible since under the Koszul duality $\Omega^{2k}\mathcal{N}^k\ (rP) = F(J^{2k}B)$ and $\dim_{\mathbb{C}} \Omega^{2k}\mathcal{N}^k\ (rP)/r\Omega^{2k}\mathcal{N}^k\ (rP) = \dim_{\mathbb{C}}(J^{2k}/J^{2k+1}) > 1$.

If $W(\lambda)$ appears in region **B**, then it would contain either rP/S or $P/\ S$, which is impossible by dimension.

The module cannot appear either in region **A** because in that case either rP or rP/S would be a quotient of $W(\lambda)$.

Now if $W(\lambda)$ appears in region **C**, then a submodule L' of $W(\lambda)$ appears in the going up diagonal containing $rP/S \to P/S$, and rP/S would be a quotient of L', which is impossible by dimensions.

Observe that in a regular component there is only region **D** and by the above arguments $W(\lambda)$ has to appear at the mouth. $\square$

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