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ELEMENTARY QUOTIENTS OF ABELIAN GROUPS, AND SINGULAR HOMOLOGY ON MANIFOLDS

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§ 0. **Introduction. The topological origin of the problem.** Let there be given a compact topological manifold M_n. If M_n admits a "triangulation" it is known that the fundamental invariants, namely the connectivities of M_n over fields, the Betti numbers and torsion coefficients over $\boldsymbol{Z}$ of the singular homology groups of M_n, are finite and calculable. However it is not known that a "triangulation" of M_n always exists when $n > 3$.

Singular homology groups are understood in the sense of Eilenberg [1].

An alternative to triangulation of M_n. When M_n is differentiable, of at least class C^2, the alternative to the hypothesis of triangulation will be understood to be the existence of a differentiable non-degenerate (ND) function[1)] f on M_n. When M_n is not known to be so differentiable the alternative will be understood to be the existence in the sense of [2] of a topologically non-degenerate (TND) function[1)] f on M_n.

We shall be concerned with the subsets

$$f_c = \{p \in M_n \mid f(p) \leq c\} \tag{0. 0}$$

of M_n where c is an arbitrary value of f on M_n and shall term f_c a *sublevel set* of M_n.

In a series of papers which make no use of a triangulation of M_n we shall show that the fundamental invariants of the singular homology groups of the sublevel sets f_c of M_n are uniquely determined by suitable relative numerical invariants associated with the respective critical points on f_c of

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1) The function f can and will be chosen so as to have different values a at different critical points of f.

a *ND* f. We shall restrict ourselves initially, for simplicity, to the differentiable case.

That the connectivities over fields of the sublevel sets f_c are so determined has already been established in [3] in the differentiable case. In this case the invariants associated with a critical point are its index and its type as "linking" or "non-linking". See §29 of [3].

The next step is to show how the Betti numbers and torsion coefficients of the sublevel sets f_c are determined by suitable relative invariants associated with the critical points. This will be done in [5], beginning with the differentiable case. An abstract of the results in [5] appears as Appendix III of [3].

The *objective* of the present paper is to solve two problems in abelian group theory, essential in [5]. These group-theoretic problems will be defined at the end of § 2.

The *existence* of *ND* functions in the differentiable case has been known and used in singular homology theory since 1927. Cf. § 6 of [3]. *TND* functions were introduced in [2]. Let C be the class of manifolds M_n on which *TND* functions exist. The class C includes all "combinatorial" manifolds whether differentiable or not. See [7]. If C^* is the class of all manifolds M_n which admit combinatorial triangulations, it is clear that $C^* \subset C$. That this inclusion is *proper* is believed to be true but has not yet been proved. Cf. [4].

However, the principal reason for the introduction of the new methods was more their directness and *relevance* than the possibility that they may apply to more general manifolds.

The critical points of *ND* functions on manifolds, and critical extremals in variational problems *must be related* to the homology groups of the underlying spaces. It is clearly preferable, when possible, to establish these relations directly without the assumption of triangulability.

It is our ultimate aim to show how the differential or topological invariants, which we associate with the critical points of a *ND* or *TND* function f, determine the singular homology groups, up to an isomorphism, of all sublevel sets f_c on M_n.

The restriction in this introduction to compact manifolds M_n is merely for simplicity.

Introductory remarks on abelian groups A. The *generation* of subgroups of A. For i on the range $1, \cdots, r$ let X_i be a subset, subgroup, or element of A. We shall denote by

$$\{X_1, X_2, \cdots, X_r\} \tag{0.1}$$

the subgroup of A which is the intersection of those subgroups of A which include or contain each X_i. If a_i is an element of A the group

$$G = \{a_1, \cdots, a_r\} \tag{0.2}$$

is said to be *finitely generated.* By virtue of this notation, if a is an element of A, $\{a\}$ denotes the cyclic subgroup generated by a.

DEFINITION 0. 1. *Free subgroups.* A subset $(u_1, \cdots, u_r)$ of elements $u_i \in A$ is termed *free*, provided a relation $n_1u_1 + \cdots + n_ru_r = 0$ in which n_i is an integer is valid if and only if $n_1 = n_2 = \cdots = n_r = 0$. If $(u_1, \cdots, u_r)$ is free the subgroup $G = \{u_1, \cdots, u_r\}$ is termed *free* and $(u_1, \cdots, u_r)$ a *base* for G. It will be convenient to term the *trivial* group free, and say that it has an *empty* base.

DEFINITION 0. 2. *The torsion subgroup of A.* The subset of elements of A of finite order form a group, termed the *torsion* subgroup of A.

DEFINITION 0. 3. *Direct sums.* An ensemble $A_1, \cdots, A_n$ of subgroups of A such that $A = \{A_1, A_2, \cdots, A_n\}$ will be said to have the *direct sum*

$$A = A_1 \oplus \cdots \oplus A_n, \tag{0.3}$$

provided a relation $x_1 + x_2 + \cdots + x_n = 0$, $x_i \in A_i$, is valid only if each $x_i = 0$. We admit null summands A_i.

If $(u_1, \cdots, u_r)$ is a base for a free subgroup $\mathscr{B}$ of A, then

$$\mathscr{B} = \{u_1\} \oplus \cdots \oplus \{u_r\}. \tag{0.4}$$

The order of each cyclic summand is infinite. We admit the case in which $r = 0$ and $\mathscr{B} = 0$.

THEOREM 0. 1. *A finitely generated abelian group A is a direct sum $A = \mathscr{B} \oplus \mathscr{T}$ of the torsion subgroup $\mathscr{T}$ of A, and a free subgroup $\mathscr{B}$ of A termed complementary to $\mathscr{T}$ in A* [8], p. 151.

A group complementary to the torsion subgroup $\mathscr{T}$ of A is uniquely determined if and only if $\mathscr{T} = 0$ or $\mathscr{T} = A$. The most general group complementary to $\mathscr{T}$ has the form

$$\{v_1 + t_1\} \oplus \cdots \oplus \{v_r + t_r\} \tag{0.5}$$

where $(v_1, \cdots, v_r)$ is an arbitrary base of $\mathscr{B}$ and $t_1, \cdots, t_r$ arbitrary elements in $\mathscr{T}$.

DEFINITION 0.4. *Betti subgroups of A.* When A is[2] FG we shall call a subgroup $\mathscr{B}$ complementary to the torsion subgroup $\mathscr{T}$ of A a Betti subgroup of A, and its dimension the *Betti number* of A.

We shall be concerned with abelian groups A which are FG.

§1. Sylow groups, elementary divisors, and torsion coefficients. We shall recall certain terms and theorems associated with decomposing a finite abelian group $\mathscr{T}$ into a direct sum of cyclic subgroups.

DEFINITION 1.1. *Prime powers.* An integer $q>1$ admits a factoring

$$q = p_1^{e_1} \cdots p_r^{e_r}, \qquad (r>0) \tag{1.1}$$

unique except for the order of the factors, in which the p_i's are distinct positive primes and the e_i's positive integers. Such a factoring will be called a *reduced prime power factoring* of q. Hereafter primes are supposed positive.

DEFINITION 1.2. *p-Primary subgroups* of $\mathscr{T}$. A subgroup g of $\mathscr{T}$ whose order is a power of a prime p is called a *p-primary subgroup* of $\mathscr{T}$. A *maximal* p-primary subgroup of $\mathscr{T}$ is the union, for a given prime p, of all p-primary subgroups of $\mathscr{T}$. It is called a *Sylow* p-subgroup of $\mathscr{T}$ and will be denoted by $\mathscr{S}_p$.

Two classical theorems follow.

THEOREM 1.1. *A finite abelian group* $\mathscr{T}$, *with an order* $q>1$ *given by* (1.1), *is the direct sum*

$$\mathscr{T} = \mathscr{S}_{p_1} \oplus \cdots \oplus \mathscr{S}_{p_r} \tag{1.2}$$

of the nontrivial Sylow subgroups of $\mathscr{T}$ [9], p. 137.

THEOREM 1.2. *The "cyclic primary decomposition" theorem.*[3] *A finite nontrivial abelian group* $\mathscr{T}$ *is the direct sum*

$$\mathscr{T} = g_1 \oplus \cdots \oplus g_m, \tag{1.3}$$

[2] FG abbreviates finitely generated.

[3] CPD shall abbreviate "cyclic primary decomposition".

of nontrivial primary cyclic subgroups g_i *of* $\mathscr{T}$ *uniquely determined by* $\mathscr{T}$ *up to isomorphisms and order of writing of the summands.* [10], pp. 60, 65.

The numerical values of the orders of the g_i's are prime powers. These prime powers form a list

$$p_1^{e_1}, \cdots, p_m^{e_m} \qquad (e_i > 0) \tag{1.4}$$

of prime powers, not necessarily distinct. We shall say that the groups g_i of (1.3) and the prime powers (1.4) are *normally ordered* if

$$p_1 \geq p_2 \geq \cdots \geq p_m \tag{1.5}$$

and if when $p_i = p_{i+1}$, then $e_i \geq e_{i+1}$.

According to Theorem 1.2, *the prime powers* (1.4), *if normally ordered, are uniquely determined by* $\mathscr{T}$.

DEFINITION 1.3. *Elementary divisors of* $\mathscr{T}$. The prime powers in (1.4) are called the *elementary divisors* of $\mathscr{T}$, and are regarded as algebraically distinct if they have distinct indices i in (1.4). By the *multiplicity* of an elementary divisor $p_i^{e_i}$ of $\mathscr{T}$ is meant the number of algebraically distinct elementary divisors of $\mathscr{T}$ with the same numerical value as $p_i^{e_i}$. Theorem 1.2 implies that the elementary divisors of $\mathscr{T}$, together with their multiplicities, are uniquely determined by $\mathscr{T}$.

DEFINITION 1.4. *Torsion coefficients of* $\mathscr{T}$. It is a classical theorem that a finite non-trivial trivial abelian group $\mathscr{T}$ is a direct sum of a finite set of cyclic subgroups of $\mathscr{T}$ which have orders

$$q_1, q_2, \cdots, q_\rho \tag{1.6}$$

exceeding 1, each of which, except q_ρ, is divisible by its successor. The integers in the sequence (1.6) are uniquely determined by $\mathscr{T}$. They are termed *torsion coefficients* of $\mathscr{T}$ and are said to be *canonically ordered.*

The matrix $\Pi_{r\mu}$. Let r be the number of distinct primes $p_1, \cdots, p_r$ in (1.1). Let these primes be taken in decreasing order. Let α_i be the number[4] of ED's which are powers of p_i, and let μ be the maximum of the numbers α_i.

The martix $\Pi_{r\mu}$ shall have r rows and μ columns. The i-th row of $\Pi_{r\mu}$ shall consist of the ED's of $\mathscr{T}$ which are powers of p_i, arranged in

[4] *ED* abbreviates "elementary divisor".

monotonically decreasing order and followed by enough 1's to make a row of μ-integers. Each element a_{ij} of $\Pi_{r\mu}$ is thus an *ED* of $\mathscr{T}$ or 1.

The *ED*'s of $\mathscr{T}$ and the torsion coefficients of $\mathscr{T}$ uniquely determine each other in accord with the following theorem.

THEOREM 1.3. *The torsion coefficients of* $\mathscr{T}$ *are* μ *in number and if "canonically" arranged, are the products of the elements in the respective columns of* $\Pi_{r\mu}$. *Conversely, the element* a_{ij} *of* $\Pi_{r\mu}$ *is the maximal prime power of* p_i *which is a factor of the* j-th *torsion coefficient of* $\mathscr{T}$.

This theorem may inferred from [9], p. 147.

DEFINITION 1.5. *A "basis" for A.* Suppose that $\mathscr{T}$ has a *CPD*

$$\mathscr{T} = \{x_1\} \oplus \cdots \oplus \{x_\rho\} \qquad (\text{order } x_i > 1;\ i = 1, \cdots, \rho) \tag{1.7}$$

and that $\mathscr{B}$ is a Betti group of A with a "base" $(v_1, \cdots, v_\beta)$. The set of elements

$$v_1, \cdots, v_\beta;\ x_1, \cdots, x_\rho \tag{1.8}$$

of A is called a *basis* for A. If $\mathscr{B}$ is trivial there are no v_i's, and if $\mathscr{T}$ is trivial no x_j's.

A *basis* for A is to be distinguished from a *base* for $\mathscr{B}$ which is free.

A basis for a non-trivial abelian group A is unique only if A is a cyclic group of order 2.

If w is an arbitrary element in A then

$$w = \mu_1 v_1 + \cdots + \mu_\beta v_\beta + m_1 x_1 + \cdots + m_\rho x_\rho \tag{1.9}$$

where the coefficients μ_i and m_j are integers. If the integers m_j are restricted to integral values such that

$$0 \leq m_j < \text{order } x_j \tag{1.10}$$

then the coefficients m_j, as well as the coefficients μ_i, are uniquely determined by w and the choice of the basis (1.8).

DEFINITION 1.6. **Prime-simple abelian groups.** A power p^e of a prime is termed prime-simple if $e = 1$. An element $x \in A$ is termed *prime-simple* if (order x) $= \infty$, or if (order x) $< \infty$ and each prime power factor of (order x) is 1 or prime-simple. A subgroup of A (including A) is termed prime-simple if each of its elements is prime-simple.

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Subgroups of prime-simple groups are prime-simple. The abelian group A is prime-simple if and only if each Sylow subgroup of A is prime-simple. One readily establishes the following theorem.

THEOREM 1.4. *A necessary and sufficient condition that a finitely generated abelian group be prime-simple is that each elementary divisor of its torsion subgroup be prime-simple.*

By the *length* of a *CPD* of a finite abelian group we mean the number of nontrivial summands in the *CPD*.

The following theorem gives three fundamental properties of a nontrivial finite cyclic group.

THEOREM 1.5 (i). *The elementary divisors of a finite cyclic group G are the maximal prime powers in a reduced prime power factoring of the order of G.* [14], p. 23.

(ii) *Apart from the order of its summands there is but one CPD of a nontrivial finite cyclic group G and this CPD is a Sylow decomposition of G.*

(iii) *A finite nontrivial abelian group G is cyclic if and only if each of its Sylow subgroups has a CPD of unit length.*

§2. **The invariants of group quotients $A/\boldsymbol{R}$.** The group A is the above finitely generated abelian group A, and $\boldsymbol{R}$ a subgroup of A. The quotient $A/\boldsymbol{R}$ is *FG* as we shall presently see. By the invariants of $A/\boldsymbol{R}$ we here mean its Betti numbers, torsion coefficients and elementary divisors.

The problem of finding the invariants of $A/\boldsymbol{R}$ depends formally upon how A and $\boldsymbol{R}$ are given. We suppose that A is an *FG* abelian group *given* as a direct sum

$$A = \mathscr{B} \oplus \mathscr{T} \tag{2.1}$$

of the torsion subgroup $\mathscr{T}$ of A and a complementary free subgroup $\mathscr{B}$ of A. We suppose that a "base" $(u_1, \cdots, u_r)$ of $\mathscr{B}$ is given and generators $x_1, \cdots, x_\rho$ of the respective summands of a *CPD* of $\mathscr{T}$, together with the order n_i of each x_i.

In a form more general than any we shall use, there is given a subgroup $\boldsymbol{R} = \{z_1, \cdots, z_\alpha\}$ where each generator z_i is given as a sum

$$(\nu_1 u_1 + \cdots + \nu_r u_r) + (m_1 x_1 + \cdots + m_\rho x_\rho) \tag{2.2}$$

with integral coefficients. The general problem is to find the "invariants" of $A/\boldsymbol{R}$ in terms of these data and to do this by a finite well-defined algorithm.

That $A/\boldsymbol{R}$ is FG is a consequence of the fact that the $\boldsymbol{R}$-cosets containing the elements $u_1, \cdots, u_r; x_1, \cdots, x_\rho$ generate $A/\boldsymbol{R}$.

The quotients A/g whose invariants will be studied in this paper readily reduce to quotients which we term correlated and define as follows.

DEFINITION 2.1. **Correlated quotients.** In such quotients A is given as a direct sum $G_1 \oplus \cdots \oplus G_r$ of subgroups G_i and g is a direct sum $g_1 \oplus \cdots \oplus g_r$ of cyclic groups g_i, where g_i, is a subgroup, possibly trivial, of G_i. Under these conditions A/g is termed a *correlated quotient.*

It is a classically known and easily proved theorem that when A/g is the above correlated quotient then

$$A/g \approx G/g_1 \oplus \cdots \oplus G_r/g_r \qquad \text{([10], p. 57)} \tag{2.3}$$

where the right member of (2.3) is the "external direct sum" of the quotients G_i/g_i and the isomorphism is a "natural" isomorphism which we shall now characterize.

External direct sums. The direct sums which we have been considering up to this point have been direct sums of *subgroups* of a given group. The group quotients on the right of (2.3) are not so given. They can be considered, a priori, as abelian groups $A_1, \cdots, A_r$ with no elements in common.

If i is on the range $1, \cdots, r$, and y_i is an arbitrary element in A_i, one can form a new abelian group K called the *external direct sum* of the A_i's. The elements in K are by definition r-tuples $(y_1, \cdots, y_r)$. These r-tuples are summed by adding their respective components y_i. The inverse of $(y_1, \cdots, y_r)$ is $(-y_1, \cdots, -y_r)$. One can identify A_i with the subgroup of K of elements whose components are 0 except the i-th. The i-th component shall be an element in A_i.

An isomorphism (2.3), termed "natural", can be characterized as follows: Let x_i be an arbitrary element in G_i, and $\hat{x}_i = x_i + g_i$ the coset of g_i containing x_i. In the isomorphism (2.3) the coset $(x_1 + x_2 + \cdots + x_n) + g$ in A shall go into the r-tuple $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_r)$ in the external direct sum of the r quotients G_i/g_i.

Because of the isomorphism (2.3), the problem of finding the "invariants" of "correlated" quotients A/g reduces to the problem of finding the invariants of quotients of the type which we call *elementary* and define as follows.

DEFINITION 2.2 **Elementary quotients** A/W. In such quotients W is a cyclic subgroup $\{w\}$ of A.

We shall recall a special type of group quotient, and the mode of reduction of such quotients to quotients of "correlated" type.

Quotients $\boldsymbol{A}/\boldsymbol{R}$ *with* $\boldsymbol{A}$ *free.* In such quotients $\boldsymbol{A}$ is a free group given as a direct sum

$$\boldsymbol{A} = \{v_1\} \oplus \cdots \oplus \{v_\sigma\} \qquad (\sigma > 0) \tag{2.4}$$

of σ infinite cyclic groups. $\boldsymbol{R}$ is given as a subgroup of $\boldsymbol{A}$ of the form

$$\boldsymbol{R} = \{z_1, \cdots, z_r\} \qquad (r > 0) \tag{2.5}$$

with generators z_i which are integral linear combinations

$$z_i = a_{ij}v_j \qquad (i = 1, \cdots, r) \tag{2.6}$$

of the base elements $(v_1, \cdots, v_\sigma)$. The terms $a_{ij}v_j$ are summed over the range $1, 2, \cdots, \sigma$ for j.

The diagonal matrix $\|d_{ij}\|$. Let $\rho > 0$ be the rank of the matrix $\|a_{ij}\|$. It is well known that by a finite sequence of "elementary" operations on the rows and columns of $\|a_{ij}\|$, the matrix $\|a_{ij}\|$ can be reduced to an "equivalent" matrix $\|d_{ij}\|$ with the same number of rows and columns, and with all elements zero, except positive diagonal elements

$$d_1, d_2, \cdots, d_\rho \qquad (\rho \leq \sigma) \tag{2.7}$$

of which each d_i, except the last, divides its successor. The d_i's obtained in this way are termed *invariant factors* of $\|a_{ij}\|$ and are uniquely determined by $\|a_{ij}\|$. [12] p. 308.

The invariants D_j. The numbers d_i can be determined as follows. For $j = 1, \cdots, \rho$ let D_j be the absolute value of the *HCF* of the j-rowed subdeterminants of the matrix $\|a_{ij}\|$. It is known that

$$d_1 = D_1,\ d_2 = \frac{D_2}{D_1}, \cdots, d_\rho = \frac{D_\rho}{D_{\rho-1}}. \qquad \text{([13] p. 178)} \tag{2.8}$$

Torsion coefficients of $\boldsymbol{A}/\boldsymbol{R}$. Let $\boldsymbol{T}$ be the torsion subgroup of $\boldsymbol{A}/\boldsymbol{R}$. We shall show how to determine the torsion coefficients of $\boldsymbol{T}$.

The reduction by elementary operations of the matrix $\|a_{ij}\|$ to the matrix $\|d_{ij}\|$ implies the existence of a unimodular transformation of the base $(v_1, \cdots, v_\sigma)$ into a base $(v_1', \cdots, v_\sigma')$, and of a unimodular transformation of the set $(z_1, \cdots, z_r)$ into a new set $(z_1', \cdots, z_r')$ generating $\boldsymbol{R}$ such that

$$z_i' = d_{ij} v_j' \qquad (i = 1, 2, \cdots, r). \tag{2.9}$$

(2.9)[5]

Because of the diagonal character of $\|d_{ij}\|$, $\boldsymbol{A}/\boldsymbol{R}$ is represented by a "correlated" quotient

$$\frac{\{v_1'\} \oplus \cdots \oplus \{v_\rho'\} \quad \oplus \{v_{\rho+1}'\} \oplus \cdots \oplus \{v_\sigma'\}}{\{d_1 v_1'\} \oplus \cdots \oplus \{d_\rho v'\} \oplus \{0\} \quad \oplus \cdots \oplus \{0\}} \quad (\rho \leq \sigma). \tag{2.10}$$

By virtue of the natural isomorphism (2.3) $\boldsymbol{A}/\boldsymbol{R}$ is thus isomorphic to the external direct sum

$$C(d_1) \oplus \cdots \oplus C(d_\rho) \oplus \{v_{\rho+1}\} \oplus \cdots \oplus \{v_\sigma\} \tag{2.11}$$

where $C(d_i)$ is a cyclic group of order d_i and the remaining summands in (2.11) have infinite orders.

This result implies a classical theorem:

THEOREM 2.1. *When $\boldsymbol{A}$ is free the Betti number of $\boldsymbol{A}/\boldsymbol{R}$ equals $\sigma - \rho$, where σ is the dimension of $\boldsymbol{A}$ and ρ the rank of $\|a_{ij}\|$, while the invariant factors of $\|a_{ij}\|$ which exceed* 1 *are the torsion coefficients of $\boldsymbol{A}/\boldsymbol{R}$.* [12] p. 308.

We state a useful corollary.

COROLLARY 2.1. *Given an abelian group*

$$\mathscr{T} = C(t_1) \oplus \cdots \oplus C(t_r) \tag{2.12}$$

where $t_1, \cdots, t_r$ are finite integers exceeding 1, *the torsion coefficients of $\mathscr{T}$ are the invariant factors exceeding* 1 *of the r-square diagonal matrix $\|t_{ij}\|$ with diagonal $t_1, \cdots, t_r$.*

The problems of this paper. In sections 3 and 4 we shall give, in simplified matrix form, an algorithm showing how to compute the invariants of elementary quotients A/W when the invariants of A are given and a generator w of W is given in form (1.9).

In §5 we shall show how the hypothesis of prime-simplicity of A simplifies the problem of determining the invariants of A/W. Necessary and sufficient conditions that A/W be prime-simple when A is prime-simple are given.

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5) Summing as to j on the range $1, 2, \cdots, \sigma$.

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§ 3. **The elementary quotients** A/W. The group A is "given" as A was given in § 2. Moreover $W = \{\boldsymbol{w}\}$, where $\boldsymbol{w} \in A$. The giving of $\boldsymbol{w}$ should be such that the data on $\boldsymbol{w}$, together with the invariants of A, determine the invariants of A/W.

We shall begin by assigning to each $\boldsymbol{w} \in A$ an integer $\boldsymbol{s} \geq 0$ termed the *free index* of $\boldsymbol{w}$. When $\boldsymbol{w} \notin \mathscr{T}$, this integer $\boldsymbol{s}$ is characterized in Lemma 3.1.

Notation. In formulating Lemma 3.1 we write $x = y \bmod \mathscr{T}$ whenever x and y are elements in A such that $x - y$ is in $\mathscr{T}$.

LEMMA 3.1 (i). *Corresponding to a* $\boldsymbol{w} \in A$ *of infinite order, there exists an integer* $\boldsymbol{s} > 0$, *such that a prescribed Betti group* $\mathscr{B}$ *of* A *has a base with first element* u_B *such that*

$$\boldsymbol{w} = \boldsymbol{s} u_B \bmod \mathscr{T}. \tag{3.1}$$

(ii) *If there is given a second Betti group* $\mathscr{B}'$ *of* A *and a positive integer* $\boldsymbol{s}'$, *such that for some element* $u_{B'}$ *in a base for* $\mathscr{B}'$,

$$\boldsymbol{w} = \boldsymbol{s}' u_{B'} \bmod \mathscr{T}, \tag{3.2}$$

then $\boldsymbol{s} = \boldsymbol{s}'$.

Proof of (i). Let the prescribed Betti subgroup $\mathscr{B}$ of A have a base $(v_1, \cdots, v_r)$, $r > 0$. For suitable choices of integers $t_1, \cdots, t_r$, not all zero,

$$\boldsymbol{w} = t_1 v_1 + \cdots + t_r v_r \bmod \mathscr{T}. \tag{3.3}$$

Let $\boldsymbol{s}$ be the *GCD* of $t_1, \cdots, t_r$. There then exist integers $k_1, \cdots, k_r$, whose *GCD* equals 1, and which are such that

$$\boldsymbol{w} = \boldsymbol{s}(k_1 v_1 + \cdots + k_r v_r) \bmod \mathscr{T}. \tag{3.4}$$

It follows from the lemma in [8], p. 145, that

$$u_B = k_1 v_1 + \cdots + k_r v_r \tag{3.5}$$

is an element u_1 in a base $(u_1, \cdots, u_r)$ for $\mathscr{B}$, thereby proving (i).

Proof of (ii). Suppose that (3.1) and (3.2) both hold so that

$$\boldsymbol{s} u_B = \boldsymbol{s}' u_{B'} \bmod \mathscr{T}. \tag{3.6}$$

In terms of the base $(u_1, \cdots, u_r)$ of $\mathscr{B}$, for suitable integers $\mu_1, \cdots, \mu_r$,

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(3.6) takes the form

$$su_1 = s'(\mu_1 u_1 + \cdots + \mu_r u_r) \bmod \mathscr{T}, \tag{3.7}$$

from which we infer that $s = s'\mu_1$, and hence that $s' \leq s$. Similarly $s \leq s'$ so that $s = s'$.

The proof of (ii) is complete.

To each non-trivial element $w \in A$ we now assign indices s and t as follows.

Definition 3.1. *The free index s of w. If $w \in \mathscr{T}$ set $s = 0$. If $w \notin \mathscr{T}$, let $s > 0$ be the integer affirmed to exist in Lemma* 3.1.

By virtue of this definition of s

$$w = su_B + \tau_B \qquad (\tau_B \in \mathscr{T}) \tag{3.8}$$

where u_B is either 0 or the first element in a base for $\mathscr{B}$, according as order w is finite or infinite.

Definition 3.2. *The torsion index t of w. Set*

$$\text{order } \tau_B = t_B \tag{3.9}$$

and

$$\min_B t_B = t \tag{3.10}$$

where $\mathscr{B}$ ranges over all Betti groups complementary to $\mathscr{T}$ in A. We term t the *torsion index* of w. When $s = 0$, $t = t_B$ for every choice of $\mathscr{B}$.

There are three theorems corresponding to the three cases $s = 0$, $s = 1$ and $s \geq 1$, showing how the invariants of A are related to those of A/W.

We begin with the simplest case, $s = 1$.

Theorem 3.1. *If $s = 1$ the torsion subgroup of A/W is isomorphic to $\mathscr{T}$, the torsion subgroup of A, and the Betti number of A/W is one less than that of A.*

Proof. If $\mathscr{B}$ is a Betti subgroup of A, Lemma 3.1 (i) implies that there exists a base $(u_1, \cdots, u_r)$ of $\mathscr{B}$ such that $w = u_B + \tau_B$, where $u_B = u_1$. A new Betti group $\mathscr{B}'$, complementary to $\mathscr{T}$, can be introduced with a base

$$(v_1, \cdots, v_r) = (u_B + \tau_B, u_2, \cdots, u_r). \tag{3.11}$$

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Then $w = v_1$ and A/W has the form

$$\frac{A}{W} = \frac{\{v_1\} \oplus \{v_2\} \oplus \cdots \oplus \{v_r\} \oplus \mathscr{T}}{\{v_1\} \oplus \{0\} \oplus \cdots \oplus \{0\} \oplus \{0\}} \tag{3.12}$$

of a "correlated" quotient. We infer from (2.3) that

$$\frac{A}{W} \approx \{v_2\} \oplus \cdots \oplus \{v_r\} \oplus \mathscr{T}. \tag{3.13}$$

Theorem 3.1 follows.

The case $s \geq 1$. The data. A is "given" as is A in § **2.** A prescribed Betti group $\mathscr{B}$ of A has a base $(u_1, \cdots, u_r)$ such that when $s \geq 1$

$$w = su_B + \tau_B \qquad (u_B = u_1, \tau_B \in \mathscr{T}). \tag{3.14}$$

We begin with the case $\mathscr{T} \neq 0$, and suppose that $\mathscr{T}$ has a *CPD*,

$$\mathscr{T} = \{x_1\} \oplus \cdots \oplus \{x_\rho\} \qquad (\rho > 0) \tag{3.15}$$

for which the orders of the respective generators x_i are the elementary divisors

$$n_1, \cdots, n_\rho \tag{3.16}$$

of $\mathscr{T}$. The element τ_B has a representation

$$\tau_B = m_1 x_1 + \cdots + m_\rho x_\rho \tag{3.17}$$

with

$$0 \leq m_i < n_i \qquad (i = 1, 2, \cdots, \rho). \tag{3.18}$$

We say that the set $(m_1, \cdots, m_\rho)$ is *coordinated* with the set $(n_1, \cdots, n_\rho)$ in that m_i multiplies x_i in (3.17), while n_i is the order of x_i.

The set $(m_1, \cdots, m_\rho)$ depends upon the choice of $\mathscr{B}$, upon the ordering of the summands in (3.15) and upon the choice of the generators $x_1, \cdots, x_\rho$, as well as upon τ_B. By rechoosing the generators x_i we shall, in § **4**, obtain sets $(m_1, \cdots, m_\rho)$, termed *canonical*, such that $m_i = 0$ or divides n_i.

Theorem 3.2, below, and Corollary 3.1 reduce the problem of determining the invariants of A/W to the problem of determining the invariant factors of a $(\rho + 1)$-square matrix with a diagonal,

$$n_1, n_2, \cdots, n_\rho, s$$

and a last row,

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$$m_1, m_2, \cdots, m_\rho, \boldsymbol{s},$$

with other elements 0.

Theorem 3.2 and Corollary 3.1 include the case $\boldsymbol{s} = 1$ even though the case $\boldsymbol{s} = 1$ has already been disposed of in a simpler way,

THEOREM 3.2. *The case* $\boldsymbol{s} \geq 1$, $\mathscr{T} \neq 0$. *If the data for* A *and* $\boldsymbol{w}$ *are conditioned as above,* A/W *has a Betti number, one less than that of* A, *and a torsion subgroup* $\hat{T}$ *isomorphic to a quotient group* $\boldsymbol{A}/\boldsymbol{R}$, *where* $\boldsymbol{A}$ *is a free group,*

$$\boldsymbol{A} = \{y_1\} \oplus \cdots \oplus \{y_\rho\} \oplus \{v\} \tag{3.19}$$

and $\boldsymbol{R}$ *a subgroup of* $\boldsymbol{A}$ *of the form*

$$\boldsymbol{R} = \{n_1 y_1, \cdots, n_\rho y_\rho;\ \boldsymbol{s}v + m_1 y_1 + \cdots + m_\rho y_\rho\}. \tag{3.20}$$

In proving Theorem 3.2 we shall make use of several lemmas. The "natural[6] homomorphism"

$$\Psi : A \longrightarrow A/W \tag{3.21}$$

with its kernel $W = \{\boldsymbol{w}\}$ is needed. [10] p. 26.

LEMMA 3.2. *When* $\boldsymbol{s} \geq 1$, *the "natural homomorphism"* Ψ *maps the torsion subgroup* $\mathscr{T}$ *of* A *isomorphically onto the subgroup* $\Psi(\mathscr{T})$ *of* A/W, *and maps the subgroup*

$$\mathscr{B}_1 = \{u_2\} \oplus \cdots \oplus \{u_r\} \tag{3.22}$$

of $\mathscr{B}$ *isomorphically onto* $\Psi(\mathscr{B}_1)$.

Proof. Lemma 3.2 follows on noting that when $\boldsymbol{s} \geq 1$

$$\ker(\Psi \mid \mathscr{T}) = \mathscr{T} \cap \ker \Psi = \mathscr{T} \cap \{\boldsymbol{w}\} = 0. \tag{3.23}$$

since each non-trivial element of $\{\boldsymbol{w}\}$ then has an infinite order, and noting that

$$\ker(\Psi \mid \mathscr{B}_1) = \mathscr{B}_1 \cap \ker \Psi = \mathscr{B}_1 \cap \{\boldsymbol{w}\} = 0. \tag{3.24}$$

LEMMA 3.3. *When* $\boldsymbol{s} \geq 1$, *the group* A/W *is a direct sum,*

$$A/W = \Psi(\mathscr{B}_1) \oplus \Psi\{u_1, \mathscr{T}\} = \hat{B} \oplus \hat{T} \qquad (u_1 \textit{ from } (3.14)) \tag{3.25}$$

[6] The "natural" homomorphism Ψ maps an element $x \in A$ into the element in A/W which represents the W-coset containing x.

in which $\Psi\{u_1, \mathscr{T}\}$ *is the torsion subgroup* $\hat{T}$ *of* A/W, *and* $\Psi(\mathscr{B}_1)$ *is a Betti subgroup* $\hat{B}$ *of* A/W.

To establish Lemma 3.3 it is sufficient to recall that $\Psi(A) = A/W$ and to verify successively the relations,

$$A = \{u_1, \mathscr{B}_1, \mathscr{T}\} \tag{3.26}$$

$$\Psi(A) = \{\Psi(u_1), \Psi(\mathscr{B}_1), \Psi(\mathscr{T})\} \tag{3.27}$$

$$\hat{T} \supset \{\Psi(u_1), \Psi(\mathscr{T})\} \tag{3.28}$$

$$0 = \Psi(\mathscr{B}_1) \cap \hat{T} \tag{3.29}$$

from which Lemma 3.3 follows.

Of these relations the first two are immediately verified.

Proof of (3.28). When $s \geq 1$

$$w = su_B + \tau_B \qquad (u_B = u_1). \tag{3.30}$$

Recall that order τ_B in $\mathscr{T}$ has been denoted by t_B. The application of Ψ to both members of (3.30) shows that

$$0 = s\Psi(u_1) + \Psi(\tau_B). \tag{3.31}$$

Since order $\Psi(\tau_B)$ divides order τ_B, we see that st_B annihilates $\Psi(u_1)$. Hence $\Psi(u_1)$ is in $\hat{T}$ and (3.28) follows.

Proof of (3.29). Relation (3.29) holds, since each element in $\hat{T}$ has a finite order and each element in $\Psi(\mathscr{B}_1)$, other than 0, has an infinite order.

Lemma 3.3 follows.

We refer to the torsion subgroup $\hat{T}$ of A/W.

LEMMA 3.4. *When* $s \geq 1$, *there is a surjective isomorphism*

$$\hat{T} \approx \frac{\{u_1\} \oplus \mathscr{T}}{W} \tag{3.32}$$

in which W*-cosets in* A *which are elements of* $\hat{T}$ *are mapped into the* W*-cosets in* $\{u_1\} \oplus \mathscr{T}$ *partitioning* $\{u_1\} \oplus \mathscr{T}$ *with which they are identical as subsets of* A.

Proof. Recall that $\hat{T} = \Psi\{u_1, \mathscr{T}\}$ and note that, as subsets of A,

$$\{u_1, \mathscr{T}\} = \{u_1\} \oplus \mathscr{T}. \tag{3.33}$$

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Moreover W is a subgroup, both of A and of $\{u_1\} \oplus \mathscr{T}$ and $\hat{T}$ is the group of W-cosets in A of elements in $\{u_1, \mathscr{T}\}$. Lemma 3.4 follows.

Notation for Lemma 3.5. Lemma 3.5, below, gives the final isomorphism leading to the isomorphism $\hat{T} \approx \boldsymbol{A}/\boldsymbol{R}$ of Theorem 3.2. In Lemma 3.5

$\mathscr{T}$ is given by a *CPD* as in (3.15).

$\boldsymbol{A}$ is the free group (3.19)

W is the subgroup $\{su_1 + m_1x_1 + \cdots + m_\rho x_\rho\}$ of A

$\boldsymbol{R}$ is the subgroup (3.20) of $\boldsymbol{A}$

The statement and proof of Lemma 3.5 will be abbreviated by setting

$$(3.34)\quad \begin{cases} (x_1, \cdots, x_\rho) = \boldsymbol{x} & \text{(see (3.15))} \\ (y_1, \cdots, y_\rho) = \boldsymbol{y} & (\;\text{''}\;\; (3.19)) \\ (m_1, \cdots, m_\rho) = \boldsymbol{m} & (\;\text{''}\;\; (3.17)) \\ (n_1, \cdots, n_\rho) = \boldsymbol{n} & (\;\text{''}\;\; (3.16)) \\ (n_1y_1, \cdots, n_\rho y_\rho) = (\boldsymbol{ny}) & \\ n_1y_1 + \cdots + n_\rho y_\rho = \boldsymbol{n}\cdot\boldsymbol{y} & \end{cases}$$

etc. We term $\boldsymbol{x}$ and $\boldsymbol{y}$ *element sets*. The set $(\boldsymbol{ny})$ is an element set. We shall introduce arbitrary sets of integers,

$$(q_1, \cdots, q_\rho) = \boldsymbol{q}$$
$$(r_1, \cdots, r_\rho) = \boldsymbol{r}$$

The cosets $Q(p, \boldsymbol{q})$ *and* $P(p, \boldsymbol{q})$. For arbitrary integer p and set $\boldsymbol{q}$ of ρ integers, we shall denote by $Q(p, \boldsymbol{q})$, the $\boldsymbol{R}$-coset in $\boldsymbol{A}$ of the element $pv + \boldsymbol{q}\cdot\boldsymbol{y}$ in $\boldsymbol{A}$. We similarly denote by $P(p, \boldsymbol{q})$, the W-coset in $\{u_1\} \oplus \mathscr{T}$ of the element $pu_1 + \boldsymbol{q}\cdot\boldsymbol{x}$ in $\{u_1\} \oplus \mathscr{T}$.

LEMMA 3.5. *When* $s \geq 1$ *there exists a surjective isomorphism*

$$(3.35)\qquad \Theta : \frac{\boldsymbol{A}}{\boldsymbol{R}} \approx \frac{\{u_1\} \oplus \mathscr{T}}{W}$$

in which the $\boldsymbol{R}$*-coset,* $Q(p, \boldsymbol{q})$ *in* $\boldsymbol{A}$*, goes into the* W*-coset,* $P(p, \boldsymbol{q})$ *in* $\{u_1\} \oplus \mathscr{T}$.

To prove Lemma 3.5 we note first that Θ is a homomorphism. That is, for arbitrary integers p and p' and arbitrary sets $\boldsymbol{q}$ and $\boldsymbol{q}'$ of ρ integers

$$(3.36)\qquad \Theta(Q(p, \boldsymbol{q}) + Q(p', \boldsymbol{q}')) = \Theta(Q(p, \boldsymbol{q})) + \Theta(Q(p', \boldsymbol{q}')).$$

The mapping Θ is surjective, since the ensemble of W-cosets $P(p, \boldsymbol{q})$ contains each W-coset in $\{u_1\} \oplus \mathscr{T}$. It ts crucial to establish Prop 3.1.

PROPOSITION 3.1. *The mapping Θ is biunique.*

Note first that, if $\mathbf{0}$ is a set of ρ zeros, then

(3.37) $$P(0, \mathbf{0}) = W, \quad Q(0, \mathbf{0}) = \boldsymbol{R}.$$

Let $Q(p, \boldsymbol{q})$ and $Q(p', \boldsymbol{q}')$ be arbitrary *distinct* $\boldsymbol{R}$-cosets in $\boldsymbol{A}$. To establish Prop 3.1 we must show that the equality

(3.38) $$P(p, \boldsymbol{q}) = P(p', \boldsymbol{q}')$$

of the Θ-images of $Q(p, \boldsymbol{q})$ and $Q(p', \boldsymbol{q}')$ is then *impossible*.

The equality (3.38) would imply that

(3.39) $$P(p - p', \boldsymbol{q} - \boldsymbol{q}') = P(0, \mathbf{0}),$$

and hence that

(3.40) $$\Theta(Q(p - p', \boldsymbol{q} - \boldsymbol{q}')) = P(0, \mathbf{0}).$$

Moreover the hypothesis

(3.41) $$Q(p, \boldsymbol{q}) \neq Q(p', \boldsymbol{q}')$$

implies that

(3.42) $$Q(p - p', \boldsymbol{q} - \boldsymbol{q}') \neq Q(0, \mathbf{0}).$$

Prop 3.2 below implies that (3.40) and (3.42) are incompatible, and hence that Prop 3.1 is true.

Let $\hat{p}$ be an arbitrary integer and $\hat{\boldsymbol{q}}$ an arbitrary set of ρ integers.

PROPOSITION 3.2. *The only $\boldsymbol{R}$-coset $Q(\hat{p}, \hat{\boldsymbol{q}})$ in $\boldsymbol{A}$ whose image under Θ is the W-coset $P(0, \mathbf{0})$ in $\{u_1\} \oplus \mathscr{T}$ is the $\boldsymbol{R}$-coset $Q(0, \mathbf{0})$ in $\boldsymbol{A}$.*

Proof of Prop 3.2. The Θ-image of $Q(\hat{p}, \hat{\boldsymbol{q}})$ is $P(\hat{p}, \hat{\boldsymbol{q}})$ by definition of Θ, and by hypothesis of Prop 3.2

(3.43) $$P(\hat{p}, \hat{\boldsymbol{q}}) = P(0, \mathbf{0}) = W.$$

For (3.43) to hold it is necessary that for some integer N

(3.44) $$\hat{p}u_1 + \hat{\boldsymbol{q}} \cdot \boldsymbol{x} = N(\boldsymbol{s}u_1 + \boldsymbol{m} \cdot \boldsymbol{x})$$

since $\boldsymbol{s}u_1 + \boldsymbol{m} \cdot \boldsymbol{x}$ is a generator of the cyclic group W. Since

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$$\{u_1\} \oplus \cdots \oplus \{u_r\} \oplus \{x_1\} \oplus \cdots \oplus \{x_\rho\} \tag{3.45}$$

is a direct sum, and order $x_i = n_i$ for $i = 1, \cdots, \rho$, (3.44) holds only if

$$\hat{p} = N\boldsymbol{s}, \quad \hat{\boldsymbol{q}} = N\boldsymbol{m} + (\boldsymbol{rn}) \tag{3.46}$$

for some set $\boldsymbol{r}$ of ρ integers. It follows from (3.46) that

$$\hat{p}v + \hat{\boldsymbol{q}} \cdot \boldsymbol{y} = N(\boldsymbol{s}v + \boldsymbol{m} \cdot \boldsymbol{y}) + (\boldsymbol{rn}) \cdot \boldsymbol{y},$$

that is, that $\hat{p}v + \hat{\boldsymbol{q}} \cdot \boldsymbol{y}$ is in $\boldsymbol{R}$.

Thus Prop 3.2 is true, Θ is biunique, and Lemma 3.5 follows.

Completion of proof of Theorem 3.2. It follows from Lemmas 3.2 and 3.3 that, when $\boldsymbol{s} \geq 1$, the Betti number of A/W is one less than the Betti number of A. That the torsion subgroup $\hat{T}$ of A/W is isomorphic to the quotient $\boldsymbol{A}/\boldsymbol{R}$ of Theorem 3.2 follows from Lemmas 3.4 and 3.5.

Theorem 3.2 is thereby established. Theorem 3.2 is *supplemented* as follows.

The case $\mathscr{T} = 0$. In stating and proving Theorem 3.2 we have excluded this special case. In this case $\tau_B = 0$ in (3.14), so that

$$\frac{A}{W} = \frac{\{u_1\} \oplus \{u_2\} \oplus \cdots \oplus \{u_r\}}{\{\boldsymbol{s}u_1\} \oplus \{0\} \oplus \cdots \oplus \{0\}}. \tag{3.47}$$

This is a "correlated" quotient. By (2.3)

$$\frac{A}{W} \approx C(\boldsymbol{s}) \oplus \{u_2\} \oplus \cdots \oplus \{u_r\} \qquad \text{(when } \boldsymbol{s} > 1) \tag{3.48}$$

where $C(\boldsymbol{s})$ is a cyclic group of order $\boldsymbol{s}$. The torsion subgroup of A/W is thus isomorphic to $C(\boldsymbol{s})$ when $\mathscr{T} = 0$ and $\boldsymbol{s} > 1$, and trivial when $\boldsymbol{s}$ is 1 or 0.

From (3.47) and (3.48) one can infer the following:

THEOREM 3.3. *When A is torsion free A/W is torsion free unless $\boldsymbol{s} > 1$, and when $\boldsymbol{s} > 1$ the first and only torsion coefficient of A/W is $\boldsymbol{s}$.*

The torsion coefficients of A/W **when** $\boldsymbol{s} \geq 1$. The subgroup $\boldsymbol{R}$ of the free group $\boldsymbol{A}$ is defined by a set of generators $z_1, \cdots, z_{\rho+1}$ given in Theorem 3.2 as the respective linear combinations,

$$n_1 y_1, \cdots, n_\rho y_\rho, \ \boldsymbol{s}v + m_1 y_1 + \cdots + m_\rho y_\rho \tag{3.49}$$

of the elements

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(3.50) $$y_1, \cdots, y_\rho, v$$

of the base of A. Cf. (2.3), (2.4) and (2.5). The matrix $\|a_{ij}\|$ of the integral linear representation of $z_1, \cdots, z_{\rho+1}$ in terms of the elements (3.50) has the form

(3.51) $$\|a_{ij}\| = \begin{Vmatrix} n_1 & & & & \\ & n_2 & & & \\ & & \ddots & & \\ & & & n_\rho & \\ m_1 & m_2 & \cdots & m_\rho & s \end{Vmatrix} \qquad (s \geq 1)$$

where unspecified elements are 0.

The rank of the matrix (3.51) is $\rho + 1$. The integers $m_1, \cdots, m_\rho$ were introduced in (3.17). They satisfy the conditions

(3.52) $$0 \leq m_i < n_i \qquad (i = 1, \cdots, \rho)$$

and are "coordinated" with the elementary divisors $n_1, \cdots, n_\rho$ of A.

From Theorem 2.1 we infer a fundamental corollary of Theorem 3.2.

COROLLARY 3.1. *When* $s \geq 1$ *the rank of the matrix* $\|a_{ij}\|$ *is* $\rho + 1$ *and the subset of invariant factors of this matrix which exceed* 1 *are the torsion coefficients of* A/W.

Making use of this corollary it is possible to make many comparisons between the elementary divisors of A and those of A/W when $s \geq 1$, or equivalently the torsion coefficients of A and those of A/W.

A first result of this type follows.

COROLLARY 3.2. *Suppose* $s \geq 1$.

(i) *If each* m_i *in the representation* (3.17) *of* τ_B *is a multiple (possibly null) of* s, *then the torsion coefficients of* A/W *are the invariant factors which exceed* 1 *of the diagonal matrix* J *with diagonal*

(3.53) $$n_1, \cdots, n_\rho, s.$$

(ii) *Equivalently, under the hypotheses of* (i) *the elementary divisors of* A/W *are those of* A *supplemented by the prime power factors of* s.

Proof of (i). Matrices which can be obtained, one from the other, by elementary operations are termed *equivalent*. Equivalent matrices have the same invariant factors.

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It is clear that under the hypotheses of (i), the matrices $\|a_{ij}\|$ and J are equivalent.

Statement (i) follows.

Proof of (ii). There exists an abstract abelian group A^* which is the external direct sum of abelian groups with the orders (3.53). By Corollary 2.1, the torsion coefficients of A^* are the invariant factors exceeding one of the matrix J, and hence by (i) the torsion coefficients of A/W. The elementary divisors of A^* are clearly the integers $n_1, \cdots, n_\rho$, supplemented by the prime power factors of $\boldsymbol{s}$. Since A^* and A/W have the same torsion coefficients they have the same elementary divisors.

Statement (ii) follows.

In case $\boldsymbol{s} \geq 1$ the quotient A/W has *more* torsion than A in the sense of the following corollary.

COROLLARY 3.3. *When $\boldsymbol{s} \geq 1$ the product of the torsion coefficients of A/W (or elementary divisors of A/W) is $\boldsymbol{s}$ times the corresponding product for A.*

Proof. We have seen in § **1** that the product of the elementary divisors of A equals the product of the torsion coefficients of A. Cf. Theorem 1.3. In the case at hand this product is $n_1 \cdots n_\rho$ by hypothesis. According to Corollary 3.1 the product, when $\boldsymbol{s} \geq 1$, of the torsion coefficients of A/W is the product of the invariant factors of the matrix $\|a_{ij}\|$. Hence this product equals

$$\det |a_{ij}| = n_1 \cdots n_\rho \boldsymbol{s}.$$

Thus Corollary 3.3 is true.

In § **5** an explicit comparison of the elementary divisors of A and A/W is made when both A and A/W are *prime-simple* groups. See Definition 1.6.

§ 4. **The elementary quotients A/W when $\boldsymbol{s} = 0$.** We continue with the analysis of § **3**. As in § **3**, A is finitely generated, W a subgroup of A of form $W = \{\boldsymbol{w}\}$, with $\boldsymbol{w} \neq 0$ in $\mathscr{T}$, the torsion subgroup of A.

We begin with a lemma.

LEMMA 4.1. *If $\boldsymbol{s} = 0$ the Betti number of A/W equals the Betti number of A, and if T_0 is the torsion subgroup of A/W, then*

$$T_0 \approx \mathscr{T}/W. \tag{4.0}$$

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Let $\mathscr{B}$ be a Betti subgroup of A. Then $A = \mathscr{B} \oplus \mathscr{T}$ and one has the correlated quotient

$$\frac{A}{W} = \frac{\mathscr{B} \oplus \mathscr{T}}{\{0\} \oplus W}, \qquad (s = 0) \tag{4.1}$$

since W is a subgroup of $\mathscr{T}$ when $s = 0$. By (2.3)

$$A/W \approx \mathscr{B} \oplus \mathscr{T}/W. \tag{4.2}$$

The quotient $\mathscr{T}/W$ is a *finite* group and hence the torsion subgroup of the right member of (4.2). The isomorphism (4.0) follows from the isomorphism (4.2).

Note. One should contrast the equality of the Betti numbers of A and A/W when $s = 0$, with the fact that the Betti number of A exceeds the Betti number of A/W by 1 when $s > 0$. See Theorems 3.1 and 3.2.

We now prepare for Theorem 4.1.

The data when $s = 0$. A is given as is A of (2.1). Since $s = 0$

$$w = \tau_B \qquad (\tau_B \in \mathscr{T}), \tag{4.3}$$

where $\tau_B \neq 0$, since we are assuming that $W = \{w\} \neq 0$. We suppose that $\mathscr{T}$ is given by a *CPD* of form (3.15) and that $(n_1, \cdots, n_\rho)$ is the corresponding set of[7] *ED*'s of $\mathscr{T}$. We suppose that τ_B is given a representation (3.17) with a set of integral coefficients $(m_1, \cdots, m_\rho)$ subject to (3.18), and "coordinated" with the ordered set $(n_1, \cdots, n_\rho)$ of *ED*'s of $\mathscr{T}$.

THEOREM 4.1. *The case* $s = 0$. *Under the conditions of the preceding paragraph the torsion subgroup* T_0 *of* A/W *is isomorphic to a quotient group* A_0/R_0, *where* A_0 *is a free group,*

$$A_0 = \{y_1\} \oplus \cdots \oplus \{y_\rho\}, \qquad (\rho > 0) \tag{4.4)'$$

and R_0 *is a subgroup of* A_0 *of the form*

$$R_0 = \{n_1 y_1, \cdots, n_\rho y_\rho;\ m_1 y_1 + \cdots + m_\rho y_\rho\}. \tag{4.4)''$$

Proof. By virtue of Lemma 4.1, it is sufficient to show that

$$\frac{\mathscr{T}}{W} \approx \frac{A_0}{R_0}. \tag{4.5}$$

Proof of (4.5). For arbitrary sets q of ρ integers, let $Q(q)$ denote the

[7] *ED* abbreviates "elementary divisor".

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R_0-coset in A_0 of the element $\boldsymbol{q}\cdot\boldsymbol{y}$ in A_0 (employing the notation introduced in (3.34)). Similarly let $P(\boldsymbol{q})$ denote the W-coset in $\mathscr{T}$ of the element $\boldsymbol{q}\cdot\boldsymbol{x}$ in $\mathscr{T}$. Let Φ be the surjective homomorphism,

$$\Phi : Q(\boldsymbol{q}) \longrightarrow P(\boldsymbol{q}) : \frac{A_0}{R_0} \longrightarrow \frac{\mathscr{T}}{W}. \tag{4.6}$$

As in the proof of Theorem 3.2 the principal step is to prove Φ biunique. Just as Proposition 3.1 followed from Proposition 3.2, so here the biuniqueness of Φ follows from the analogue, Proposition 4.1, of Proposition 3.2.

PROPOSITION 4.1. *The only R_0-coset, $Q(\hat{\boldsymbol{q}})$ in A_0, whose image under Φ is the W-coset $P(\mathbf{0})$ in $\mathscr{T}$, is the R_0-coset $Q(\mathbf{0})$ in A_0.*

Proof of Proposition 4.1. As in the proof of Proposition 3.2, by definition and hypothesis,

$$\Phi(Q(\hat{\boldsymbol{q}})) = P(\hat{\boldsymbol{q}}) = P(\mathbf{0}) = \{\boldsymbol{w}\}. \tag{4.7}$$

For (4.7) to hold, it is necessary that for some integer N

$$\hat{\boldsymbol{q}}\cdot\boldsymbol{x} = N(\boldsymbol{m}\cdot\boldsymbol{x}) \qquad (\text{in } \mathscr{T}) \tag{4.8}$$

since $\boldsymbol{m}\cdot\boldsymbol{x} = \boldsymbol{w}$ by (3.17) and (3.14). The relation (4.8) implies that

$$\hat{\boldsymbol{q}} = N\boldsymbol{m} + (\boldsymbol{rn}) \tag{4.9}$$

for some set $\boldsymbol{r}$ of ρ integers. It follows from (4.9) that

$$\hat{\boldsymbol{q}}\cdot\boldsymbol{y} = N(\boldsymbol{m}\cdot\boldsymbol{y}) + (\boldsymbol{rn})\cdot\boldsymbol{y},$$

that is, that $\hat{\boldsymbol{q}}\cdot\boldsymbol{y}$ is in R_0.

Thus Proposition 4.1 is true, hence Φ is biunique. It is a surjective isomorphism.

This completes the proof of Theorem 4.1.

The torsion coefficients of A/W when $s=0$. The subgroup R_0 of the free group A_0 of Theorem 4.1 is defined by a set of generators $z_1, \cdots, z_{\rho+1}$, given in Theorem 4.1 as the respective linear combinations

$$n_1 y_1, \cdots, n_\rho y_\rho;\ m_1 y_1 + \cdots + m_\rho y_\rho \tag{4.10}$$

of the elements $y_1, \cdots, y_\rho$ of the base of A_0. The matrix $\|b_{ij}\|$ of these integral linear combinations of the elements $y_1, \cdots, y_\rho$, has the form

$$(4.11) \qquad \|b_{ij}\| = \left\| \begin{matrix} n_1 & & & \\ & n_2 \cdot & & \\ & & \cdot \cdot \cdot \cdot & \\ & & & n_\rho \\ m_1 m_2 \cdots & & \cdots & m_\rho \end{matrix} \right\|$$

where unspecified elements are 0.

From Theorem 2.1 we infer the following corollary of Theorem 4.1.

COROLLARY 4.1. *When* $s = 0$ *the rank of the matrix* $\|b_{ij}\|$ *is* ρ *and the invariant factors of* $\|b_{ij}\|$ *which exceed* 1 *are the torsion coefficients of* A_0/R_0 *and hence of* A/W.

Canonical sets $(m_1, \cdots, m_\rho)$. We refer to the representation

$$(4.12) \qquad \tau_B = m_1 x_1 + \cdots + m_\rho x_\rho \qquad (\tau_B \in \mathscr{T})$$

of τ_B. The coefficients m_i which are not zero will be altered, in general, by a change of the generators x_i of the summands $\{x_i\}$ of the *CPD* (3.15) of $\mathscr{T}$. However, one does not alter $\{x_i\}$ if x_i is replaced by rx_i, provided r is relatively prime to order $\{x_i\}$. The following lemma shows how this freedom of choice of the generators can simplify the set $(m_1, \cdots, m_\rho)$.

LEMMA 4.2. *If generators* x_i *of the summands* $\{x_i\}$ *of the CPD* (3.15) *of* τ_B *are suitably chosen, each coefficient,* $m_i \neq 0$, *in the representation* (4.12) *of* τ_B *will become a proper divisor (termed canonical) of* n_i, *the associated ED of* $\mathscr{T}$ *and as such will be unique.*

Proof. The *ED* n_i equals $p_i^{e_i}$ for some prime factor p_i of order $\mathscr{T}$ and positive integer e_i. If $m_i \neq 0$ there exists a *GCD*

$$(4.13) \qquad (n_i, m_i) = p_i^{a_i}$$

of n_i and m_i, such that $0 \leq a_i < e_i$. Hence $m_i = \mu_i p_i^{a_i}$, where μ_i and p_i are relatively prime. The generator x_i can be replaced by a generator $x_i' = \mu_i x_i$. Then in (4.12) one has

$$(4.14) \qquad m_i x_i = \mu_i p_i^{a_i} x_i = x_i' p_i^{a_i} \qquad (\text{with } p_i^{a_i} \text{ canonical}).$$

so that one can replace x_i by x_i' as generator of $\{x_i\}$ and m_i by $m_i' = p_i^{a_i}$ in (3.17).

When $m_i \neq 0$, and $p_i^{a_i}$ is conditioned as above, $m_i = \mu_i p_i^{a_i}$ where μ_i is prime to p_i so that

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(4.15) $$\text{order } (m_i x_i) = \text{order } (\mu_i p_i^{a_i} x_i) = \text{order } p_i^{a_i} x_i = p_i^{e_i - a_i}.$$

Hence the replacement $m_i' = p_i^{a_i}$ of m_i is *uniquely* determined by the requirement that $\{x_i\} = \{x_i'\}$, $m_i x_i = m_i' x_i'$ and that m_i' be a proper divisor of n_i.

The computation of the ED**'s of** A/W **when** $\boldsymbol{s} = 0$. As has been seen in §**1** the ED's of A/W are immediate if the torsion coefficients of A/W are known. According to Corollary 4.1, when $\boldsymbol{s} = 0$ the torsion coefficients of A/W are the invariant factors exceeding 1 of the matrix $\|b_{ij}\|$ of (4.11). However, this computation of the ED's of A/W can be simplified as follows.

Let $p_1, \cdots, p_r$ be the distinct primes which are factors of (order $\mathscr{T}$) when $\boldsymbol{s} = 0$. The subgroup W of $\mathscr{T}$ can be given a Sylow decomposition $W = W_1 \oplus \cdots \oplus W_r$ into a direct sum of the Sylow p_i-subgroups W_i of W, some of which can be trivial. If $\mathscr{S}_{p_i}$ is the Sylow p_i-subgroup of $\mathscr{T}$ then

(4.16) $$\frac{\mathscr{T}}{W} \approx \frac{\mathscr{S}_{p_1}}{W_1} \oplus \cdots \oplus \frac{\mathscr{S}_{p_r}}{W_r}.$$

We infer the following.

LEMMA 4.3. *When* $\boldsymbol{s} = 0$ *the set of* ED*'s of* $\mathscr{T}/W$ *is the union of the* ED*'s of such of the quotients* $\mathscr{S}_{p_\mu}/W_\mu$ *in* (4.16) *as are nontrivial groups.*

Matrices J_μ. We are accordingly led to the problem of computation of the torsion quotients of a non-trivial quotient $\mathscr{S}_{p_\mu}/W_\mu$ taken from the quotients in the right member of (4.16). With such a quotient we associate that submatrix J_μ of the matrix $\|b_{ij}\|$ which consists of the columns of $\|b_{ij}\|$ whose non-null elements are powers of P_μ. If the rows of J_μ are properly rearranged J_μ will have the general form of $\|b_{ij}\|$ with a diagonal composed of ED's of $\mathscr{S}_{p_\mu}$, say ω in number, and an $(\omega+1)$-th row of integers $m_1, \cdots, m_\omega$ each of which is 0 or divides the ED of $\mathscr{S}_{p_\mu}$ in the same column.

The invariant factors of J_μ. The computation of the invariant factors $d_1, \cdots, d_\omega$ of J_μ is very simple by virtue of the fact that

(4.17) $$d_1 = D_1, \quad d_2 = \frac{D_2}{D_1}, \cdots, d_\omega = \frac{D_\omega}{D_{\omega-1}} \qquad \text{(cf. (2.8))}$$

where for i on the range, $1, \cdots, \omega$, D_i is the *smallest* in absolute value of the non-vanishing i-rowed subdeterminants Δ_i of J_μ. One observes that each non-vanishing Δ_i is the product of i elements of J_μ.

DEFINITION 4.1. **Profiles of w.** If the generator w of W is given as in (1.9), subject to (1.10), the set of integers

$$\mu_1, \cdots, \mu_\beta, m_1, \cdots, m_\rho \tag{4.18}$$

will be called the *profile* of w relative to the "basis" (1.8) of A.

We shall prove the following theorem.

THEOREM 4.2. *A "profile" of a non-null element $w \in A$ uniquely determines the "free index" s, and together with the coordinated ED's of $\mathcal{T}$ uniquely determines the "torsion index" of w when $s = 0$.*

To establish this theorem a second definition is needed.

DEFINITION 4.2. If the generators x_i of the summands of the CPD of $\mathcal{T}$ are suitably chosen the non-null integers m_i in the set $m_1, \cdots, m_\rho$ are "canonical" in the sense of Lemma 4.2, and the "profile" (4.18) relative to the basis (1.8) will be termed *canonical.*

According to Lemma 4.2 a profile (4.18) of w relative to a "basis" (1.8) uniquely determines a "canonical profile" of w relative to a modified basis (1.8). Theorem 4.2 is accordingly equivalent to the following lemma.

LEMMA 4.4. *A canonical profile*

$$\mu_1, \cdots, \mu_\beta; m_1, \cdots, m_\rho \tag{4.19}$$

of a non-null element $w \in A$ uniquely determines the free index s of w and together with the coordinated ED's of $\mathcal{T}$ uniquely determines order w when $s = 0$.

The free index s. If $\beta = 0$ in (4.19), or if $\mu_1 = \cdots = \mu_\beta = 0$, then $s = 0$. In any other case s is the GCD of the elements $\mu_1, \cdots, \mu_\beta$, as the proof of Lemma 3.1(i) shows.

The torsion index t. If $s = 0$, $t =$ order w in accord with the definition of t. If $\mathcal{T} = 0$, or if $m_1 = \cdots = m_\rho = 0$, $t = 1$.

Suppose then that $s = 0$ and $w \neq 0$. Let $n_1, \cdots, n_\rho$ be the ED's of $\mathcal{T}$.

Suppose first that $\mathcal{T}$ is a Sylow p-subgroup of A. The ED's of $\mathcal{T}$ then form a list

$$p^{e_1}, \cdots, p^{e_\rho} \qquad (e_i > 0) \tag{4.20}$$

of powers of p. By hypothesis (4.19) is a "canonical profile" of w relative to a basis (1.8) of A. We suppose that the ED's (4.20) are so ordered that for some integer r such that $1 \leq r \leq \rho$, $m_1, \cdots, m_r$ is the subset of the m_i's in (4.19) which do not vanish. Since (4.19) is by hypothesis a "canonical profile" of w the first r elements in the reordered list $m_1, \cdots, m_\rho$ have the form

$$p^{a_1}, \cdots, p^{a_r} \qquad (0 \leq a_j < e_j) \tag{4.21}$$

where j has the range $1, 2, \cdots, r$.

Moreover

$$w = p^{a_1}x_1 + \cdots + p^{a_r}x_r. \tag{4.22}$$

We shall show that

$$\text{order } w = \max_j \; p^{e_j - a_j} = p^e, \tag{4.23}$$

introducing p^e.

Proof of (4.23). One establishes (4.23) by verifying the following.

(a) The order of $p^{a_j}x_j$ in $\{x_j\}$ is $p^{e_j - a_j}$.

(b) The integer p^e annihilates w.

(c) The order t of w is a divisor p^a of p^e.

(d) Were $0 \leq a < e$, p^a could not annihilate w.

Statements (a), (b) and (c) are readily verified. Were $a < e$ and p^a an annihilator of w it would follow from the representation (4.22) of (w) that

$$\{x_1, \cdots, x_r\}$$

could not be a direct sum $\{x_1\} \oplus \cdots \oplus \{x_r\}$. From this contradiction (d) follows.

Thus (4.23) is true and Lemma 4.4 follows when $\mathscr{T}$ is a Sylow p-subgroup of A.

In any case $\mathscr{T}$ is a direct sum of its Sylow subgroups. The order of w is then the product of the orders of its non-null components in the respective Sylow subgroups of $\mathscr{T}$.

Lemma 4.4 follows and implies Theorem 4.2.

§ 5. A/W **when** A **is prime-simple.** We shall give an explicit determination in this section of how the ED's of A/W differ from those of

A when both A and A/W are prime-simple in the sense of Def 1.6. To this end a general lemma with no assumption of prime-simplicity will be useful.

Notation for Lemma 5.1. We refer again to the natural homomorphism $\Psi : A \longrightarrow A/W$ of A onto A/W, where $W = \{\boldsymbol{w}\}$, and as in (3.8),

$$\boldsymbol{w} = \boldsymbol{s}u_B + \tau_B \qquad (\tau_B \in \mathscr{T}). \tag{5.1}$$

Here u_B is an element in A which is 0 or a first element in a base for a Betti group of A, according as $\boldsymbol{w}$ is in $\mathscr{T}$ or not in $\mathscr{T}$.

LEMMA 5.1. *If order τ_B in A is denoted by t_B, order $\Psi(u_B)$ in A/W equals $\boldsymbol{s}t_B$ or 1 according as $u_B \neq 0$ or $u_B = 0$.*

The lemma is trivial if $u_B = 0$.

Suppose then that $u_B \neq 0$ and $\boldsymbol{s} > 0$. Since $\ker \Psi = \{\boldsymbol{w}\}$ it follows from (5.1) that

$$\Psi(\boldsymbol{w}) = \boldsymbol{s}\Psi(u_B) + \Psi(\tau_B) = 0. \tag{5.2}$$

Moreover, $t_B\Psi(\tau_B) = 0$, since $t_B\tau_B = 0$. Hence $\boldsymbol{s}t_B\Psi(u_B) = 0$, so that order $\Psi(u_B)$ is a divisor m of $\boldsymbol{s}t_B$.

Since $m\Psi(u_B) = 0$, mu_B must be in $\ker \Psi$. Hence

$$mu_B = \mu(\boldsymbol{s}u_B + \tau_B) \tag{5.3}$$

for some integer μ. Since $u_B \neq 0$ by hypothesis, $m = \mu\boldsymbol{s}$. Hence $\mu\tau_B = 0$. It follows that μ is a multiple of t_B. We conclude that m must be a multiple of $\boldsymbol{s}t_B$ and so equal $\boldsymbol{s}t_B$.

This establishes Lemma 5.1.

We now suppose that A is *prime-simple.* When A is finitely generated and $W = \{\boldsymbol{w}\}$, we have seen in Lemma 4.1 that the torsion subgroup, T_0 of A/W, when $\boldsymbol{s} = 0$, is such that

$$T_0 \approx \mathscr{T}/W \tag{5.4}$$

while Lemma 3.4 implies that the torsion subgroup, $\hat{T}$ of A/W, when $\boldsymbol{s} \geq 1$, is such that

$$\hat{T} \approx \frac{\{u_1\} \oplus \mathscr{T}}{W}. \tag{5.5}$$

We include the case $\boldsymbol{s} = 1$, although this case is adequately covered by Theorem 3.1.

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Corresponding to the cases $\boldsymbol{s} = 0$ and $\boldsymbol{s} \geq 1$, Theorems 5.1 and 5.2 are the principal results of this section. We are assuming that A is finitely generated and that $W = \{\boldsymbol{w}\}$ is non-trivial. By hypothesis each ED of A is a prime.

A convention. To say that a prime p is an ED of A/W of *multiplicity* 0 shall mean that p is *not* an ED of A/W.

THEOREM 5.1. *When $\boldsymbol{s} = 0$ and A is prime-simple, each ED of A which is not a factor of* (order $\boldsymbol{w}$) *is an ED of A/W, while each ED of A, of multiplicity μ which is a factor of* (order $\boldsymbol{w}$) *is an ED of A/W of multiplicity $\mu - 1$.*

These are the only ED's of A/W and A/W is accordingly prime-simple (Th 1.4).

We refer to the "torsion index" $\boldsymbol{t}$ of $\boldsymbol{w}$. Def 3.2.

THEOREM 5.2. *When $\boldsymbol{s} \geq 1$ and A is prime-simple, the following is true:*

(i) *Necessary and sufficient conditions that A/W be prime-simple are that $\boldsymbol{s}$ be prime-simple and that $\boldsymbol{t} = 1$.*

(ii) *When $\boldsymbol{s}$ is prime-simple and $\boldsymbol{t} = 1$, the ED's of A, supplemented by the prime factors of $\boldsymbol{s}$, are the ED's of A/W.*

Proof of Theorem 5.1. It follows from Def 1.6 of a prime-simple group that each subgroup of a prime-simple group is prime-simple. Hence $W = \{\boldsymbol{w}\}$ is prime-simple. Since $\boldsymbol{s} = 0$ by hypothesis, $\boldsymbol{w}$ has a finite order. Since W is nontrivial by hypothesis and is finite and prime-simple, order $\boldsymbol{w}$ is a product

$$p_1 p_2 \cdots p_r \qquad (r > 0) \tag{5.6}$$

of distinct primes. Lemma 5.2 is needed.

Notation for Lemma 5.2. Let p be one of the primes in the product (5.6). The Sylow subgroup $\mathscr{S}_p$ of A is prime-simple since A is prime-simple. Let μ be the "length" of $\mathscr{S}_p$. Each ED of $\mathscr{S}_p$ must equal p in magnitude. $\mathscr{S}_p$ must then have a CPD of the form

$$\mathscr{S}_p = \{e_1\} \oplus \cdots \oplus \{e_\mu\} \tag{5.7}$$

where each generator e_i has the order p.

We shall verify the following lemma.

LEMMA 5.2. *If an element $a \in W$ has the prime order p and $\mathscr{S}_p$ has the length μ, then the quotient group $\mathscr{S}_p/\{a\}$ has exactly $\mu-1$ ED's, each numerically p.*

Proof. If the generators e_i of the cyclic groups $\{e_i\}$ in (5.7) are properly chosen and ordered, then

$$a = e_1 + e_2 + \cdots + e_n \qquad \text{(cf. Lemma 4.2)} \tag{5.8}$$

for some integer n such that $1 \leq n \leq \mu$. The subgroups

$$\{a\}, \ \{e_2\}, \cdots, \{e_\mu\} \tag{5.9}$$

of $\mathscr{S}_p$ generate $\mathscr{S}_p$. They also sum *directly* to $\mathscr{S}_p$; for a nontrivial relation of integral linear dependence between the generators $a, e_2, \cdots, e_\mu$ would imply a nontrivial relation of integral linear dependence between the elements $e_1, \cdots, e_\mu$. Hence

$$\frac{\mathscr{S}_p}{\{a\}} = \frac{\{a\} \oplus \{e_2\} \oplus \cdots \oplus \{e_\mu\}}{\{a\} \oplus \{0\} \oplus \cdots \oplus \{0\}}. \tag{5.10}$$

This is a "correlated quotient". It follows from (2.3) that

$$\mathscr{S}_p/\{a\} \approx \{e_2\} \oplus \cdots \oplus \{e_\mu\},$$

thereby establishing Lemma 5.2.

Completion of proof of Theorem 5.1. Starting with the primes (5.6) let

$$p_1, \cdots, p_r; \ p_{r+1}, \cdots, p_m \tag{5.11}$$

be the list of *distinct* primes which are the ED's of the torsion subgroup $\mathscr{T}$ of A. The complete set of ED's of $\mathscr{T}$ is obtained by listing each of the primes in (5.11) a number of times equal to its multiplicity. One has the Sylow decomposition

$$\mathscr{T} = \mathscr{S}_{p_1} \oplus \cdots \oplus \mathscr{S}_{p_m} \tag{5.12}$$

of $\mathscr{T}$. The subgroup W of A admits a CPD

$$W = \{a_1\} \oplus \cdots \oplus \{a_r\} \tag{5.13}$$

where the generator a_i has the order p_i. From (5.12) and (5.13) we obtain a correlated quotient

$$\frac{\mathscr{T}}{W} = \frac{\mathscr{S}_{p_1} \oplus \cdots \oplus \mathscr{S}_{p_r} \oplus \mathscr{S}_{p_{r+1}} \oplus \cdots \oplus \mathscr{S}_{p_m}}{\{a_1\} \oplus \cdots \oplus \{a_r\} \oplus \{0\} \oplus \cdots \oplus \{0\}}. \tag{5.14}$$

It follows from (5.14) and (5.4) that T_0 is isomorphic to the external direct sum

$$\frac{\mathscr{S}_{p_1}}{\{a_1\}} \oplus \cdots \oplus \frac{\mathscr{S}_{p_r}}{\{a_r\}} \oplus \mathscr{S}_{p_{r+1}} \oplus \cdots \oplus \mathscr{S}_{p_m}. \tag{5.15}$$

Theorem 5.1 follows on applying Lemma 5.2 to each of the quotients in (5.15).

Proof of Theorem 5.2. It follows from Def 3.2 of the "torsion index" $\boldsymbol{t}$ of $\boldsymbol{w}$ that when $\boldsymbol{s} > 0$, there *exists* a Betti subgroup $\mathscr{B}$ of A with a base $(u_1, \cdots, u_r)$ such that

$$\boldsymbol{w} = \boldsymbol{s}u_B + \tau_B \qquad (u_B = u_1, \text{ order } \tau_B = \boldsymbol{t}). \tag{5.16}$$

If $\Psi : A \longrightarrow A/W$ is the natural homomorphism of A onto A/W it follows from Lemma 5.1 that

$$\text{order } \Psi(u_B) = \boldsymbol{st} \qquad (\text{when order } \tau_B = \boldsymbol{t}). \tag{5.17}$$

Hypothesis. *We assume that when* $\boldsymbol{s} \geq 1$, $\boldsymbol{w}$ *is given a representation of form* (5.16). *This is a condition on* τ_B *and not on* $\boldsymbol{w}$.

The following lemma is needed to establish the necessity of the conditions of Theorem 5.2 (i).

Lemma 5.3. *If the free index* $\boldsymbol{s}$ *and torsion index* $\boldsymbol{t}$ *of* $\boldsymbol{w}$ *are relatively prime and* $\boldsymbol{s} > 0$, *then* $\boldsymbol{t} = 1$.

Starting with (5.16) we note that

$$\tau_B \in \{\tau_B\} = \{\boldsymbol{s}\tau_B\} \subset \boldsymbol{s}\mathscr{T}. \tag{5.18}$$

The equality in (5.18) is valid since $\boldsymbol{s}$ is prime to (order τ_B). From (5.18) we infer the existence of an element $x \in \mathscr{T}$ such that $\tau_B = \boldsymbol{s}x$. We rewrite (5.16) in the form

$$\boldsymbol{w} = \boldsymbol{s}(u_B + x) + (\tau_B - \boldsymbol{s}x) = \boldsymbol{s}(u_B + x). \tag{5.19}$$

From the given Betti group $\mathscr{B}$ one can obtain a second Betti group $\mathscr{B}'$ of A by replacing the generator u_B of $\mathscr{B}$ by $u_B + x = u_{B'}$. In terms of $u_{B'}$, $\boldsymbol{w} = \boldsymbol{s}u_{B'}$. This shows that $\boldsymbol{t} = 1$.

Thus Lemma 5.3 is true.

Proof of (ii) *of Theorem* 5.2. According to Lemma 3.4, when $s \geq 1$ the torsion subgroup $\hat{T}$ of A/W satisfies the relation,

$$\hat{T} \approx \frac{\{u_1\} \oplus \mathscr{T}}{W}. \qquad (W = \{w\}) \tag{5.20}$$

Since $t = 1$ by hypothesis of (ii), (5.16) gives w the form $w = su_B = su_1$. Hence

$$\hat{T} \approx \frac{\{u_1\} \oplus \mathscr{T}}{\{su_1\} \oplus \{0\}} \approx C(s) \oplus \mathscr{T} \qquad \text{(by (2.3))} \tag{5.21}$$

where $C(s)$ is a cyclic group of order s.

Statement (ii) of Theorem 5.2 follows from (5.21).

Necessity of the conditions of Theorem 5.2 (i). By Lemma 5.1, $\Psi(u_B)$ is an element of A/W of order st. Hence if A/W is a prime-simple group, st is a prime-simple number, that is, s must be prime-simple and relatively prime to t. According to Lemma 5.3 it is accordingly necessary that $t = 1$.

Sufficiency of the conditions of Theorem 5.2 (i). The sufficiency follows from Theorem 5.2 (ii) and Theorem 1.4.

This completes the proof of Theorem 5.2.

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