Canad. Math. Bull. Vol. **45** (1), 2002 pp. 154–160

On the Poisson Integral of Step Functions and Minimal Surfaces

Allen Weitsman

Abstract. Applications of minimal surface methods are made to obtain information about univalent harmonic mappings. In the case where the mapping arises as the Poisson integral of a step function, lower bounds for the number of zeros of the dilatation are obtained in terms of the geometry of the image.

1 Introduction

Let f be a univalent harmonic mapping of the unit disk U. By this it is meant not only that f is $1-1$ and harmonic, but also that f is sense preserving. Then f can be written

$$f = h + \bar{g} \tag{1.1}$$

where h and g are analytic in U. If $a(\zeta)$ is defined by

$$a(\zeta) = \overline{f_{\bar{\zeta}}(\zeta)}/f_{\zeta}(\zeta) = g'(\zeta)/h'(\zeta), \tag{1.2}$$

then $a(\zeta)$ is analytic and $|a(\zeta)| < 1$ in U. We shall refer to $a(\zeta)$ as the *analytic dilatation* of f. General function theoretic properties of univalent harmonic mappings may be found in [CS-S]. The case where $a(\zeta)$ is a finite Blaschke product is of special interest since this case arises in taking the Poisson integrals of step functions [S-S]. This connection has been studied in [HS] and [S-S]. In [W], a method was developed using the theory of minimal surfaces to study univalent harmonic mappings. We shall continue this study.

In Section 2 we shall review the definitions of the height function and conjugate height function introduced in [W], along with their relevant properties. In Section 3 we shall prove the comparison principle for the height function. In Section 4 we collect some results from the theory of minimal surfaces which enable us to use the conjugate height function as a combinatorial tool. In Section 5 we give some applications to the Poisson integrals of step functions.

2 The Height Function and Conjugate Height Function

Using the Weierstrass representation [O, p. 63], we associate with f a minimal surface given parametrically in a simply connected subdomain $N \subseteq U$ where $a(\zeta)$ does not have a zero of odd order.

Received by the editors February 11, 2000.
AMS subject classification: 30C62, 31A05, 31A20, 49Q05.
Keywords: harmonic mappings, dilatation, minimal surfaces.

With g and h as in (1.1) we define up to an additive constant, a branch of

(2.1)
$$F(\zeta) = 2i \int \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta = 2i \int h'(\zeta)\sqrt{a(\zeta)}\, d\zeta = 2i \int f_\zeta(\zeta)\sqrt{a(\zeta)}\, d\zeta.$$

Then, by (1.2) it follows that a branch of F can be defined in N, and for $\zeta \in N$,

$$\zeta \to \big(f(\zeta), \operatorname{Re} F(\zeta)\big) \tag{2.2}$$

gives a parametric representation of a minimal surface. Here we have identified $\mathbb{R}^2$ with $\mathbb{C}$ by $(x, y) \leftrightarrow (\operatorname{Re} f, \operatorname{Im} f)$.

Let $\hat{U}$ be the Riemann surface of the function $\sqrt{a(\zeta)}$. Then $\hat{U}$ has algebraic branch points corresponding to those points $\zeta \in U$ for which $a(\zeta)$ has a zero of odd order. Specifically, $\hat{U}$ can be concretely described (the *analytic configuration* [S, pp. 69–74]) in terms of function elements (α, F_α) where $\alpha \in U$, and F_α is a power series expansion of a branch of F in a neighborhood of α if $a(\zeta)$ does not have a zero of odd order at $\zeta = \alpha$, and F_α a power series in $\sqrt{\zeta - \alpha}$ otherwise. The mapping $p\colon (\alpha, F_\alpha) \to \alpha$ is the *projection* of the surface so realized. The mapping F may now be lifted to a mapping $\hat{F}$ on $\hat{U}$.

By continuation, we may induce a mapping $\hat{U} \to \tilde{U}$ to a surface $\tilde{U}$ with a real analytic structure defined in terms of elements $(\beta, \tilde{F}_\beta)$ with $\beta \in f(U)$ by $\alpha = f^{-1}(\beta)$ and $\tilde{F}_\beta = F_\alpha \circ f^{-1}$. We again define a projection by $\pi\colon (\beta, \tilde{F}_\beta) \to \beta$.

We refer to a point $\hat{\zeta} \in \hat{U}$ to be over ζ, if $p(\hat{\zeta}) = \zeta$, and $\tilde{z} \in \tilde{U}$ to be over z if $\pi(\tilde{z}) = z$.

The harmonic mapping $f\colon U \to f(U)$ lifts to a mapping $\hat{f}\colon \hat{U} \to \tilde{U}$ which is $1-1$, onto, and satisfies the condition $\pi\big(\hat{f}(\hat{\zeta})\big) = f\big(p(\hat{\zeta})\big)$ for all $\zeta \in \hat{U}$. With these notations, we extend the meaning of (2.2). Thus

$$\hat{\zeta} \to \big(\hat{f}(\hat{\zeta}), \operatorname{Re} \hat{F}(\hat{\zeta})\big) \tag{2.3}$$

gives a parametric representation of a minimal surface in the sense that in a neighborhood of $\hat{\zeta} \in \hat{U} \setminus \mathcal{B}$ where $\mathcal{B}$ is the branch set, that is, the points above the zeros of a of odd order, then (2.2) is the same as (2.3) computed in terms of local coordinates given by projection.

We may also define the surface nonparametrically on $\tilde{U} \setminus \tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}} = \hat{f}(\mathcal{B})$, as follows. Let D be an open disk in $f(U)$ such that $f^{-1}(D)$ contains no zeros of a of odd multiplicity. Let $w = \varphi(x, y)$ be the nonparametric description of the minimal surface corresponding to (2.2), that is, for $\zeta \in f^{-1}(0)$ (*cf.* [HS3, p. 87]),

$$\begin{gathered} x = \operatorname{Re} f(\zeta) \quad y = \operatorname{Im} f(\zeta), \\ \varphi(x, y) = \operatorname{Re} F(\zeta). \end{gathered} \tag{2.4}$$

Then, by continuation φ lifts to a function $\tilde{\varphi}$ on $\tilde{U}$ which satisfies the minimal surface equation when computed in local coordinates given by projection off the branch set $\tilde{\mathcal{B}}$. We call $\tilde{\varphi}(\tilde{z})$ a *height function* corresponding to f. We define a *conjugate height function* $\tilde{\psi}(z)$ by solving locally

$$\psi_y = \varphi_x/W, \psi_x = -\varphi_y/W \quad \left(W = \sqrt{1 + \varphi_x^2 + \varphi_y^2}\right) \tag{2.5}$$

(*cf.* [JS, p. 326]), and lifting to $\tilde{U} \setminus \tilde{\mathcal{B}}$ as was done for φ. Let $\tilde{F} = \tilde{\varphi} + i\tilde{\psi}$. Then, as shown in [W], $\tilde{F} = \hat{F} + c$ is well defined on $\tilde{U} \setminus \tilde{B}$. Finally, we extend $\hat{F}$ and $\tilde{F}$ to $\hat{U}$ and $\tilde{U}$ respectively by continuity to the branch points.

A glossary of terminology is given schematically in Figure 1.

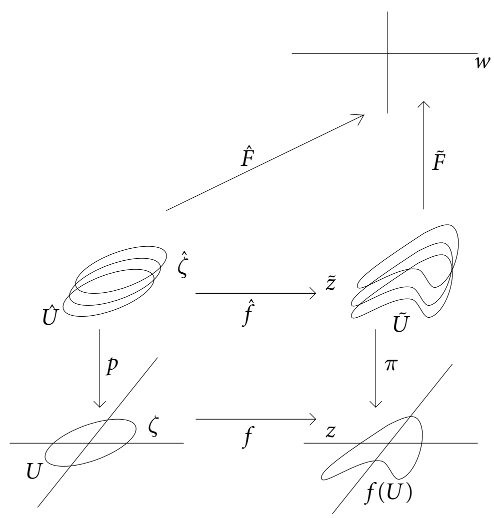

Figure 1

3 The Comparison Principle for the Height Function

In this section we point out that the comparison principle for solutions to the minimal surface equation

$$\operatorname{div} \frac{\nabla u}{W} = 0 \quad \left(W = \sqrt{1 + |\nabla u|^2}\right) \tag{3.1}$$

carries over to the height function corresponding to univalent harmonic mappings.

Let f_1 and f_2 be univalent harmonic mapping in U, and suppose $U_0 \subseteq f_1(U) \cap f_2(U) \neq \phi$, where U_0 is open, connected and bounded. Let $z_0 \in U_0$ not be the image of a branch point of f_1 or f_2, and in a neighborhood N of z_0 define $\varphi_1(x, y)$ and $\varphi_2(x, y)$ corresponding to f_1 and f_2 respectively, as is done for $\varphi(x, y)$ in (2.4). Let

https://doi.org/10.4153/CMB-2002-018-5 Published online by Cambridge University Press

$\Phi(x, y) = \varphi_1(x, y) - \varphi_2(x, y)$. We now consider continuations of Φ along all curves in U_0 emanating from N.

Theorem 1 *Suppose that* $\limsup \Phi \leq 0$ *for any continuation along curves from* N *in* U_0*, tending to points on* ∂U_0*. Then, the supremum* M *of any continuation of* Φ *along curves in* U_0 *also satisfies* $M \leq 0$.

Proof We may assume that f_1 and f_2 are univalent harmonic in $\bar{U}$ by considering $f_1(rz)$ and $f_2(rz)$ where $r < 1$, and letting $r \to 1$. Thus we may assume the analytic dilatations for f_1 and f_2 have finitely many zeros, so their height functions have only a finite number of different function elements over each point. Let $z = z(t)$, $0 \leq t < 1$, $z(0) = z_0$ be a curve along which the continuation of Φ tends to its supremum on some sequence. We assume, contrary to the theorem, that M is positive. By hypothesis there exists $z^* \in U_0$ such that $z(t_k) \to z^*$ and $\Phi\big(z(t_k)\big) \to M$ for some sequence $t_k \to 1$. In order to analyze this, we consider the individual continuations of φ_1 and φ_2 along the curve and recall that φ_1 and φ_2 have only a finite number of function elements over each point of U_0. Thus, at least for some subsequence $t_{k_n} \to 1$, the continuation of Φ corresponds to fixed function elements of φ_1 and φ_2 over z^*. For these branches we then consider $\Phi = \varphi_1 - \varphi_2$, and show that this determination of Φ cannot have a negative relative maximum at z^*. The theorem will then follow.

If φ_1 and φ_2 do not have a branch point at z^*, then the result follows from the usual comparison principle for solutions to (2.1) (*cf.* [O, p. 91]).

If φ_1 or φ_2 do have a branch point at z^*, then we analyze Φ on a two sheeted surface $\tilde{D}$ over a disk $D = \{|z - z^*| < \rho$ where we have assumed ρ is small enough so that z^* is the only branch point in D, and $\bar{D} \subseteq U_0$.

Following Nitsche [N], let m and ϵ be positive constants, with m large and ϵ small. In particular $\epsilon < \rho$. On $\tilde{D}$ define

$$\tau = \begin{cases} m - \epsilon & \text{if } \varphi_1 - \varphi_2 \geq m \\ \Phi - \epsilon & \text{if } \epsilon < \varphi_1 - \varphi_2 < m \\ 0 & \text{if } \varphi_1 - \varphi_1 \leq \epsilon \end{cases}. \tag{3.2}$$

Let $\tilde{C}_\epsilon = \partial \tilde{D}_\epsilon$ be oriented positively, where $\tilde{D}_\epsilon$ is the subset of $\tilde{D}$ over $D \setminus \{|z - z^*| < \epsilon\}$. Then, using coordinates given by projection on $\tilde{D}$ and W_1, W_2 as defined in (3.1) for φ_1 and φ_2 in place of u, we have by (3.1) and the divergence theorem that

$$\oint_{\tilde{C}_\epsilon} \tau\Big(\frac{\nabla\varphi_1}{W_1} \cdot \nu - \frac{\nabla\varphi_2}{W_2} \cdot \nu\Big)\, ds = \iint_{\tilde{D}_\epsilon} \nabla\tau \cdot \Big(\frac{\nabla\varphi_1}{W_1} - \frac{\nabla\varphi_2}{W_2}\Big)\, dA, \tag{3.3}$$

where ν is the outward unit normal.

It follows from (3.3) that

$$8\pi m\epsilon \geq \iint_{\tilde{D}_\epsilon} \nabla\tau \cdot \Big(\frac{\nabla\varphi_1}{W_1} - \frac{\nabla\varphi_2}{W_2}\Big)\, dA \geq 0.$$

Here we have also used the observation that with (3.2), the integrand is nonnegative. As $\epsilon \to 0$ the sets $\tilde{D}_\epsilon$ expand and we obtain the fact that $\nabla\varphi_1 = \nabla\varphi_2$ in the set $\{0 <$

https://doi.org/10.4153/CMB-2002-018-5 Published online by Cambridge University Press

$\varphi_1 - \varphi_2 < m\}$. Thus $\varphi_1 \equiv \varphi_2 +$ constant in any component of $\tilde{D}$ where $\varphi_1 > \varphi_2$. Since φ_1 and φ_2 are real analytic off the branch set, we obtain a contradiction unless the set where $\varphi_1 > \varphi_2$ is empty.

4 Combinatorial Properties of the Conjugate Height Function

In this section we shall collect the relevant facts which enable us to give combinatorial arguments using the conjugate height functions.

Let $\tilde{\psi}$ be a conjugate height function for a univalent harmonic mapping f in U. We assume in Theorems A, B, C below that $f(U)$ is bounded, and its analytic dilatation has finitely many zeros of odd order so that $\tilde{U}$ is finite sheeted. In the present context, [JS, p. 327] gives

Theorem A *Let $\tilde{C}$ be a simple piecewise smooth curve in the closure of $\tilde{U}$. Then, using the coordinates given by projection,*

$$\Big| \int_{\tilde{C}} d\tilde{\psi} \Big| \leq \text{length}\,(\tilde{C})$$

with strict inequality if any portion of $\tilde{C}$ lies over points in $\tilde{U}$.

Moreover [JS, Lemma 2], we have

Theorem B *If $\tilde{C}$ is a simple, piecewise smooth closed curve in the closure of $\tilde{U}$, then $\int_{\tilde{C}} d\tilde{\psi} = 0$.*

The companion to Theorems A and B is provided by Lemma 4 of [JS]. Again paraphrasing, in the current setting we have

Theorem C *Suppose that $\partial f(U)$ contains a line segment T, and $\tilde{T}$ is a segment over T in $\tilde{U}$. If $\tilde{T}$ is oriented so that the right hand normal to $\tilde{T}$ is the outer normal to $\tilde{U}$, and $\tilde{\varphi} = +\infty$ on $\tilde{T}$, then in the coordinates given by projection,*

$$\int_{\tilde{T}} d\tilde{\psi} = \text{length}\,(T).$$

The value of Theorems A, B, and C when applied to Poisson integrals of step functions stem from [W, Theorem 2]

Theorem D *Let $\mathcal{P}$ be a polygon having vertices $c_1, \dots, c_n$ given cyclically on $\partial\mathcal{P}$, and ordered by a positive orientation on $\partial\mathcal{P}$. Let f be a univalent harmonic mapping of U such that f is the Poisson integral of a step function having the ordered sequence $c_1, \dots, c_n$ as its values. Then the analytic dilatation $a(\zeta)$ of f is a finite Blaschke product of order at most $n-2$, $f(U) = \mathcal{P}$, with equality if $\mathcal{P}$ is convex. If $\tilde{\varphi}$ is a height function for f, then $\tilde{\varphi}$ tends to $+\infty$ or $-\infty$ at points over the open segments making up the sides of $\mathcal{P}$. At any vertex c_j of $\mathcal{P}$ at which the interior angle is less than π, then $+\infty$ and $-\infty$ alternate on adjacent sides having c_j as the common vertex.*

The proofs of Theorems A, B, and C are just as given in [JS]. The statement of Theorem D differs in the last sentence in [W, Theorem 2], but the statement given above is actually what is proved there.

https://doi.org/10.4153/CMB-2002-018-5 Published online by Cambridge University Press

5 Poisson Integrals of Step Functions

Throughout this section we use the notations of Theorem D. The classical Scherk surface arises from taking $c_1 = 0$, $c_2 = 1$, $c_3 = 1 + i$, $c_4 = i$ as vertices, and with the Poisson integral f taking those values on respective intervals of equal length, and ordered positively around ∂U. Then $a(\zeta) = c\zeta^2$, and the height function can be taken as a saddle with heights $\pm\infty$ alternately over the sides of the square. As forecast by Theorem D, $a(\zeta)$ has two zeros (counting multiplicity). In order to motivate the general phenomenon, suppose we extend the top and bottom sides of the square to make a rectangle R, and stretch or shrink the corresponding intervals for the new f in ∂U in any fashion. Still, since R is convex, the analytic dilatation will have two zeros ζ_1, ζ_2. As we shall see in the proof of Theorem 2, the images $z_1 = f(\zeta_1)$, $z_2 = f(\zeta_2)$ of these points lie cannot both be to the left or right of the center of R, regardless of the relative sizes of the intervals in ∂U corresponding to the vertices of R.

In general, if f comes from the Poisson integral of a step function mapping U onto a polygon $\mathcal{P}$ with the values of the step function being vertices $c_1, \dots, c_n$, then its analytic dilatation $a(\zeta)$ has at most $n - 2$ zeros in U. Theorem 2 shows that if we have some knowledge of $\partial\mathcal{P}$, we can say more.

Theorem 2 *With f, a, $\mathcal{P}$, and $c_1, \dots, c_n$ as above, suppose that the interior angles at c_j and c_{j+1} are less than π. Let d_j, d_{j+1} be points on the segments $c_{j-1}c_j$ and $c_{j+1}c_{j+2}$ respectively, and assume that the open quadrilateral $\mathcal{P}_j$ with vertices d_j, c_j, c_{j+1}, d_{j+1} is contained in $f(U)$. If* length (d_jd_{j+1}) + length (c_jc_{j+1}) < length (d_jc_j)+ length $(c_{j+1}d_{j+1})$, *then $\mathcal{P}_j$ contains the image of at least one zero of $a(\zeta)$ of odd order. Thus, in particular, if $\mathcal{P}$ has k disjoint such quadrilaterals, then $a(\zeta)$ has at least k zeros.*

Proof Suppose that $\mathcal{P}_j$ were such a quadrilateral without a branch point. Then there would exist a single valued branch of the height function $\tilde{\varphi}$ over $\mathcal{P}_j$ whose values over d_jc_j, c_jc_{j+1}, $c_{j+1}d_{j+1}$ would alternate $\pm\infty$ by Theorem D. Let $\tilde{\gamma}$ be the boundary of a quadrilateral $\tilde{\mathcal{P}}_j$ over $\mathcal{P}_j$ in $\tilde{U}$, oriented positively. We have from Theorem B that

$$\int_{\tilde{\gamma}} d\tilde{\psi} = 0. \tag{5.1}$$

By the alternation of signs, if $\tilde{\gamma}_1$ is the edge over d_jc_j, and $\tilde{\gamma}_3$ is over $c_{j+1}d_{j+1}$, then by Theorem C,

$$\Big|\int_{\tilde{\gamma}_1} d\tilde{\psi} + \int_{\tilde{\gamma}_3} d\tilde{\psi}\Big| = \text{length}\,(d_jc_j) + \text{length}\,(c_{j+1}d_{j+1}). \tag{5.2}$$

From (5.1) and (5.2) we then obtain

$$\text{length}\,(d_jc_j) + \text{length}\,(c_{j+1}d_{j+1}) \le \Big|\int_{\tilde{\gamma}_2} d\tilde{\psi}\Big| + \Big|\int_{\tilde{\gamma}_4} d\tilde{\psi}\Big|, \tag{5.3}$$

where $\tilde{\gamma}_2$ is over c_jc_{j+1}, and $\tilde{\gamma}_4$ is over d_jd_{j+1}.

Again, by Theorem C,

$$\Big|\int_{\tilde{\gamma}_2} d\tilde{\psi}\Big| = \text{length}\,(c_j c_{j+1}), \tag{5.4}$$

and by Theorem A,

$$\Big|\int_{\tilde{\gamma}_4} d\tilde{\psi}\Big| \le \text{length}\,(d_j d_{j+1}). \tag{5.5}$$

Combining (5.3)–(5.5), we contradict the hypothesis of the theorem. Thus, $\mathcal{P}_j$ must contain the image of at least one zero of odd order of $a(\zeta)$.

References

[C] G. Choquet, *Sur un type de transformation analytique généralisant la représentation conforme et définie an moyen de fonctions harmoniques*. Bull. Sci. Math. **69**(1945), 156–165.

[CS-S] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*. Ann. Acad. Sci. Fenn. Ser. A I Math. **9**(1984), 3–25.

[HS] W. Hengartner and G. Schober, *On the boundary behavior of orientation-preserving harmonic mappings*. Complex Variables **5**(1986), 197–208.

[JS] H. Jenkins and J. Serrin, *Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation*. Arch. Rat. Mech. Anal. **21**(1965/66), 321–342.

[K] H. Kneser, *Lösung der Aufgabe 41*. Jahresber. Deutsch. Math.-Verein. **35**(1925), 123–4.

[N] J. C. C. Nitsche, *Über ein verallgemeinertes Dirichletsches Problem für die Minimalflächengleichung und hebbare Unstetigkeiten ihrer Lösungen*. Math. Ann. **158**(1965), 203–214.

[O] R. Osserman, *A Survey of Minimal Surfaces*. Dover, 1986.

[S] G. Springer, *Introduction to Riemann Surfaces*. Addison-Wesley, 1957.

[S-S] T. Sheil-Small, *On the Fourier series of a step function*. Michigan Math. J. **36**(1989), 459–475.

[W] A. Weitsman, *On Univalent Harmonic Mappings and Minimal Surfaces*. Pacific Math. J. **192**(2000), 191–200.

Department of Mathematics
Purdue University
West Lafayette, Inidiana 47907
U.S.A.

https://doi.org/10.4153/CMB-2002-018-5 Published online by Cambridge University Press