The centre of perspective for the two tetrahedra $A_1B_1C_1D_1 : A_3B\,C_3D_3$ is the point where coordinates are $\alpha = \frac{1}{\rho_1^2+\rho^2}\cdot\left[\rho^2\cdot\frac{3V_1}{A}+\rho_1^2\cdot\frac{3(V-2V_1)}{2A}\right]$ etc. This point is the pole of the plane whose equation is $\Sigma A\,\alpha\,(b^2+c^2-a^2)=O$ with respect to the sphere circumscribing the tetrahedron $ABCD$.

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Some properties of the paraboloid $z = x^2 + y^2$

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In a recent paper[1], I showed how the properties of algebraic systems of circles in the (x, y) plane could be investigated by means of a representation in which to the circle $x^2 + y^2 - 2px - 2qy + r = 0$ there corresponds the point (p, q, r) in space of three dimensions. The plane of (x, y) may be considered to lie in the space (x, y, z), so that the centre of the mapped circle is the orthogonal projection of the representative point.

Point-circles, or circles of zero radius, are mapped by points on the paraboloid of revolution $z = x^2 + y^2$, and this quadric, which we call Ω, plays a fundamental part in the representation. An algebraic curve C in the three-dimensional space S_3 represents an algebraic system of circles in the (x, y) plane, and it was shown that the envelope of this system of circles is found by projecting orthogonally on to the (x, y) plane the curve in which the polar lines with regard to Ω of the tangents to C meet Ω.

This representation offers a convenient method for obtaining what has recently been called the "circle-tangential equation" of a given plane curve[2]. Suppose we are given a plane algebraic curve, of equation $f(x, y) = 0$. This curve is touched by an infinity of circles through the point $(0, 0)$. If such a circle is $x^2 + y^2 - 2px - 2qy = 0$, a relation $g(p, q) = 0$ holds; *i.e.* the centres of these contact circles lie on an algebraic curve of equation $g(x, y) = 0$. This is called the *circle-tangential equation* of the given curve.

As a trivial example, the circle-tangential equation of a point (a, b) is that of the perpendicular bisector of the join of this point to the origin. That of a line gives the parabola for which the origin is a focus and the line the directrix. When we come to the circle $x^2 + y^2 - 2px - 2qy + r = 0$ there are two cases to consider. If the origin lies inside the circle, the circle-tangential locus is an ellipse with real foci at the points $(0, 0)$ and (p, q), but if the origin lies outside, the locus is an hyperbola with the same two foci. These two cases may be distinguished by the condition $r \lessgtr 0$.

The representation described above gives a geometric method for obtaining the circle-tangential locus of a curve $f(x, y) = 0$. The locus $g(x, y) = 0$ represents an algebraic system of circles with envelope f. The tangents to g have for polar lines with respect to Ω a cone of lines, vertex $(0, 0)$. This cone meets Ω in a curve of which f is the orthogonal projection on to $z = 0$. So that, to obtain g, we project f up on to Ω, obtaining a curve D, say. The tangent planes to Ω at the points of D meet $z = 0$ in the tangents of the required circle-tangential locus. We obtain a number of simple but interesting properties of Ω when we apply this general construction to the simple cases we have mentioned.

For the circle-tangential locus of a point (a, b), we consider the intersection with $z = 0$ of the tangent plane to Ω at the point $(a, b, a^2 + b^2)$. This intersection must be the perpendicular bisector of the join of (a, b) to the origin.

If we suppose (a, b) to move on a curve f, our construction for the circle-tangential locus g yields the theorem "The locus of the centres of circles through the origin which touch the curve f is also the envelope of the perpendicular bisectors of the joins of the origin to points on f." An analytical proof is given in the paper mentioned above[2].

The projection on to Ω of the circle $x^2 + y^2 - 2px - 2qy + r = 0$ is the intersection with Ω of the plane $2px + 2qy - z - r = 0$. Tangent planes to Ω at the points of this plane section all pass through the point (p, q, r), and meet $z = 0$ where the tangent cone to Ω from (p, q, r) meets this plane. So that the circle-tangential locus of the circle is obtained by putting $z = 0$ in the equation

$$4(p^2 + q^2 - r)(x^2 + y^2 - z) - (2px + 2qy - z - r)^2 = 0.$$

The curve obtained is a conic, with foci at $(0, 0)$ and (p, q). The presence of the focus at the origin is manifest, since the equation is

$$x^2 + y^2 = \frac{(p^2 + q^2)}{(p^2 + q^2 - r)} \left\{\frac{2px + 2qy - r}{2\sqrt{p^2 + q^2}}\right\}^2.$$

The corresponding directrix is the intersection with $z = 0$ of the polar plane of the point (p, q, r). The eccentricity of the conic is $\gtrless 1$ according as $r \gtrless 0$, in accordance with our previous remark. We have found then that the tangent cone to Ω from the point (p, q, r) meets $z = 0$ in a conic with foci at $(0, 0)$ and (p, q), and this conic is an ellipse or hyperbola according as r is negative or positive.

2. Another interesting application of the envelope theorem mentioned at the beginning of this note is to the circles of curvature of a given plane curve $f(x, y) = 0$. The circles of curvature form an algebraic system of circles with envelope f. Suppose that this system is represented by the space curve C. Then since a circle of curvature at a general point P of f touches f at P and at no other point, the polar line of t, the tangent to C at a general point Q of C, will not intersect Ω in two distinct points, but will *touch* Ω at a point P' whose orthogonal projection on the (x, y) plane is P. It follows that t also touches Ω at P'. The orthogonal projections of t and of the polar line of t are the tangent and normal of f at P respectively. To obtain the system of circles of curvature of f we need only project f up on to Ω, obtaining a space curve f', and consider the system of tangent planes to Ω at points of f'. The envelope of this system is a developable surface, and the *edge of regression* on this surface represents the system of the circles of curvature of the curve f. Now a general point on the edge of regression is the intersection of three "consecutive" tangent planes to Ω at the point P' on f'. It is easily verified that the points of the tangent plane to Ω at P' represent the circles of the (x, y) plane which pass through P. So that the point on the edge of regression corresponding to P represents the circle through three "consecutive" points of f.

Applying the above conclusions to simple cases we obtain examples of developable surfaces circumscribed to Ω. For example, the curve

$$x = \frac{(a^2-b^2)}{a}\cos^3\phi,\ y = -\frac{(a^2-b^2)}{b}\sin^3\phi,\ z = (a^2-2b^2)\cos^2\phi + (b^2-2a^2)\sin^2\phi$$

represents the system of the circles of curvature of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The tangent to this curve at 'ϕ' must touch the paraboloid $z = x^2 + y^2$ at the point $(a\cos\phi,\ b\sin\phi,\ a^2\cos^2\phi + b^2\sin^2\phi)$. But points on the tangent have coordinates

$$\left(x + \frac{\lambda}{a}\cos\phi,\ y + \frac{\lambda}{b}\sin\phi,\ z + 2\lambda\right)$$

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where λ is a variable parameter. The quadratic $p\lambda^2 + 2q\lambda + r = 0$ where

$$p = \frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}, \; q + 1 = (a^2 - b^2)\left[\frac{\cos^4\phi}{a^2} - \frac{\sin^4\phi}{b^2}\right]$$

$$r = (a^2 - b^2)^2\left(\frac{\cos^6\phi}{a^2} + \frac{\sin^6\phi}{b^2}\right) + (2b^2 - a^2)\cos^2\phi + (2a^2 - b^2)\sin^2\phi$$

must therefore be a perfect square. Similar examples are easily constructed.

3. Points which are inverse points with respect to the circle $x^2 + y^2 - 2ax - 2\beta y + \gamma = 0$ are represented by points on Ω collinear with the point (α, β, γ)[1]. This fact enables us to find the equations of curves of any given order anallagmatic (self-inverse) in the given circle. A plane through (α, β, γ) is

$$p(x - \alpha) + q(y - \beta) + r(z - \gamma) = 0.$$

This plane intersects Ω in a conic, the orthogonal projection of which on the (x, y) plane is the circle

$$p(x - \alpha) + q(y - \beta) + r(x^2 + y^2 - \gamma) = 0.$$

This circle is anallagmatic in, and therefore orthogonal to the given circle. A quadric cone with vertex at (α, β, γ) is given by the equation

$$a(x - \alpha)^2 + 2h(x - \alpha)(y - \beta) + b(y - \beta)^2$$
$$+ 2g(x - \alpha)(z - \gamma) + 2f(y - \beta)(z - \gamma) + c(z - \gamma)^2 = 0$$

and therefore the quartic curves anallagmatic in the given circle are obtained by merely substituting $z = x^2 + y^2$ in this equation. They are, of course, bicircular quartics.

To obtain the anallagmatic cubics, the quartic curve of intersection of the quadric cone vertex (α, β, γ) with Ω must have one point at infinity on the axis of Ω. Therefore one generator of the cone must be parallel to $x = y = 0$. We therefore put $c = 0$ in the above equation to obtain anallagmatic cubics. This process can easily be extended.

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