A NOTE ON INDUCED MODULES

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1. Introduction. In this paper, A denotes a ring with an identity element 1, and B a subring of A containing 1 such that B satisfies the left and right minimum conditions, and A is a finitely generated left and right B-module. The identity element 1 is required to act as the identity operator on all modules which we shall consider. For any left B-module V, there is a standard construction of a left A-module which is, roughly speaking, the smallest A-module containing V. Namely, we form the tensor product group $A \otimes_B V$, and define the module operations in this group according to the rule

$$(1) \qquad a\Big(\sum a_i \otimes v_i\Big) = \sum aa_i \otimes v_i, \qquad a, a_i \in A, \qquad v_i \in V.$$

If a is taken from B instead of A, then (1) defines the structure of a left B-module on $A \otimes_B V$, and there is a natural B-homomorphism

$$(2) \qquad \epsilon: v \longrightarrow 1 \otimes v, \qquad v \in V,$$

of V into $A \otimes_B V$, such that $A \otimes_B V$ is generated, as an A-module, by $\epsilon(V)$. In case A is the group algebra of a finite group, and B is the group algebra of a subgroup H of G, the representation of G afforded by the module $A \otimes_B V$ is the induced representation, defined first by Frobenius, of the representation of H afforded by V. The theory of *induced modules* $A \otimes_B V$ in general has been treated extensively by Higman **(3)**, **(4)**, and Hochschild **(5)**.

The purpose of this note is to investigate the following question.

(I) Let V be a left B-module. Does there exist a B-homomorphism π of $A \otimes_B V$ onto V such that $\pi\epsilon = 1$, where ϵ is given by (2)?

The existence of π is clearly equivalent to the requirement that ϵ map V monomorphically (that is, with kernel zero) onto a B-direct summand of $A \otimes_B V$.

The condition (I) is satisfied for all left B-modules V in case B is a semi-simple ring. Higman has observed in **(2)** that (I) holds for all left B-modules V whenever A is the group algebra of a finite group over an arbitrary field, and B the group algebra of a subgroup of G. The question (I) for general non-semi-simple rings B is of interest for the following reason. Following Jans **(6)**, we say that a ring A with left minimum condition has *unbounded representation type* if there exist indecomposable left A-modules with arbitrarily long composition series. Jans **(6)** and others have discovered criteria for A to be of unbounded representation type in case A is a finite dimensional

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algebra over a field. Little seems to be known, however, about the following question.

(II) Let B be a subring of A, and suppose that B has unbounded representation type. Does A then have unbounded representation type also?

In § 3 we shall give examples to show that in general the answers to both questions are negative. Our contribution to the study of (II) is the remark, already made in the group algebra case by Higman **(2)** that an affirmative answer to (I) for all left B-modules V implies an affirmative answer to (II). To prove this assertion, let V be an indecomposable left B-module which possesses a composition series. Then $A \otimes_B V$ is a finitely generated left B-module, and has a composition series both as a left B-module and as a left A-module. Then $A \otimes_B V$ is a direct sum of a finite number of indecomposable left A-modules $U_1, \ldots, U_r$. Each U_i in turn is a direct sum of indecomposable B-submodules. By (I), $A \otimes_B V$ has a B-direct summand isomorphic to V. Therefore, by the Krull–Schmidt Theorem, some U_{i_0} has a B-direct summand isomorphic to V, and hence U_{i_0} has a composition series as a B-module at least as long as a composition series for V. Because any irreducible left A-module is a homomorphic image of the finitely generated left B-module A, we see that the B-composition length of all irreducible left A-modules is bounded by the B-composition length of A. Combining our remarks, we conclude that if B has indecomposable modules with arbitrarily long composition series, the same assertion holds for A, as we wished to prove.

2. Main results.

We give some sufficient conditions for condition (I) to hold. The first is rather trivial, but it includes the group algebra case. By a *projection* of A upon B we shall mean an endomorphism ϕ of the additive group of A such that $\phi(A) = B$, and $\phi(b) = b$ for all $b \in B$.

THEOREM 1. *Suppose there exists a projection τ of A upon B which is both a left and right B-homomorphism. Then condition (I) holds for every left B-module V.*

Proof. Let ϵ be the homomorphism of V into $A \otimes_B V$ given by (2). Then $\pi = \tau \otimes 1$ defines a homomorphism of $A \otimes_B V$ onto V such that $\pi\epsilon = 1$, because τ is a right B-homomorphism of A upon B which reduces to the identity on B. Because τ is also a left B-homomorphism of A onto B, π is a B-homomorphism, and the Theorem is proved.

COROLLARY. (*Higman* **(2)**.) *Let $A = KG$ be the group algebra of a finite group G over a field K, and let $B = KH$ be the group algebra of a subgroup H of G. Then (I) holds for every left B-module V.*

Proof. Let $g_1 = 1, g_2, \ldots, g_r$ be a set of representatives of the left cosets g_iH of H in G. Then A is a free right B-module with basis $g_1, \ldots, g_r$. Then the mapping $\tau: \sum g_i b_i \to b_1$, $b_i \in B$, is a projection of A upon B which satisfies the hypothesis of Theorem 1.

For the rest of the section we assume that there exists a projection λ of A upon B which is assumed only to be a left B-homomorphism. This hypothesis is automatically satisfied, for example, whenever B is a quasi-Frobenius ring, since a quasi-Frobenius ring B is an injective left (as well as right) B-module (see **(1)**). With the projection λ we shall associate a two-sided ideal I_λ in B which measures the extent to which λ fails to be a right B-homomorphism. We begin by defining for each $a \in A$ and $b \in B$ the element

$$f(a, b) = \lambda(ab) - \lambda(a)b$$

of B. We then define I_λ to be the set of all finite sums $\sum f(a_i, b_i)$, with a_i in A and b_i in B. The function f satisfies the conditions

$$f(-a, b) = f(a, -b) = -f(a, b),$$

$$bf(a_i, b_i) = f(ba_i, b_i),$$

and

$$f(a_i, b_i)b = f(a_i, b_ib) - f(a_ib_i, b),$$

for all $a, a_i \in A$ and $b, b_i \in B$. From these formulas it follows at once that I_λ is a two-sided ideal.

Our main result can be stated as follows.

THEOREM 2. *Let B be a subring of A such that A is a projective right B-module, and let λ be a left B-projection of A upon B with associated ideal I_λ in B. Then for every left B-module V such that $I_\lambda V = 0$, the mapping $\epsilon : v \to 1 \otimes v$ maps V monomorphically onto a B-direct summand of $A \otimes_B V$.*

Proof. We have to prove that there exists a B-homomorphism π of $A \otimes_B V$ onto V such that $\pi\epsilon = 1$. Because A is a projective right B-module, the first theorem of **(7)** implies that A is a direct sum of submodules a_iB, $1 \leqslant i \leqslant s$, such that each a_iB is B-isomorphic to a right ideal e_iB in B generated by an idempotent e_i. The isomorphism θ_i of a_iB onto e_iB can be chosen so that $\theta_i(a_i) = e_i$, $1 \leqslant i \leqslant s$, and it follows that $a_ie_i = a_i$ for each i. From the properties of tensor products, it follows that $A \otimes_B V$ can be expressed as a direct sum

$$A \otimes_B V = \sum_{i=1}^{s} \oplus\ (a_iB \otimes V) = \sum_{i=1}^{s} \oplus\ (a_i \otimes V). \tag{3}$$

For any element $x = \sum a_i \otimes v_i$, $v_i \in V$, of $A \otimes_B V$, we define

$$\pi(x) = \sum_{i=1}^{s} \lambda(a_i)e_iv_i. \tag{4}$$

First we check that π is well-defined. If $x = 0$, then because of the direct sum decomposition (3) we have $a_i \otimes v_i = 0$ for $i = 1, \ldots, s$. Moreover, since $\theta_i(a_ib)v = \theta_i(a_i)bv$, $b \in B$, $v \in V$, there exists a homomorphism σ_i of $a_iB \otimes V$ into V such that $\sigma_i(a_ib \otimes v) = \theta_i(a_ib)v$, $b \in B$, $v \in V$, and we have

$$0 = \sigma_i(a_i \otimes v_i) = \theta_i(a_i)v_i = e_iv_i.$$

Then from (4) we have $\pi(x) = 0$. π is obviously additive. We next verify that $\pi\epsilon = 1$. We can express the identity element 1 in A in the form $1 = \sum a_i b_i$, where we may assume that $e_i b_i = b_i$, $1 \leqslant i \leqslant s$. Then for $v \in V$, we have

$$\pi\epsilon(v) = \pi(1 \otimes v) = \pi\Big(\sum a_i \otimes b_i v\Big) = \sum \lambda(a_i) e_i b_i v = \sum \lambda(a_i) b_i v$$
$$= \sum (\lambda(a_i b_i) - f(a_i, b_i))v = \sum \lambda(a_i b_i) v = \lambda(1) v = v$$

since $\lambda(1) = 1$, and $I_\lambda V = 0$. Finally we prove that π is a B-homomorphism. Let $b \in B$; then we have for each i,

$$b a_i = \sum_{j=1}^{s} a_j \beta_{ji},$$

where the $\beta_{ji} \in B$, and we may assume that $e_j \beta_{ji} e_i = \beta_{ji}$ for all i and j. Then we have

$$b\pi\Big(\sum a_i \otimes v_i\Big) = b \sum \lambda(a_i) e_i v_i = \sum \lambda(b a_i) e_i v_i,$$

while on the other hand we have

$$\pi\Big(b \sum a_i \otimes v_i\Big) = \pi\Big(\sum_i \sum_j a_j \beta_{ji} \otimes v_i\Big) = \pi\Big(\sum_j a_j \otimes \sum_i \beta_{ji} v_i\Big)$$
$$= \sum_j \lambda(a_j) e_j \Big(\sum_i \beta_{ji} v_i\Big) = \sum_j \sum_i \lambda(a_j) \beta_{ji} e_i v_i$$
$$= \sum_j \sum_i (\lambda(a_j \beta_{ji}) - f(a_j, \beta_{ji})) e_i v_i = \sum_i \lambda\Big(\sum_j a_j \beta_{ji}\Big) e_i v_i$$
$$= \sum_i \lambda(b a_i) e_i v_i = b\pi\Big(\sum a_i \otimes v_i\Big),$$

since $f(a_j, \beta_{ji}) \in I_\lambda$ and $I_\lambda V = 0$. This completes the proof of the theorem.

From Theorem 2 and the remarks in § 1, we have the following corollary.

COROLLARY 1. *Let A and B satisfy the hypothesis of Theorem* 2, *and suppose that B/I_λ has unbounded representation type. Then A has unbounded representation type.*

COROLLARY 2. *Let A be a projective right B-module, where B is a quasi-Frobenius ring. Moreover, for some left B-projection λ of A upon B, let $NI_\lambda = 0$, where N is the radical of B. Then for every finitely generated left B-module V, $\epsilon: v \to 1 \otimes v$ maps V monomorphically onto a B-direct summand of $A \otimes_B V$.*

Proof. Because V is finitely generated, V is a direct sum of indecomposable left B-modules V_i, $1 \leqslant i \leqslant t$. If we can prove Corollary 2 for each V_i, then it is clear that Corollary 2 will hold for V. Therefore we may assume that V is indecomposable. The hypothesis that $NI_\lambda = 0$ implies that I_λ is a sum of minimal left ideals in B. We prove first that $I_\lambda V \neq 0$ implies that V is injective. This result is a familiar one in the theory of quasi-Frobenius rings (see **(8)**), but for the sake of completeness we sketch the proof. We have $I_\lambda = \sum I_\lambda e_i$, where the e_i are primitive idempotents in B. Because $NI_\lambda = 0$, $I_\lambda e_i \neq 0$

implies that $I_\lambda e_i$ is the unique minimal subideal in the indecomposable left ideal Be_i of B. Now $I_\lambda V \neq 0$ implies that for some e_i, $I_\lambda e_i V \neq 0$, and there exists $v \in V$ such that $I_\lambda e_i v$ is a non-zero submodule of V. The mapping $be_i \to be_i v$ is a B-homomorphism of Be_i onto $Be_i v$, and since Be_i has a unique minimal subideal not contained in the kernel of the homomorphism, it follows that $Be_i v$ is isomorphic to Be_i. On the other hand, Be_i is an injective left B-module; hence $Be_i v$ is an injective submodule of V. Since V is indecomposable we must have $Be_i v = V$, and V is injective. We have now shown that for a given indecomposable left B-module V, either $I_\lambda V = 0$ and Theorem 2 applies to V, or V is injective. It remains to prove that if V is injective, then $\epsilon: v \to 1 \otimes v$ maps V monomorphically onto a B-direct summand of $A \otimes_B V$. Because B is quasi-Frobenius, B is also injective as a right B-module, and there exists a right B-projection ρ of A upon B. Then $(\rho \otimes 1)\epsilon = 1$, and it follows that ϵ is a monomorphism of V into $A \otimes_B V$. Because $\epsilon(V)$ is injective, $\epsilon(V)$ is a B-direct summand of $A \otimes_B V$, and Corollary 2 is proved.

3. Examples. First we give an example to show that in general the answer to (II) is "no" even when both A and B are quasi-Frobenius rings. Let K be any field of characteristic $p > 0$, and let B be the group algebra over K of any finite group with a non-cyclic p-Sylow subgroup. Higman has proved in **(2)** that B has unbounded representation type. Moreover B is quasi-Frobenius (in fact a symmetric algebra), and can be imbedded in the algebra A consisting of all n by n matrices over K, where n is the dimension of B over K. But A is a simple algebra, and has only one indecomposable module. Therefore the answer to (II) is negative in this case, and there must also exist left B-modules for which (I) does not hold either.

Finally we give an example of a pair (A, B), with B quasi-Frobenius, and A a free right B-module, such that for some left B-projection λ of A onto B, we have $I_\lambda = B$. We show, furthermore, that in this case there do exist left B-modules V such that $\epsilon(V)$ is not a direct summand of $A \otimes_B V$.

Let K be an arbitrary field, and let n be an even integer, $n \geqslant 2$. Let I denote the n by n identity matrix, and J the n by n matrix $e_{21} + e_{32} + \ldots + e_{n,n-1}$, where e_{ij} denotes the matrix with a 1 in the (i, j) position and zeros elsewhere. Let A be the algebra of $2n$ by $2n$ matrices generated by the identity matrix 1, and the matrices

$$a = \begin{pmatrix} 0 & I \\ I & I \end{pmatrix}, \qquad b = \begin{pmatrix} J & I + J \\ 0 & -J \end{pmatrix},$$

where the entries in a and b stand for n by n blocks. Then we have

$$a^2 = a + 1, \qquad b^{n-1} \neq 0, \qquad b^n = 0, \qquad ab + ba = b + 1.$$

Let $B = K[b]$; then B is a quasi-Frobenius subalgebra of A, and A is a free right B-module with basis $\{1, a\}$. The elements $\{1, a\}$ form also a basis for A as a free left B-module, so that every element in A can be expressed uniquely in the form $\beta_0 + \beta_1 a$, with β_0 and β_1 in B. Define a mapping $\lambda: A \to B$ by setting $\lambda(\beta_0 + \beta_1 a) = \beta_0$. Then λ is a left B-homomorphism of A onto B whose restriction to B is the identity mapping. The ideal I_λ defined by λ contains

$$\lambda(ab) - \lambda(a)b = \lambda(-ba + b + 1) = b + 1,$$

which is an invertible element in B since b is nilpotent. Therefore $I_\lambda = B$. Now let L be a left ideal in B; then $AL \cong A \otimes_B L$, and the mapping $\epsilon: l \to l$ is a B-homomorphism of $L \to AL$. In particular, let $L = Kb^{n-1}$; then $AL = Kb^{n-1} + Kab^{n-1}$. Let M be a left B-submodule of AL not contained in L. Then M must contain an element

$$m = \xi b^{n-1} + \eta ab^{n-1}, \qquad \xi, \eta \in K, \qquad \eta \neq 0.$$

Then

$$bm = \eta(-ab + b + 1)b^{n-1} = \eta b^{n-1} \in M,$$

and $M \cap \epsilon(L) \neq 0$. Therefore $\epsilon(L)$ is not a left B-direct summand of AL.

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