J. Aust. Math. Soc. **92** (2012), 385–390
doi:10.1017/S144678871200033X

A SHORT NOTE ON THE THOM–BOARDMAN SYMBOLS OF DIFFERENTIABLE MAPS

YULAN WANG, JIAYUAN LIN[✉] and MAORONG GE

(Received 19 April 2011; accepted 19 September 2012; first published online 4 March 2013)

Communicated by M. K. Murray

Abstract

It is well known that Thom–Boardman symbols are realized by nonincreasing sequences of nonnegative integers. A natural question is whether the converse is also true. In this paper we answer this question affirmatively, that is, for any nonincreasing sequence of nonnegative integers, there is at least one map-germ with the prescribed sequence as its Thom–Boardman symbol.

2010 *Mathematics subject classification*: primary 58K40, 58K20; secondary 32S10, 14J17.
Keywords and phrases: Thom–Boardman symbol, Jacobian extension, critical extension.

1. Introduction

Thom–Boardman symbols were first introduced by R. Thom and later generalized by J. M. Boardman to classify singularities of differentiable maps. They are realized by nonincreasing sequences of nonnegative integers. Although Thom–Boardman symbols have been around for over 50 years, computing those numbers is extremely difficult in general. Before Lin and Wethington [3] proved R. Varley's conjecture on the Thom–Boardman symbols of polynomial multiplication maps, there were only sporadic known results. Lin and Wethington [3] provide infinitely many examples of map-germs with distinct Thom–Boardman symbols. Since Thom–Boardman symbols are given by nonincreasing sequences of nonnegative integers, a natural question is whether the converse is also true. In this paper we answer this question affirmatively. We prove that for any given nonincreasing sequence of nonnegative integers, there is at least one map-germ with the prescribed sequence as its Thom–Boardman symbol.

For the reader's convenience, let us briefly recall the definition of Thom–Boardman symbols from [2].

Let $x_1, \dots, x_m$ be local coordinates on a differential manifold M of dimension m. Denote by $\mathcal{A}$ the local ring of germs of differentiable functions at a point $x \in M$. For any ideal B in $\mathcal{A}$, the *Jacobian extension*, $\Delta_k B$, is the ideal spanned by B and all the minors of order k of the *Jacobian matrix* $(\partial\phi_i/\partial x_j)$, denoted by δB, formed from partial derivatives of functions ϕ_i in B. We say that $\Delta_i B$ is *critical* if $\Delta_i B \neq \mathcal{A}$ but $\Delta_{i-1} B = \mathcal{A}$ (just $\Delta_1 B \neq \mathcal{A}$ when $i = 1$). That is, the *critical extension* of B is B adjoined with the least-order minors of the Jacobian matrix of B for which the extension does not coincide with the whole algebra.

Suppose that N is another differential manifold of dimension n and let $y_1, \ldots, y_n$ be local coordinates on it. For a differential map $F : M \to N$, $F = (f_1, f_2, \ldots, f_n)$, we denote by J the ideal generated by $f_1, \ldots, f_n$ in $\mathcal{A}$. Then $\Delta_k J$ is spanned by J and all the minors of order k of the *Jacobian matrix* $\delta J = (\partial f_i/\partial x_j)$.

Now we shift the lower indices to upper indices of the critical extensions by the rule $\Delta^i J = \Delta_{m+1-i} J$. We repeat the process described above with the resulting ideals until we have a sequence of critical extensions of J,

$$J \subseteq \Delta^{i_1} J \subseteq \Delta^{i_2}\Delta^{i_1} J \subseteq \cdots \subseteq \Delta^{i_k}\Delta^{i_{k-1}} \cdots \Delta^{i_1} J \subseteq \cdots .$$

The nonincreasing sequence $(i_1, i_2, \ldots, i_k, \ldots)$ is called the *Thom–Boardman symbol* of J, denoted by $TB(J)$. The purpose of switching the indices is that doing so allows us to express $TB(J)$ as follows:

$$i_1 = \operatorname{corank}(J), i_2 = \operatorname{corank}(\Delta^{i_1} J), \ldots, i_k = \operatorname{corank}(\Delta^{i_{k-1}} \cdots \Delta^{i_1} J), \ldots$$

where the rank of an ideal is defined to be the maximal number of independent coordinates from the ideal and the corank is the number of variables minus the rank.

We also need the following construction from [3]. Let M_n be the set of monic complex polynomials in one variable of degree n. $M_n \cong \mathbb{C}^n$ by the map sending $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ to the n-tuple $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$.

If we take $f(x)$ of degree n as above and $g(x) = x^r + b_{r-1}x^{r-1} + \cdots + b_0$ of degree r, then the product $h(x) = f(x)g(x)$ is a monic polynomial of the form $h(x) = x^{n+r} + c_{n+r-1}x^{n+r-1} + \cdots + c_0$, where the c_j are polynomials in the coefficients of f and g. This gives us maps

$$\mu_{n,r} : \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^{n+r}$$

defined by

$$(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{r-1}) \to (c_{n+r-1}, \ldots, c_0).$$

Assume that $n \geq r$ and consider the Euclidean algorithm applied to n and r:

$$\begin{aligned} n &= q_1 r + r_1, \quad 0 < r_1 < r, \\ r &= q_2 r_1 + r_2, \quad 0 < r_2 < r_1, \\ &\vdots \\ r_{k-1} &= q_{k+1} r_k, \quad 0 < r_k < r_{k-1}. \end{aligned}$$

Let $I(n, r)$ be the tuple given by the Euclidean algorithm on n and r:

$$I(n, r) = (r, \ldots, r, r_1, \ldots, r_1, \ldots, r_k, \ldots, r_k, 0, \ldots)$$

where r is repeated q_1 times, and r_i is repeated q_{i+1} times.

https://doi.org/10.1017/S144678871200033X Published online by Cambridge University Press

Let $I(\mu_{n,r})$ be the ideal in the algebra $\mathcal{A}$ of germs at the origin generated by the c_j in the map $\mu_{n,r} : \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^{n+r}$. Denote by $TB(I(\mu_{n,r}))$ the Thom–Boardman symbol of this ideal. Robert Varley conjectured that $TB(I(\mu_{n,r})) = I(n, r)$ for any $n \geq r$. In [3], Lin and Wethington confirmed Varley's conjecture. For the reader's convenience, let us state their result here.

Theorem 1.1. *$TB(I(\mu_{n,r})) = I(n, r)$ for any $n \geq r$.*

Professor J. M. Boardman provides us the following example which has constant Thom–Boardman symbol.

Example 1.2. The zero map-germ at origin $F : \mathbb{C}^a \to \mathbb{C}^b$ has Thom–Boardman symbol $(a, a, \ldots, a, \ldots)$.

Denote the map-germ in Example 1.2 by $\mu_{\infty,a}$. We have the following theorem.

Theorem 1.3. *Let $(i_0, \ldots, i_0, i_1, \ldots, i_1, \ldots, i_k, \ldots, i_k, \ldots)$ be a nonincreasing sequence of nonnegative integers, where $i_0 > i_1 > \cdots > i_k \geq 0$ for some k, i_j repeats l_j times for each $j < k$ and i_k repeats infinitely many times.*

(1) *Denote by μ the Cartesian product of the map-germs of*

$$\mu_{l_0(i_0-i_1),i_0-i_1} \times \mu_{(l_0+l_1)(i_1-i_2),i_1-i_2} \times \cdots \times \mu_{(l_0+l_1+\cdots+l_{k-1})(i_{k-1}-i_k),i_{k-1}-i_k} \times \mu_{\infty,i_k}$$

at the origin. Then $(i_0, \ldots, i_0, i_1, \ldots, i_1, \ldots, i_k, \ldots, i_k, \ldots)$ is the Thom–Boardman symbol of the map-germ μ.

(2) *Denote by ψ the Cartesian product of the map-germs of*

$$\psi_{l_0,i_0-i_1} \times \psi_{(l_0+l_1),i_1-i_2} \times \cdots \times \psi_{(l_0+l_1+\cdots+l_{k-1}),i_{k-1}-i_k} \times \mu_{\infty,i_k}$$

at the origin, where $\psi_{l,i} : \mathbb{C}^i \to \mathbb{C}^i$ is given by $y_j = x_j^{l+1}$, $j = 1, \ldots, i$. Then ψ has the same Thom–Boardman symbol as μ.

Remark. Theorem 1.3(2) was communicated to us by an anonymous referee. His/her example is much simpler because the Thom–Boardman symbol $(\overbrace{i, \ldots, i}^{l}, 0, \ldots)$ of $\psi_{l,i}$ is easy to compute. Historically there were only sporadic known Thom–Boardman symbols for some map-germs. Now Theorem 1.3 provides at least two complete sets of representatives of map-germs classified by their Thom–Boardman symbols. However, we should warn the reader that the classification of map-germs by their Thom–Boardman symbols is not complete. Two left–right nonequivalent map-germs may have the same Thom–Boardman symbols. Under such a circumstance, other invariants are needed (see, for example, [1] or [2, p. 67]).

Remark. Although Theorem 1.3 is stated in the complex context, it also holds true in the smooth real case. We state it in the complex context because Varley's conjecture was originally motivated by the classification of the singularities of Gauss maps of theta divisors of the Jacobian variety of complex curves. Theorem 1.1 and Example 1.2 are also true even if one replaces the complex numbers $\mathbb{C}$ with the real numbers $\mathbb{R}$.

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Note that the maps we are looking for are not generic in any sense. The Thom–Boardman symbol of a generic map has only finite many nonzero entries. So if $i_k \neq 0$, the map will never be generic.

The proof of Theorem 1.3 is very simple. We first prove that Thom–Boardman symbols have the additive property under Cartesian product. Theorem 1.3(2) follows immediately from this property. Combining this property with Theorem 1.1, we obtain Theorem 1.3(1).

2. Proof of Theorem 1.3

We first prove that Thom–Boardman symbols have the additive property under Cartesian product.

PROPOSITION 2.1. *Let $F_t : M_t \to N_t, t = 1, 2,$ be two differentiable maps. Then the Thom–Boardman symbol $TB(F)$ of $F = (F_1, F_2) : M_1 \times M_2 \to N_1 \times N_2$ is equal to the sum of $TB(F_1)$ and $TB(F_2)$, where $F = (F_1, F_2) : M_1 \times M_2 \to N_1 \times N_2$ is the Cartesian product of $F_t : M_t \to N_t, t = 1, 2$.*

Let $x_1^{(t)}, \ldots, x_{m_t}^{(t)}$ be local coordinates of points p_t on the differential manifold $M_t, t = 1, 2$. Denote by $\mathcal{A}_t$ the local ring of germs of differentiable functions at $p_t, t = 1, 2$, and $\mathcal{A}$ the local ring of germs of differentiable functions at (p_1, p_2) on $M_1 \times M_2$. Proposition 2.1 follows easily from a corollary of the following lemmas.

LEMMA 2.2. *Given ideals $B_t \subseteq \mathcal{A}_t, t = 1, 2,$ the Jacobian extension $\Delta_k(B_1 + B_2)$ is generated by B_1, B_2 and products $|D_1||D_2|$ with $|D_1| \in J_{q_1}(\delta B_1)$, $|D_2| \in J_{q_2}(\delta B_2)$ and $q_1 + q_2 = k$, where $(B_1 + B_2)$ is the ideal generated by B_1 and B_2 in $\mathcal{A}$, $J_q(\delta B)$ denotes the set of all $q \times q$ minors of the Jacobian matrix δB, and we stipulate that the 0×0 minor has value 1.*

PROOF. By the definition, $\Delta_k(B_1 + B_2)$ is generated by the ideal $(B_1 + B_2) \subseteq \mathcal{A}$ and the $k \times k$ minors of the Jacobian matrix $\delta(B_1 + B_2)$. A key observation is that the generators of B_t in $\mathcal{A}_t, t = 1, 2$, still can be used as the generators of $(B_1 + B_2)$ in $\mathcal{A}$. So the Jacobian matrix $\delta(B_1 + B_2)$ has the form

$$\delta(B_1 + B_2) = \begin{pmatrix} \delta B_1 & 0 \\ 0 & \delta B_2 \end{pmatrix}.$$

Now if a $k \times k$ minor of $\delta(B_1 + B_2)$ consists of p_1 rows and q_1 columns from δB_1 and p_2 rows and q_2 columns from δB_2, it must be zero unless $p_1 = q_1, p_2 = q_2$ and $q_1 + q_2 = k$. So the Jacobian extension $\Delta_k(B_1 + B_2)$ is generated by B_1, B_2 and products $|D_1||D_2|$ with $|D_1| \in J_{q_1}(\delta B_1)$, $|D_2| \in J_{q_2}(\delta B_2)$ and $q_1 + q_2 = k$. □

LEMMA 2.3. *Suppose that $B_1 \subset \Delta^{i_1} B_1 \subseteq \mathcal{A}_1$ and $B_2 \subset \Delta^{i_2} B_2 \subseteq \mathcal{A}_2$ are critical extensions. Then:*

(1) $\Delta^{i_1+i_2}(B_1 + B_2) = (\Delta^{i_1} B_1 + \Delta^{i_2} B_2) \neq \mathcal{A}$;
(2) $\Delta^{i_1+i_2+1}(B_1 + B_2) = \mathcal{A}$.

So $\Delta^{i_1+i_2}(B_1 + B_2)$ is the critical extension of $(B_1 + B_2)$ in $\mathcal{A}$.

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Proof. Take $k = m_1 + m_2 + 1 - i_1 - i_2$ in Lemma 2.2. If $q_1 \geq m_1 + 1 - i_1$, then $|D_1| \in \Delta^{i_1} B_1$; if $q_1 \leq m_1 - i_1$, then $q_2 \geq m_2 + 1 - i_2$ and hence $|D_2| \in \Delta^{i_2} B_2$. In either case we have $\Delta^{i_1+i_2}(B_1 + B_2) \subseteq (\Delta^{i_1} B_1 + \Delta^{i_2} B_2)$. We will show that the opposite inclusion is also true. By the definition of critical extension, we have $\Delta^{i_2+1} B_2 = \mathcal{A}_2$, so there exists an $(m_2 - i_2) \times (m_2 - i_2)$ minor $|D|$ of δB_2 which is a unit in $\mathcal{A}_2$. For any $|D_1| \in J_{m_1+1-i_1}(\delta B_1)$, the determinant of the diagonal matrix $[D_1, D]$ is a $k \times k$ minor of $\delta(B_1 + B_2)$. Thus $(\Delta^{i_1} B_1) \subseteq \Delta^{i_1+i_2}(B_1 + B_2)$. Similarly, we can prove that $(\Delta^{i_2} B_2) \subseteq \Delta^{i_1+i_2}(B_1 + B_2)$. This gives $(\Delta^{i_1} B_1 + \Delta^{i_2} B_2) \subseteq \Delta^{i_1+i_2}(B_1 + B_2)$. Therefore $\Delta^{i_1+i_2}(B_1 + B_2) = (\Delta^{i_1} B_1 + \Delta^{i_2} B_2)$.

The last inequality $(\Delta^{i_1} B_1 + \Delta^{i_2} B_2) \neq \mathcal{A}$ in (1) can be proved as follows. Suppose that $(\Delta^{i_1} B_1 + \Delta^{i_2} B_2) = \mathcal{A}$. Then by Lemma 2.2 and

$$\Delta^{i_1+i_2}(B_1 + B_2) = (\Delta^{i_1} B_1 + \Delta^{i_2} B_2),$$

we have

$$1 = \sum \lambda_i b_{1i} + \sum \kappa_j b_{2j} + \sum \chi_w |D_{1w}||D_{2w}|,$$

where $\lambda_i, \kappa_j, \chi_k \in \mathcal{A}$, $b_{1i} \in B_1$, $b_{2j} \in B_2$, $|D_{1w}| \in J_{m_1+1-i_1}(\delta B_1)$ and $|D_{2w}| \in J_{m_2+1-i_2}(\delta B_2)$. We claim that $\sum \kappa_j b_{2j}$ is not a unit in $\mathcal{A}$. Otherwise $\sum \kappa_j b_{2j}$ evaluated at some values for $x_1^{(1)}, \ldots, x_{m_1}^{(1)}$ will produce a unit in B_2, which implies that $B_2 = \mathcal{A}_2$, in contradiction to $B_2 \subset \Delta^{i_2} B_2 \neq \mathcal{A}_2$. So $\sum \kappa_j b_{2j}$ is not a unit in $\mathcal{A}$. Because $\mathcal{A}$ is a local ring, that $\sum \kappa_j b_{2j}$ is not a unit implies $1 - \sum \kappa_j b_{2j}$ is invertible. Rewriting the equation

$$1 = \sum \lambda_i b_{1i} + \sum \kappa_j b_{2j} + \sum \chi_k |D_{1k}||D_{2k}|$$

as

$$1 - \sum \kappa_j b_{2j} = \sum \lambda_i b_{1i} + \sum \chi_k |D_{1k}||D_{2k}|$$

and dividing both sides by $1 - \sum \kappa_j b_{2j}$, renaming the coefficients of $b_{1i}, |D_{1k}||D_{2k}|$ as λ_i, χ_k, we obtain that $1 = \sum \lambda_i b_{1i} + \sum \chi_k |D_{1k}||D_{2k}|$. Evaluating this equation at some values for $x_1^{(2)}, \ldots, x_{m_2}^{(2)}$ gives that $1 \in \Delta^{i_1} B_1$, a contradiction to $\Delta^{i_1} B_1 \neq \mathcal{A}_1$. This proves that $(\Delta^{i_1} B_1 + \Delta^{i_2} B_2) \neq \mathcal{A}$ and (1) follows.

Take $k = m_1 + m_2 - i_1 - i_2$, $q_1 = m_1 - i_1$ and $q_2 = m_2 - i_2$. By Lemma 2.2 and the definition of critical extensions, we can choose a $|D_1| \in J_{q_1}(\delta B_1)$ and a $|D_2| \in J_{q_2}(\delta B_2)$ to be units, then $|D_1||D_2| \in \Delta^{i_1+i_2+1}(B_1 + B_2)$ generates $\mathcal{A}$. So $\Delta^{i_1+i_2+1}(B_1 + B_2) = \mathcal{A}$. □

Repeatedly applying Lemma 2.3, we immediately have the following corollary.

Corollary 2.4. *Given sequences of critical extensions for $B_1 \in \mathcal{A}_1$ and $B_2 \in \mathcal{A}_2$, we get a sequence of critical extensions for $(B_1 + B_2)$ in $\mathcal{A}$ by adding the indices.*

Applying Corollary 2.4 to the ideals $B_1 = J_{F_1}$ and $B_2 = J_{F_2}$, we immediately get Proposition 2.1. Theorem 1.3(2) follows directly from Proposition 2.1. Combining Proposition 2.1 with Theorem 1.1, we prove Theorem 1.3(1).

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REMARK. As an easy consequence of Proposition 2.1 and Theorem 1.1, we also obtain that the map-germs $\mu_{kr,r}$ and Cartesian product of r copies of $\mu_{k,1}$ have the same Thom–Boardman symbols. So Theorem 1.3(1) can be restated in term of the $\mu_{k,1}$ and $\mu_{\infty,1}$. A similar conclusion holds true for Theorem 1.3(2). Those special map-germs form building blocks for map-germs with arbitrary Thom–Boardman symbols.

Acknowledgements

We thank Professor J. M. Boardman for providing us with Example 1.2, which is not well known even among experts. We are grateful for his generosity in allowing us to use his example in Theorem 1.3. Without it, Theorem 1.3 would be incomplete. We are also indebted to the anonymous referee for communicating Theorem 1.3(2) to us. His/her comments and suggestions greatly improve the clarity of this paper.

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YULAN WANG, Anhui Economic Management Institute,
Hefei, Anhui 230059, China
and
Department of Mathematics, SUNY Canton,
34 Cornell Drive, Canton, NY 13617, USA

JIAYUAN LIN, Department of Mathematics, SUNY Canton,
34 Cornell Drive, Canton, NY 13617, USA
e-mail: linj@canton.edu

MAORONG GE, School of Mathematics and Computational Science,
Anhui University, Hefei, Anhui 230039, China