SOME POLYNOMIALS OF TOUCHARD CONNECTED WITH THE BERNOULLI NUMBERS

LEONARD CARLITZ

In a recent paper **(4)** Touchard has constructed a set of polynomials $\Omega_n(z)$ such that

$$(1) \qquad B^r\,\Omega_n(B) = \begin{cases} 0 & (0 \leqslant r < n) \\ K_n & (r = n), \end{cases}$$

where after expansion of the left member B^m is replaced by B_m,

$$e^{Bx} = \sum_{m=0}^{\infty} B_m x^m/m! = \frac{x}{e^x - 1},$$

and

$$(2) \qquad K_n = \frac{(-1)^n}{2n+1}\,\frac{1}{2^n}\,\frac{(n!)^4}{[1.3.5\ldots(2n-1)]^2}.$$

(Touchard writes $Q_n(z)$ in place of $\Omega_n(z)$; we have changed the notation in order to avoid a clash with the Legendre function of the second kind.) It is proved by Touchard that

$$(3) \qquad \Omega_{n+1}(z) = (2z+1)\,\Omega_n(z) + \frac{n^4}{4n^2-1}\,\Omega_{n-1}(z).$$

Using (3), Wyman and Moser **(5)** showed that

$$(4) \qquad \Omega_n(z) = 2^n n! \binom{2n}{n}^{-1} \sum_{2r \leqslant n} \binom{2z+n-2r}{n-2r}\binom{z}{r}^2.$$

In the usual notation of generalized hypergeometric functions (4) may be written

$$(5) \qquad \Omega_n(z) = 2^n n! \binom{2n}{n}^{-1}\binom{2z+n}{n} \cdot {}_4F_3\left[\begin{matrix} -\frac{n}{2}, \frac{1}{2} - \frac{n}{2}, & -z, -z; \\ 1, -z - \frac{n}{2}, & -z + \frac{1}{2} - \frac{n}{2} \end{matrix}\right].$$

Now Bateman **(1; 2)** has introduced a polynomial

$$(6) \qquad F_n(z) = {}_3F_2\left[\begin{matrix} -n, n+1, \frac{1}{2}(1+z); \\ 1, 1 \end{matrix}\right]$$

such that

$$(7) \qquad F_n(-z) = (-1)^n F_n(z);$$

also in place of (6) there is the alternate expansion

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$$(8) \qquad F_n(z) = (-1)^n \binom{z}{n} {}_4F_3\left[\begin{matrix} -\frac{n}{2}, \frac{1}{2}-\frac{n}{2}, \frac{1+z}{2}, \frac{1+z}{2}; \\ 1, \frac{z+1-n}{2}, \frac{z+2-n}{2} \end{matrix}\right].$$

Using (7), (8) becomes

$$(9) \qquad F_n(2z+1) = (-1)^n \binom{2z+n}{n} {}_4F_3\left[\begin{matrix} -\frac{n}{2}, \frac{1}{2}-\frac{n}{2}, -z, -z; \\ 1, -z-\frac{n}{2}, -z+\frac{1}{2}-\frac{n}{2} \end{matrix}\right].$$

Comparison of (9) with (5) yields at once

$$(10) \qquad \Omega_n(z) = (-1)^n 2^n n! \binom{2n}{n}^{-1} F_n(2z+1).$$

In the next place we recall Bateman's formula **(2)**

$$(11) \qquad F_n\left(\frac{d}{dz}\right) z \operatorname{cosech} z = \operatorname{cosech} z . Q_n(\coth z),$$

where $Q_n(z)$ denotes the Legendre function of the second kind. We have also in the notation of Nörlund **(3**, Ch. 2**)**

$$z \operatorname{cosech} z = \sum_0^\infty D_m z^m/m! \qquad (D_{2m+1} = 0),$$

where (symbolically)

$$(12) \qquad D_m = (2B+1)^m.$$

Expanding the left member of (11) we get

$$\sum_{r=0}^\infty \frac{z^r}{r!} D^r F_n(D);$$

in view of (10) and (12) this is equal to

$$(-1)^n (2^n n!)^{-1} \binom{2n}{n} \sum_{r=0}^\infty \frac{z^r}{r!} (2B+1)^r \Omega_n(B).$$

Since

$$Q_n(z) = \frac{2^n (n!)^2}{(2n+1)!} \frac{1}{z^{n+1}} F\left(\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; \frac{n}{2}+\frac{3}{2}; z^{-2}\right) \qquad (|z| > 1)$$

we accordingly get

$$(13) \qquad \sum_{r=0}^\infty \frac{z^r}{r!} (2B+1)^r \Omega_n(B)$$
$$= (-1)^n \frac{2^{2n}(n!)^5}{(2n+1)!\,(2n)!} \sinh^n z \operatorname{sech}^{n+1} z F\left(\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; n+\frac{3}{2}; \tanh^2 z\right).$$

From (13) it is clear that

$$(2B+1)^r \Omega_n(B) = 0 \qquad (0 \leqslant r < n),$$

and therefore

$$B^r\,\Omega_n(B) = 0 \qquad (0 \leqslant r < n). \tag{14}$$

As for $r = n$, we have

$$(2B+1)^n\,\Omega_n(B) = (-1)^n \frac{2^{2n}(n!)^6}{(2n+1)!\,(2n)!};$$

using (14) this becomes

$$B^n\,\Omega_n(B) = (-1)^n \frac{(n!)^4}{2^n(2n+1)[1.3.5\ldots(2n-1)]^2} = K_n \tag{15}$$

This evidently completes the proof of (1).

It may be of interest to remark that (1) can be *verified* rapidly in the following way. Using (6) and (10) we see that

$$\begin{aligned}\Omega_n(B) &= (-1)^n\,2^n\,n!\binom{2n}{n}^{-1}{}_3F_2\begin{bmatrix}-n,\ n+1,\ B+1;\\ 1,\ 1\end{bmatrix}\\ &= (-1)^n\,2n!\binom{2n}{n}^{-1}\sum_{s=0}^{n}(-1)^s\binom{n}{s}\binom{n+s}{s}\binom{B+s}{s}.\end{aligned}$$

But (**3**, p. 149)

$$(B+1)(B+2)\ldots(B+s) = \frac{s!}{s+1},$$

so that

$$\begin{aligned}\sum_{s=0}^{n}(-1)^s\binom{n}{s}\binom{n+s}{s}\binom{B+s}{s} &= \sum_{s=0}^{n}(-1)^s\binom{n}{s}\binom{n+s}{s}\frac{1}{s+1}\\ &= F(-n,\ n+1;\ 2;\ 1) = 0.\end{aligned}$$

Thus $\Omega_n(B) = 0$. Repeated use of the recurrence (3) now completes the proof of (14). Finally (3) yields

$$\begin{aligned}(2B+1)^n\,\Omega_n(B) &= (-1)^n \frac{n^4(n-1)^4\ldots 1^4}{(4n^2-1)(4(n-1)^2-1)\ldots(4-1)}\\ &= (-1)^n \frac{(n!)^4}{(2n+1)[(2n-1)\ldots 3.1]^2},\end{aligned}$$

which gives (15).

References

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Duke University