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A SHORT NOTE ON ENHANCED DENSITY SETS

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Abstract. We give a simple proof of a statement extending Fu's (J.H.G. Fu, Erratum to 'some remarks on legendrian rectiable currents', Manuscripta Math. **113**(3) (2004), 397–401) result: *'If Ω is a set of locally finite perimeter in $\mathbb{R}^2$, then there is no function $f \in C^1(\mathbb{R}^2)$ such that $\nabla f(x_1, x_2) = (x_2, 0)$ at a.e. $(x_1, x_2) \in \Omega$'*. We also prove that every measurable set can be approximated arbitrarily closely in L^1 by subsets that do not contain enhanced density points. Finally, we provide a new proof of a Poincaré-type lemma for locally finite perimeter sets, which was first stated by Delladio (S. Delladio, Functions of class C^1 subject to a Legendre condition in an enhanced density set, to appear in *Rev. Mat. Iberoamericana*).

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1. Introduction. In [**3**] we introduced the notions of enhanced density point and enhanced density set.

DEFINITION 1.1. Let Ω be a measurable subset of $\mathbb{R}^n$. Then $x \in \mathbb{R}^n$ is said to be a 'point of enhanced density of Ω' if

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^n(B(x,r)\backslash\Omega)}{r^{n+1}} = 0.$$

By $\Omega_\bullet$ we denote the set of all the points of enhanced density of Ω. We say that 'Ω is an enhanced density set' whenever $\mathcal{L}^n(\Omega\backslash\Omega_\bullet) = 0$.

The family of enhanced density sets includes locally finite perimeter sets (we refer the reader to [**2**, Section 3.3] or [**5**, Definition 7.5.4] for the definition). In fact, even stronger density property proved by Delladio [**3**] holds.

THEOREM 1.1. *Let Ω be a locally finite perimeter subset of $\mathbb{R}^n$. Then*

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^n(B(x,r)\backslash\Omega)}{r^{n+\frac{n}{n-1}}} = 0$$

at a.e. $x \in \Omega$. In particular, Ω is an enhanced density set.

The following result, given in [**3**], provides a generalisation of the classical Poincare's Lemma.

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THEOREM 1.2. *Let λ and μ be differential forms of class C^1 in $\mathbb{R}^n$, respectively, of degree h and $h+1$ (with $h \geq 0$). If*

$$K := \{x \in \mathbb{R}^n \mid d\lambda(x) = \mu(x)\},$$

then $K_\bullet \subset K$ and $(d\mu)|_{K_\bullet} = 0$.

REMARK 1.1. Let $f \in C^1(\mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and

$$K := \{x \in \mathbb{R}^n \mid \nabla f(x) = g(x)\}.$$

If we apply Theorem 1.2 with

$$\lambda := f, \qquad \mu := \sum_{j=1}^{n} g_j \, dx_j$$

and observe that

$$K = \{x \in \mathbb{R}^n \mid d\lambda(x) = \mu(x)\}, \quad d\mu = \sum_{\substack{i,j=1 \\ i<j}}^{n} \left(\frac{\partial g_j}{\partial x_i} - \frac{\partial g_i}{\partial x_j} \right) dx_i \wedge dx_j,$$

then we obtain $K_\bullet \subset K$ and

$$(\operatorname{curl} g)|_{K_\bullet} = 0,$$

where curl is defined as in [**1**].

This note collects the following three independent results related to enhanced density sets.

- The first one extends [**4**, Corollary 2], which states, '*If Ω is a set of locally finite perimeter in $\mathbb{R}^2$, then there is no function $f \in C^1(\mathbb{R}^2)$ such that $\nabla f(x_1, x_2) = (x_2, 0)$ at a.e. $(x_1, x_2) \in \Omega$*'. Observe that this fact, proved by Fu [**4**] through a quite technical argument based on integral currents, is an immediate and GMT-free consequence of Remark 1.1 and Theorem 1.1.
- The second one states that every measurable set can be approximated arbitrarily closely in L^1 by subsets that do not contain enhanced density points.
- The third one is a direct proof, using nothing but Stokes for Whitney's flat chains, of a Poincaré-type lemma for locally finite perimeter sets.

2. Statements and proofs of the results

2.1. Generalisation of [4, Corollary 2]

THEOREM 1.3. *Let Ω be an enhanced density set, e.g. a locally finite perimeter subset of $\mathbb{R}^n$. Let μ be a differential form of class C^1 in $\mathbb{R}^n$ (and degree $h \geq 1$) and assume that*

$$\mathcal{L}^n(\Omega \cap E) > 0, \qquad E := \{x \in \mathbb{R}^n \mid d\mu(x) \neq 0\}.$$

Then there is no differential form of class C^1 in $\mathbb{R}^n$ (and degree $h-1$) such that $d\lambda = \mu$ a.e. in $\Omega \cap E$.

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We will obtain Theorem 1.3 as a trivial corollary of Theorem 1.4 below. In order to prove the latter, we state a simple proposition.

PROPOSITION 1.1. *The following facts hold:*

(i) Let Ω and Ω' be measurable subsets of $\mathbb{R}^n$ such that 'Ω is a subset of Ω' in measure', i.e. $\mathcal{L}^n(\Omega\backslash\Omega') = 0$. Then one has $\Omega_\bullet \subset \Omega'_\bullet$.

(ii) Let Ω and A be, respectively, a measurable subset of $\mathbb{R}^n$ and an open subset of $\mathbb{R}^n$. Then $\Omega_\bullet \cap A \subset (\Omega \cap A)_\bullet$ holds, while the opposite inclusion is false in general.

Proof. The assertion (i) is obvious. As for (ii), let $x \in \Omega_\bullet \cap A$ and observe that since A is open, then

$$B(x, r)\backslash(\Omega \cap A) = (B(x, r)\backslash\Omega) \cup (B(x, r)\backslash A) = B(x, r)\backslash\Omega$$

provided r is small enough. Hence,

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^n(B(x, r)\backslash(\Omega \cap A))}{r^{n+1}} = \lim_{r\downarrow 0} \frac{\mathcal{L}^n(B(x, r)\backslash\Omega)}{r^{n+1}} = 1,$$

namely $x \in (\Omega \cap A)_\bullet$. The assertion about the opposite inclusion is proved by the following example, where we assume $n = 2$. Let

$$\Omega := \mathbb{R}^2\backslash\{(0, 0)\}, \qquad A := \Omega.$$

Then one has

$$\Omega_\bullet \cap A = A, \qquad (\Omega \cap A)_\bullet = \mathbb{R}^2.$$

□

THEOREM 1.4. *Let Ω be a measurable subset of $\mathbb{R}^n$, μ be a differential form of class C^1 in $\mathbb{R}^n$ (and degree $h \geq 1$) and assume that*

$$\mathcal{L}^n(\Omega \cap E) > 0, \qquad E := \{x \in \mathbb{R}^n \mid d\mu(x) \neq 0\}.$$

If there exists a differential form λ of class C^1 in $\mathbb{R}^n$ (and degree $h - 1$) such that $d\lambda = \mu$ a.e. in $\Omega \cap E$, then Ω is not an enhanced density set.

Proof. Let K be defined as in Theorem 1.2. Since E is open, it follows from Proposition 1.1 that

$$\Omega_\bullet \cap E \subset (\Omega \cap E)_\bullet \subset K_\bullet.$$

Then the set $\Omega_\bullet \cap E$ has to be empty, by Theorem 1.2. Hence,

$$\mathcal{L}^n(\Omega\backslash\Omega_\bullet) \geq \mathcal{L}^n((\Omega \cap E)\backslash(\Omega_\bullet \cap E)) = \mathcal{L}^n(\Omega \cap E) > 0,$$

namely Ω is not an enhanced density set. □

2.2. Approximation by sets without enhanced density points

THEOREM 1.5. *Let $\varepsilon > 0$ be fixed arbitrarily. Then there exists an open subset A of $\mathbb{R}^n$ such that $\mathcal{L}^n(A) \leq \varepsilon$ and $(\Omega\backslash A)_\bullet$ is empty for all measurable subsets Ω of $\mathbb{R}^n$.*

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Proof. Let $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be such that $\operatorname{curl} g(x) \neq 0$ at all $x \in \mathbb{R}^n$. Consider the open subsets of $\mathbb{R}^n$

$$\Gamma_0 := \{|x| < 1\}, \qquad \Gamma_j := \{j < |x| < j+1\} \quad (j = 1, 2, \dots).$$

Then, for all j we can find

(i) an open neighbourhood A'_j of $\partial\Gamma_j$ such that

$$\mathcal{L}^n(A'_j) \leq \frac{\varepsilon}{2^{j+2}},$$

(ii) an open subset A''_j of Γ_j and a function $f_j \in C^1_0(\Gamma_j)$ such that

$$\mathcal{L}^n(A''_j) \leq \frac{\varepsilon}{2^{j+2}},$$

and

$$(\nabla f_j)|_{\Gamma_j \setminus A''_j} = g|_{\Gamma_j \setminus A''_j} \tag{1.1}$$

by [**1**, Theorem 1].

From (1.1) and Remark 1.1, it follows that (for $j = 0, 1, \dots$) there are no points of enhanced density of the set

$$R_j := \Gamma_j \setminus (A'_j \cup A''_j) \subset \Gamma_j \setminus A''_j.$$

Since $R_j \subset\subset \Gamma_j$ for all j, there are no points of enhanced density of

$$\bigcup_{j=0}^{\infty} R_j = \bigcup_{j=0}^{\infty} \Gamma_j \setminus (A'_j \cup A''_j) = \mathbb{R}^n \setminus A,$$

where A is the open set defined by

$$A := \bigcup_{j=0}^{\infty} (A'_j \cup A''_j).$$

The conclusion follows from Proposition 1.1(i) and by observing that

$$\mathcal{L}^n(A) \leq \sum_{j=0}^{\infty} [\mathcal{L}(A'_j) + \mathcal{L}(A''_j)] \leq \sum_{j=0}^{\infty} \frac{\varepsilon}{2^{j+1}} = \varepsilon.$$

□

2.3. A Poincaré-type lemma for locally finite perimeter sets. The following fact is an immediate consequence of Theorems 1.1 and 1.2. It has first been stated in [**3**]. Here we give an alternative short proof using nothing but Stokes for Whitney's flat chains [**6**]. This new proof is based on a global argument that could reveal to be useful for treating similar issues in the context of integral currents.

THEOREM 1.6. *Let λ and μ be C^1 forms of degree h and $h+1$, respectively, with $0 \leq h \leq n-2$. Assume that $d\lambda = \mu$ almost everywhere in a locally finite perimeter set Ω. Then one also has $d\mu = 0$ almost everywhere in Ω.*

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Proof. Let ω be any smooth form of degree $n-h-2$ with compact support. Then one has (adopting the notation of [**5**, Section 7.2])

$$\begin{aligned}\int_\Omega d\mu\wedge\omega &= \int_\Omega d(\mu\wedge\omega) + (-1)^h\int_\Omega \mu\wedge d\omega\\ &= \partial[\![\Omega]\!](\mu\wedge\omega) + (-1)^h\int_\Omega \mu\wedge d\omega\\ &= \partial[\![\Omega]\!](d\lambda\wedge\omega) + (-1)^h\int_\Omega \mu\wedge d\omega\\ &= \partial[\![\Omega]\!](d(\lambda\wedge\omega)) + (-1)^{h+1}\partial[\![\Omega]\!](\lambda\wedge d\omega) + (-1)^h\int_\Omega \mu\wedge d\omega.\end{aligned}$$

Now, by [**5**, Remark 7.5.6 and Theorem 7.9.2], a locally finite perimeter set is a flat chain in the sense of Whitney. We get

$$\partial[\![\Omega]\!](d(\lambda\wedge\omega)) = 0$$

by [**6**, Chapter V, Section 3]. Hence,

$$\begin{aligned}\int_\Omega d\mu\wedge\omega &= (-1)^{h+1}\int_\Omega d(\lambda\wedge d\omega) + (-1)^h\int_\Omega \mu\wedge d\omega\\ &= -(-1)^h\int_\Omega d\lambda\wedge d\omega + (-1)^h\int_\Omega \mu\wedge d\omega\\ &= 0.\end{aligned}$$

The conclusion follows from the arbitrariness of ω. □

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