On the Failure of Heilermann's Theorem.

By HARIPADA DATTA, Research Student, Edinburgh University.

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The theorem of Heilermann* can be stated thus:—
If the series

$$\frac{\alpha_0}{x} + \frac{\alpha_1}{x^2} + \frac{\alpha_2}{x^3} + \dots\dots\dots\dots \quad (1)$$

is converted into a continued fraction of the form

$$\frac{a_1}{x+b_1} + \frac{a_2}{x+b_2} + \frac{a_3}{x+b_3} + \dots\dots\dots\dots \quad (2)$$

then the elements of the continued fraction are

$$b_n = \frac{{}^1K_{n-1}}{K_{n-1}} - \frac{{}^1K_n}{K_n}$$

$$a_n = \frac{K_{n-2}K_n}{K_{n-1}^2}$$

where

$$K_{n+1} = \begin{vmatrix} \alpha_n & \alpha_{n-1} \dots\dots \alpha_1 & \alpha_0 \\ \alpha_{n+1} & \alpha_n \dots\dots \alpha_2 & \alpha_1 \\ \dots & \dots\dots\dots & \dots \\ \alpha_{2n} & \alpha_{2n+1} \dots\dots \alpha_{n+1} & \alpha_n \end{vmatrix}$$

and ${}^rK_n$ is obtained from this determinant by deleting the $(r+1)^{\text{th}}$ column and the last row. Moreover, if $f_n(x)$ and $\phi_{n-1}(x)$ are respectively the denominator and the numerator of the n^{th} convergent, then

$$\left.\begin{aligned} f_n(x) &= x^n - \frac{{}^1K_n}{K_n}x^{n-1} + \frac{{}^2K_n}{K_n}x^{n-2} - \dots + (-1)^n\frac{{}^nK_n}{K_n} \\ &\text{and} \\ \phi_{n-1}(x) &= \gamma_{n-1}^{(n)}x^{n-1} + \gamma_{n-2}^{(n)}x^{n-2} + \dots + \gamma_1^{(n)}x + \gamma_0 x \end{aligned}\right\} \dots\dots\dots \quad (3)$$

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where

$$\left.\begin{aligned}
\gamma_0^{(n)} &= \alpha_{n-1} - \frac{{}^1K_n}{K_n}\alpha_{n-2} + \frac{{}^2K_n}{K_n}\alpha_{n-3} - \ldots + (-1)^{n-1}\frac{{}^{n-1}K_n}{K_n}\alpha_0 \\
\gamma_1^{(n)} &= \alpha_{n-2} - \frac{{}^1K_n}{K_n}\alpha_{n-3} + \ldots\ldots\ldots\ldots\ldots + (-1)^{n-2}\frac{{}^{n+2}K_n}{K_n}\alpha_0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\gamma_{n-2}^{(n)} &= \alpha_1 - \frac{{}^1K_n}{K_n}\alpha_0 \\
\gamma_{n-1}^{(n)} &= \alpha_0
\end{aligned}\right\} \quad (4)$$

The successive convergents to the continued fraction (2) have the property that if the n^{th} convergent is expanded as a power-series in $\frac{1}{x}$, the first $2n$ terms of this expansion will be, term for term, the same as the first $2n$ terms of the series (1).

1. Now if any of the a's be zero (say $a_{n+1} = 0$), then the continued fraction terminates at the n^{th} convergent. And from the recurrence-formulae connecting three successive convergents it would appear that all the convergents after the $(n-1)^{\text{th}}$ are then equal to one another. But if the numerators and denominators of the convergents are evaluated by the formulae (3), we find that in general these higher convergents are not equal to each other.

Consider for example the series

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \frac{2}{x^5} + \frac{3}{x^6} + \frac{\alpha_6}{x^7} + \frac{\alpha_7}{x^8} + \ldots .$$

The determinant K_2 vanishes, therefore $a_2 = 0$.

$$\text{The first convergent} = \frac{1}{x-1}.$$

$$\text{,, second ,,} = \frac{1}{x-1}.$$

$$\text{,, third ,,} = \frac{4x^2 + 10x + 21}{4x^3 + 6x^2 + 11x - 13}.$$

We see that the second convergent is equal to the first, while the third convergent is different from the first. The expansion of either the first or the second convergent coincides with the given series as far as the third term, whereas the expansion of the third

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convergent coincides with the given series as far as the sixth term. This illustrates the abnormality with which we are dealing: for in the normal cases of Heilermann's Theorem, the expansions of the first, second, and third convergents coincide with the given series so far as the 2nd, 4th, and 6th terms respectively. Formulae (3) furnish a regular sequence of convergents, whereas the recurrence-formulae would give all the convergents equal to $\dfrac{1}{x-1}$.

2. We shall now examine the phenomenon from a somewhat different stand-point, by enquiring what is the relation between the series and the continued fraction if K_{n+1} vanishes (and $K_n \neq 0$).

We shall now show that *if* $K_{n+1}=0$, *and if* $K_n \neq 0$, *the first* $(2n+1)$ *terms of the expansion of the* n^{th} *convergent will be, term for term, the same as the first* $(2n+1)$ *terms of the series, and vice-versa.* (In the normal case there is equivalence of $2n$ terms only).

Let us consider the second convergent, which is

$$\frac{\gamma_1^{(2)} x + \gamma_0^{(2)}}{x^2 + \beta_1^{(2)} x + \beta_0^{(2)}} = \frac{\alpha_0}{x} + \frac{\alpha_1}{x^2} + \frac{\alpha_2}{x^3} + \frac{\alpha_3}{x^4} + \frac{A_1}{x^5} + \frac{A_2}{x^6} + \dots .$$

Multiplying up and equating to zero the coefficients of $\dfrac{1}{x}$, $\dfrac{1}{x^2}$, etc., we obtain the following relations:—

$$\left.\begin{array}{l} \alpha_2 + \beta_1^{(2)} \alpha_1 + \beta_0^{(2)} \alpha_0 = 0 \\ \alpha_3 + \beta_1^{(2)} \alpha_2 + \beta_0^{(2)} \alpha_1 = 0 \\ A_1 + \beta_1^{(2)} \alpha_3 + \beta_0^{(2)} \alpha_2 = 0 \\ A_2 + \beta_1^{(2)} A_1 + \beta_0^{(2)} \alpha_3 = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \dots\dots\dots\dots\dots\dots\dots \quad (5)$$

Eliminating β's from the first three of these relations, we have

$$\begin{vmatrix} \alpha_2 & \alpha_1 & \alpha_0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ A_1 & \alpha_3 & \alpha_2 \end{vmatrix} = 0.$$

And hence if $K_2 \neq 0$ and $K_3 = 0$, then $A_1 = \alpha_4$. And if $A_1 = \alpha_4$, then $K_3 = 0$; which establishes the theorem stated above.

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3. In this case it is to be shown that *the* $(n+1)^{\text{th}}$ *convergent is equal to the* n^{th} *convergent*, and its expansion, as a power-series in $\frac{1}{x}$, will agree with the original series as far as the $(2n+1)^{\text{th}}$ term (inclusive) instead of as far as the $(2n+2)^{\text{th}}$ term (inclusive). But the expansion of the $(r+2)^{\text{th}}$ convergent can, in general, be made to agree as far as the $(2r+4)^{\text{th}}$ term (inclusive) of the series, where $r \geqq n$.

From the third convergent we have the relations

$$\left.\begin{array}{l} \alpha_3 + \beta_2^{(3)}\alpha_2 + \beta_1^{(3)}\alpha_1 + \beta_0^{(3)}\alpha_0 = 0 \\ \alpha_4 + \beta_2^{(3)}\alpha_3 + \beta_1^{(3)}\alpha_2 + \beta_0^{(3)}\alpha_1 = 0 \\ \alpha_5 + \beta_2^{(3)}\alpha_4 + \beta_1^{(3)}\alpha_3 + \beta_0^{(3)}\alpha_2 = 0 \end{array}\right\} \quad \dots\dots\dots\dots\dots\dots \quad (6)$$

As $A_1 = \alpha_4$, so we can reduce the first two equations of (6) by means of the first (3) equations of (5) to

$$\left.\begin{array}{l} (\beta_2^{(3)} - \beta_1^{(2)})\,\alpha_2 + (\beta_1^{(3)} - \beta_0^{(2)})\,\alpha_1 + \beta_0^{(3)}\alpha_0 = 0 \\ (\beta_2^{(3)} - \beta_1^{(2)})\,\alpha_3 + (\beta_1^{(3)} - \beta_0^{(2)})\,\alpha_2 + \beta_0^{(3)}\alpha_1 = 0 \end{array}\right\}.$$

Now, comparing these two equations with the first two equations of (5), we have

$$\frac{\beta_2^{(3)} - \beta_1^{(2)}}{1} = \frac{\beta_1^{(3)} - \beta_0^{(2)}}{\beta_1^{(2)}} = \frac{\beta_0^{(3)}}{\beta_0^{(2)}} = p \text{ (say)}.$$

Therefore

$$\left.\begin{array}{l} \beta_2^{(3)} = \beta_1^{(2)} + p \\ \beta_1^{(3)} = \beta_0^{(2)} + p\,\beta_1^{(2)} \\ \beta_0^{(3)} = p\,\beta_0^{(2)} \end{array}\right\}.$$

Now, substituting these values of β's in the third relation of (6), we have

$$(\alpha_5 + \beta_1^{(2)}\alpha_4 + \beta_0^{(2)}\alpha_3) + p\,(\alpha_4 + \beta_1^{(2)}\alpha_3 + \beta_0^{(2)}\alpha_2) = 0.$$

As $\alpha_4 + \beta_1^{(2)}\alpha_3 + \beta_0^{(2)}\alpha_2 = 0$, therefore if the third relation of (6) hold, then

$$\alpha_5 + \beta_1^{(2)}\alpha_4 + \beta_0^{(2)}\alpha_3$$

must vanish. That is, the expansion of the 2nd convergent will agree with the series as far as the 6th term (*i.e.* $A_2 = \alpha_5$), but by hypothesis this is not the case. Hence the third equation of (6) is not valid, and the expansion of the third convergent will agree with the series as far as the 5th term only.

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Moreover, we see that the denominator of the third convergent is

$$f_3(x) = x^3 + \beta_2^{(3)} x^3 + \beta_1^{(3)} x + \beta_0^{(3)}$$
$$= (x+p) f_2(x),$$

and the numerator

$$\phi_2(x) = (x+p)\,\phi_1(x).$$

Thus $$^{*}\ \phi_2(x)/f_3(x) = \phi_1(x)/f_2(x).$$

It is to be noted that whatever the value of p may be, the first two equations of (6) are satisfied. Therefore the denominator and the numerator of the third convergent, taken separately, are not unique. But the third convergent, as a whole, is unique, and is equal to the second convergent.

4. We have seen that if the third relation of (6) holds, then A_2 becomes α_5. That is, the expansion of the second convergent agrees with the series as far as the 6th term.

In this case the 4th convergent will be equal to the 2nd convergent.

From the 4th convergent we have the relations

$$\left.\begin{array}{l} \alpha_4 + \beta_3^{(4)}\alpha_3 + \beta_2^{(4)}\alpha_2 + \beta_1^{(4)}\alpha_1 + \beta_0^{(4)}\alpha_0 = 0 \\ \alpha_5 + \beta_3^{(4)}\alpha_4 + \ \dots\dots\dots\dots\dots\ + \beta_0^{(4)}\alpha_1 = 0 \\ \alpha_6 + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots = 0 \\ \alpha_7 + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots = 0 \end{array}\right\} \dots\dots\dots\dots (7)$$

As $A_1 = \alpha_4$ and $A_2 = \alpha_5$, we can reduce the first two equations of (7) by means of (5) and obtain

$$\left.\begin{array}{l} \beta_2^{(4)} = p_1 + q\,\beta_1^{(2)} + \beta_0^{(2)} \\ \beta_1^{(4)} = p_1\,\beta_1^{(2)} + q\,\beta_0^{(2)} \\ \beta_0^{(4)} = p_1\,\beta_0^{(2)} \end{array}\right\} \text{ where } q = \beta_3^{(4)} - \beta_1^{(2)}.$$

Hence

$$f_4(x) = (x^2 + qx + p_1) f_2(x)$$
$$\phi_3(x) = (x^2 + qx + p_1)\phi_1(x),$$
$$\therefore\quad \phi_3(x)/f_4(x) = \phi_1(x)/f_2(x).$$

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* Thus, even if p be ∞, the third relation of (6) will not hold unless $A_2 = \alpha_5$.

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It can also be independently shown that the third equation of (7) can not hold.

Similarly, *if* $K_n \neq 0$, *and if all the determinants of order* $(n+1)$ *of the type*

$$\begin{vmatrix} \alpha_0 & \alpha_1 & \cdots\cdots & \alpha_n \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-2} & \alpha_{n-1} & \cdots & \alpha_{2n-2} \\ \alpha_{n-1} & \alpha_n & \cdots\cdots & \alpha_{2n-1} \\ \alpha_p & \alpha_{p+1} & \cdots & \alpha_{p+n} \end{vmatrix} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (8)$$

(where $p = n,\ n+1,\ \ldots\ n+r$)

vanish, then the expansion of the n^{th} *convergent will agree with the series as far as the* $(2n+r)^{\text{th}}$ *term, and each of the* $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, ... $(n+r)^{\text{th}}$ *convergents will be equal to the* n^{th} *convergent.*

5. If in (8) r is infinitely large, then the expansion of the n^{th} convergent will give the whole series. It is evident that the series in this case is a recurring one. But it is also evident that this is really an exceptional case, happening only when an infinite number of conditions are satisfied: in general, if one of the numerator-elements of the continued fraction vanishes, the continued fraction terminated does not represent the whole series, and Heilermann's Theorem may in this sense be said to fail. But as we have seen, the formulae (3) furnish convergents which do give a continually-improving approximation to the series.

We shall consider the three following cases of recurring series:—

(I) $\quad \Pi\,(a,\ a+n-2)\dfrac{1}{x} + \Pi\,(a+1,\ a+n-1)\dfrac{1}{x^2} + \ldots$

where

$\Pi\,(a+r,\ a+r+n-2) = (a+r)\,(a+r+1)\ldots(a+r+n-2)$

(J) $\quad \Pi\,(1+q,\ 1+yq^{n-2})\dfrac{1}{x} + (1+yq,\ 1+yq^{n-1})\dfrac{1}{x^2} + \ldots$

(I′) $\quad a^{n-1}\dfrac{1}{x} + (a+1)^{n+1}\dfrac{1}{x^2} + (a+2)^{n-1}\dfrac{1}{x^3} + \ldots$

where n is a positive integer.

(I).

The series (I) is a case of the series

$$\frac{D}{a-1}\,\frac{1}{n} + \frac{D\,(D+1)}{(a-1)\,a}\,\frac{1}{n^2} + \ldots.$$

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For if we put first $(a+n-2)$ for D, and then multiply this transformed series by $\Pi(a-1, a+n-2)$, we obtain the series (I). The above series has been considered in the section (C) of the first paper.*

As all the determinants of order higher than n and of the type (8) contain $(D-a-n+2)$ as a factor, hence they vanish when $D=a+n-2$. Therefore the n^{th} convergent must represent the whole series. We shall now find this convergent.

From the section (C) we obtain

$$b_m = \frac{(m-1)(a+n+m-3)}{a+2m-4} - \frac{m(a+n+m-2)}{a+2m-2},$$

$$a_m = \frac{(m-1)(n-m+1)(a+m-3)(a+n+m-3)}{(a+2m-5)(a+2m-4)^2(a+2m-3)},$$

and

$$f_m(x) = x^m - \binom{m}{1}\frac{a+m+n-2}{a+2m-2}x^{m-1}$$
$$+\binom{m}{2}\frac{(a+m+n-2)(a+m+n-3)}{(a+2m-2)(a+2m-3)}x^{m-2} - \dots$$
$$\dots + (-1)^m\frac{(a+m+n-2)\dots(a+n-1)}{(a+2m-2)\dots\dots(a+m-1)}$$

Hence $f_n(x) = (x-1)^n$.

To find the expression for $\phi_{n-1}(x)$, we have

$$\gamma_{n-1}^{(n)} = \Pi(a, a+n-2)$$

$$\gamma_{n-2}^{(n)} = \Pi(a+1, a+n-1) - \binom{n}{1}\Pi(a, a+n-2)$$
$$= -\binom{n-1}{1}\frac{1}{a}\Pi(a-1, a+n-2)$$

$$\gamma_{n-3}^{(n)} = \left\{\Pi(a+2, a+n) - \binom{n}{1}\Pi(a+1, a+n-1)\right\}$$
$$+\binom{n}{2}\Pi(a, a+n-2).$$

The quantities within { } become $\gamma_{n-2}^{(n)}$, if $(a-1)$ is put for a.

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Therefore

$$\gamma^{(n)}_{n-3} = -\binom{n-1}{1}\frac{1}{a+1}\Pi(a, a+n-1) + \binom{n}{2}\Pi(a, a+n-2)$$

$$=\binom{n-1}{2}\frac{1}{a+1}\Pi(a-1, a+n-2)$$

....

$$\gamma^{(n)}_1 = (-1)^{n-2}\binom{n-1}{1}\frac{1}{a+n-3}\Pi(a-1, a+n-2)$$

$$\gamma^{(n)}_0 = (-1)^{n-1}\Pi(a-1, a+n-3).$$

Hence $\phi_{n-1}(x) = \Pi(a-1, a+n-2)$

$$\times\left\{\frac{x^{n-1}}{a-1} - \binom{n-1}{1}\frac{x^{n-2}}{a} + \binom{n-1}{2}\frac{x^{n-3}}{a+1} - \dots + (-1)^{n-1}\frac{1}{a+n-2}\right\}.$$

And thus the n^{th} convergent, which is the sum* of the whole series is determined.

We can find expressions for higher convergents, but they are all equal to the n^{th} convergent, as has been already shown. Here we have

$$f^{(x)}_{n+r} = x^{n+r} f_n(x)\, F\left(-r,\ 1-a-n-r,\ 1-a-n-2r,\ \frac{1}{x}\right)$$

and

$$\phi^{(x)}_{n+r-1} = x^{n+r}\,\phi_{n-1}(x)\, F\left(-r,\ 1-a-n-r,\ 1-a-n-2r, \frac{1}{x}\right)$$

We can also convert the same series,

$$\Pi(a, a+n-2)\,x + \Pi(a+1, a+n-2)\,x^2 + \dots$$

into a continued fraction of a different form, namely,

$$\frac{c_1 x}{1+}\ \frac{c_3 x}{1+}\ \frac{c_3 x}{1+ \dots},$$

the constants c's being given by the equations

$$c_1 = \Pi(a, a+n-2),$$

$$c_{2m} = -\frac{(a+m-2)(a+m+n-2)}{(a+2m-2)(a+2m-2)},$$

$$c_{2m+1} = \frac{m(n-m)}{(a+2m-2)(a+2m-1)}.$$

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* It is also evident from the relation

$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x)$ due to Euler.

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Hence in this case the continued fraction can be written as

$$\frac{c_1 x}{1-}\ \frac{(a+n-1)x}{a\ \ +}\ \frac{1.(n-1)x}{a+1\ -}\ \frac{a(a+n)x}{a+2+\ldots+}\ \frac{(n-1)1.x}{a+2n-3-}\ \frac{(a+n-2)x}{1}.$$

Expressing the denominator of the continued fraction as a continuant, we obtain a theorem in factorizable continuants, namely,

$$\begin{vmatrix} 1 & x & & & & & \\ a+n-1 & a & x & & & & \\ & 1.(1-n) & a+1 & x & & & \\ & & a(a+n) & a+2 & x & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & (1-n).1 & a+2n-3 & x \\ & & & & & a+n-2 & 1 \end{vmatrix}_{2n}$$

$$=\{a(a+1)\ldots\ldots(a+2n-3)\}\ (1-x)^n.$$

That is equal to the product of the principal diagonal terms multiplied by $(1-x)^n$.

(J).

The series (J) is a case of the series (H) (*loc. cit.*), namely,

$$\frac{1+y}{1+z}\,\frac{1}{x}+\frac{(1+y)(1+qy)}{(1+z)\,(1+qz)}\,\frac{1}{x^2}+\frac{(1+y)(1+qy)(1+q^2y)}{(1+z)\,(1+qz)\,(1+q^2z)}\,\frac{1}{x^3}+\ldots$$

As in the previous case, the n^{th} convergent represents the sum of the entire series (J).

It can be shown that

$$f_n(x)=(x-1)\,(n-q)\,(n-q^2)\ldots(n-q^{n-1}),$$

and

$$\phi_{n-1}(x)=\Pi\,(1+q^{-1}y,\ 1+q^{n-2}y)$$
$$\times\left\{\frac{x^{n-1}}{1+q^{-1}y}-\begin{bmatrix}n-1\\1\end{bmatrix}\frac{1}{1+y}qx^{n-2}+\begin{bmatrix}n-1\\2\end{bmatrix}\frac{1}{1+qy}q^2x^{n-3}=\ldots\right.$$
$$\left.\ldots+(-1)^{n-1}\begin{bmatrix}n-1\\n-1\end{bmatrix}\left[\frac{1}{1+q^{n-2}y}\,q^{n-1}\right]\right\},$$

where $$\begin{bmatrix}n\\r\end{bmatrix}=\frac{(q^n-1)\,(q^{n-1}-1)\ldots\ldots(q^{n-r+1}-1)}{(q-1)\,\,(q^2-1)\,\,\ldots\ldots\,\,(q^r-1)}\,q^{\frac{1}{2}(r-1)r}.$$

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If the series is expressed as a continued fraction of the second form, we have

$$c_{2m} = -\frac{(1+q^{m-2})(1+q^{m+n-2}y)}{(1+q^{2m-3}y)(1+q^{2m-2}y)}q^{m-1}$$

$$c_{2m+1} = -\frac{(q^m-1)(q^{n-m}-1)}{(1+q^{2m-2}y)(1+q^{2m-1}y)}q^{2m-2}y.$$

Therefore

$$\Pi(1+y,\ 1+q^{n-2}y)x+\Pi(1+qy,\ 1+q^{n-1}y)x^2+\ldots$$

$$=\frac{c_1 x}{1-}\ \frac{(1+q^{n-1}y)x}{1+y\ \ -}\ \frac{(q-1)(q^{n-1}-1)yx}{1+qy\qquad -\ldots.}\ \frac{(1+yq^{n-2})q^{n-1}x}{1}.$$

Hence if

$$e_{2m} = (1+q^{m-2}y)\ 1+q^{m+n-2}y)q^{m-1}$$

$$e_{2m+1} = (q^m-1)(q^{n-m}-1)yq^{2m-2}$$

and

$$gr = 1+q^{r-3}y,$$

we have another theorem in factorisable continuants, namely,

$$\left|\begin{matrix} 1 & x & & & & \\ e_2 & g_2 & x & & & \\ & e_3 & g_3 & x & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & e_{2n-1} & g_{2n-1} & x \\ & & & & e_{2n} & 1 \end{matrix}\right|_{2n}$$

$$=\{(1+y)(1+qy)\ldots(1+q^{2n-3}y)\}\ \{(1-x)(1-qx)\ldots(1-q^{n-1}x)\}.$$

It can also be shown that if σ is a primitive root of $\rho^n=1$, then

$$\left\{\frac{\alpha+y}{\alpha+z}\frac{1}{x}+\frac{(\alpha+y)(\alpha+\sigma y)}{(\alpha+z)(\alpha+\sigma z)}\frac{1}{x^2}+\ldots+\frac{\Pi(\alpha+y,\ \alpha+\sigma^{n-1}y)}{\Pi(\alpha+z,\ \alpha+\sigma^{n-1}z)}\frac{1}{x^2}\right\}$$

$$\times\frac{\Pi(\alpha+z,\ \alpha+\sigma^{n-1})x^n}{\Pi(\alpha+z,\ \alpha+\sigma^{n-1}z)x^n-\Pi(\alpha+y,\ \alpha+\sigma^{n-1}y)}$$

$$=\frac{a_1}{x+b_1+}\ \frac{a_2}{x+b_2+}\ldots+\frac{a_n}{x+b_n},$$

where

$$b_m=\frac{\sigma^{m-1}-1}{\sigma-1}\frac{\alpha+\sigma^{m-1}y}{\alpha+\sigma^{2m-3}z}-\frac{\sigma^m-1}{\sigma-1}\frac{\alpha+\sigma^m y}{\alpha+\sigma^{2m-1}z},$$

and

$$a_m=\frac{\alpha\sigma^{2m-3}(\sigma^{m-1}-1)(\alpha+\sigma^{m-2}z)(\alpha+\sigma^{m-1}y)(z\sigma^{m-2}-y)}{(\alpha+\sigma^{2m-4}z)(\alpha+\sigma^{2m-3}z)^2(\alpha+\sigma^{2m-2}z)}.$$

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(I′).

By a well-known theorem in the Calculus of Finite Differences we have

$$0=\Delta^n a^m=(a+n)^m-\binom{n}{1}(a+n-1)^m+\binom{n}{2}(a+n-2)^m-\dots$$
$$\dots+(-1)^n a^m \dots(6)$$

where m and n are both positive integers and $m<n$. Hence it can be shown that all the determinants (8) vanish, $K_n=\{(n-1)!\}^n$ and the denominator of the n^{th} convergent to the series (I′) is

$$f_n(x)=(x-1)^n.$$

Therefore by (4) we have

$$\gamma_0^{(n)}=(a+n-1)^{n-1}-\binom{n}{1}(a+n-2)^{n-1}+\dots+(-1)^{n-1}\binom{n}{1}a^{n-1}$$

$$\gamma_1^{(n)}=(a+n-2)^{n-1}-\binom{n}{1}(a+n-3)^{n-1}+\dots+(-1)^{n-2}\binom{n}{2}a^{n-1}$$

....

$$\gamma_{n-2}^{(n)}=(a+1)^{n-1}-\binom{n}{1}a^{n-1}$$
$$\gamma_{n-1}^{(n)}=a^{n-1}.$$

Hence by the identity (6) we have

$$\gamma_0^{(n)}=(1-a)^{n-1}$$
$$\gamma_1^{(n)}=(2-a)^{n-1}-\binom{n}{1}(1-a)^{n-1}$$
$$\gamma_2^{(n)}=(3-a)^{n-1}-\binom{n}{1}(2-a)^{n-1}+\binom{n}{2}(1-a)^{n-1}$$

....

$$\gamma_{n-3}^{(n)}=(a+2)^{n-1}-\binom{n}{1}(a+1)^{n-1}+\binom{n}{2}a^{n-1}$$
$$\gamma_{n-2}^{(n)}=(a+1)^{n-1}-\binom{n}{1}a^{n-1}$$
$$\gamma_{n-1}^{(n)}=a^{n-1}.$$

(i) If $a=1$, then we have

$$\gamma_0^{(n)}=0$$
$$\gamma_1^{(n)}=\gamma_{n-1}^{(n)}=1$$
$$\gamma_2^{(n)}=\gamma_{n-2}^{(n)}=2^{n-1}-\binom{n}{1}$$
$$\gamma_3^{(n)}=\gamma_{n-3}^{(n)}=3^{n-1}-\binom{n}{1}2^{n-1}+\binom{n}{2}$$

....

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Therefore (if $n-1=2s$) $\phi_{2s}(x)$ contains $(x+1)$ as a factor.

(ii) If $a=\frac{1}{2}$, we have

$$\gamma_0^{(n)}=\gamma_{n-1}^{(n)}=2^{1-n}$$
$$\gamma_1^{(n)}=\gamma_{n-2}^{(n)}=2^{1-n}\left\{3^{n-1}-\binom{n}{1}\right\}$$
$$\gamma_2^{(n)}=\gamma_{n-3}^{(n)}=2^{1-n}\left\{5^{n-1}-\binom{n}{2}3^{n-1}+\binom{n}{1}\right\}$$
....

Therefore in this case $\phi_{2s+1}(x)$ contains $(x+1)$ as a factor.

Up to this we have considered only infinite series. The finite series can be treated exactly in a similar manner. For every finite series we obtain a factorizable continuant.

As for example, in connection with the series

$$\frac{1}{x}-\frac{n}{x^2}+\frac{n(n-1)}{x^3}-\ldots+(-1)^n\frac{n!}{x^{n+1}},$$

the elements of the continued fraction are

$$b_1=n$$
$$b_m=(n-2m+2)$$
$$a_m=(m-1)(n-m+2).$$

And the continuant is that of Painvin of the year 1858.*

6. In case of the quotient of two series the theorem of Heilermann is

$$\frac{A_0+A_1x+A_2x^2+\ldots}{B_0+B_1x+B_2x^2+\ldots}=\frac{b_0}{1+}\,\frac{b_1x}{1+}\,\frac{b_2x}{1+\ldots}$$

where
$$b_0=A_0/B_0$$
$$b_{2n}=-\frac{\Delta_{2n-3}}{\Delta_{2n-2}}\cdot\frac{\Delta_{2n}}{\Delta_{2n-1}}$$
$$b_{2n+1}=-\frac{\Delta_{2n-2}}{\Delta_{2n-1}}\cdot\frac{\Delta_{2n+1}}{\Delta_{2n}}$$

and
$$\Delta_3=\begin{vmatrix} A_3 & A_2 & B_2 & B_3 \\ A_2 & A_1 & B_1 & B_2 \\ A_1 & A_0 & B_0 & B_1 \\ A_0 & 0 & 0 & B_0 \end{vmatrix}$$

* Sur un certain systéme d'équations linéares. *Journ.* (*de Liouville*) *de Math.* (2), iii., p. 46.

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$$\Delta_4 = \begin{vmatrix} A_4 & A_3 & A_2 & B_3 & B_4 \\ A_3 & A_2 & A_1 & B_2 & B_3 \\ A_2 & A_1 & A_0 & B_1 & B_2 \\ A_1 & A_0 & 0 & B_0 & B_1 \\ A_0 & 0 & 0 & 0 & B_0 \end{vmatrix}$$

It can be shown that if $\Delta_3 \neq 0$, and if all the bigradients

$$\begin{vmatrix} A_p & A_{p-1} & A_{p-2} & B_{p-1} & B_p \\ A_3 & A_2 & A_1 & B_2 & B_3 \\ A_2 & A_1 & A_0 & B_1 & B_2 \\ A_1 & A_0 & 0 & B_0 & B_1 \\ A_0 & 0 & 0 & 0 & B_0 \end{vmatrix} \qquad \ldots\ldots\ldots\ldots\ldots (9)$$

(where $p = 4, 5, \ldots$)

vanish, the 4th convergent to the continued fraction will be equal to the quotient of the two series.

The bigradients (9) may be denoted by

$$\begin{vmatrix} A_p & B_p \\ (A_3)_3 & (B_3)_2 \end{vmatrix}$$

where 3 denotes the number of A-columns and 2 denotes the number of B-columns.

In the case of the quotient of the two hypergeometric series, namely,

$$F(\alpha, \beta+1, \gamma+1, x) \div F(\alpha, \beta, \gamma, x),$$

the bigradients of the type (9) and of orders higher than $(2n+1)$ contain $(\alpha - \gamma - n)$ * as a factor.

Therefore for $\alpha = \gamma + n$ all these bigradients vanish, and the $2n^{\text{th}}$ convergent is the quotient. Thus we obtain

$$F(\gamma+n, \beta+\alpha, \gamma+1, x) \div F(\gamma+n, \beta, \gamma, x)$$

$$= \frac{\gamma}{\gamma+} \, \frac{(\gamma+n)(\beta-\gamma)\,x}{\gamma+1 \quad +} \, \frac{(\beta+1)(n-1)\,x}{\gamma+2 \ + \ldots +} \, \frac{(\beta+n-1).1x}{\gamma+2n-2 \ +} \, \frac{(\beta-\lambda-n+1)x}{1}$$

$$\ldots\ldots(10)$$

* See Section (*O*), "On the Theory of Continued Fractions" (2nd Paper), *Proc. Edin. Math. Soc.* **35** (Part I.), 1916–17, p. 48.

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Denoting the continued fraction (10) by $\dfrac{1}{g_{2n}(\beta,\gamma)}$, we obtain

$$\left.\begin{aligned}
F(\gamma+1,\beta,\gamma,x) &= (1-x)^{-(\beta+1)}\, g_2(\beta,\gamma)\\
F(\gamma+2,\beta,\gamma,x) &= (1-x)^{-(\beta+2)}\, g_2(\beta+1,\gamma+1)\, g_4(\beta,\gamma)\\
&\cdots\cdots\cdots\cdots\\
F(\gamma+n,\beta,\gamma,x) &= (1-x)^{-(\beta+n)}\, g_2(\beta-n-1,\gamma+n-1)\ldots g_{2n}(\beta,\gamma)\\
&= (1-x)^{-(\beta+n)}\, F(-n,\gamma-\beta,\gamma,x)
\end{aligned}\right\}\ldots(11)$$

If $x=1$, the continued fraction (10) automatically reduces to the value γ/β.

Therefore, when $x=1$,

$$g_{2n}(\beta,\gamma)=\beta/\gamma.$$

And then from the last of the relations (11) we obtain

$$Lt\, x=1\,[F(\gamma+n,\beta,\gamma,x)\,(1-x)^{\beta+n}]^{*} = \frac{(\beta+n-1)(\beta+n-2)\ldots\beta}{(\gamma+n-1)(\gamma+n-2)\ldots\gamma}$$
$$=F(-n,\gamma-\beta,\gamma,1).$$

Similarly in the case of the generalised hypergeometric series of Heine, namely, $\phi(\alpha,\beta,\gamma,q,x)$†, we have

$$\phi(\gamma+n,\beta,\gamma+1,q,1)\div\phi(\gamma-n,\beta,\gamma,q,1)=(1-q^{\gamma})\div(1-q^{\beta})$$

and $\phi(\gamma+n,\beta,\gamma,q,x)\div(\beta+n,q,x)$

$$=\{(q^{\beta+n-1}-1)\ldots(q^{\beta}-1)\}\div\{(q^{\gamma+n-1}-1)\ldots(q^{\gamma}-1)\}$$

$$=1-\begin{bmatrix}n\\1\end{bmatrix}\frac{q^{\gamma-\beta}-1}{q^{\gamma}-1}\,q^{\beta}x+\begin{bmatrix}n\\2\end{bmatrix}\frac{(q^{\gamma+\beta}-1)(q^{\gamma-\beta+1}-1)}{(q^{\gamma}-1)(q^{\gamma+1}-1)}\,q^{2\beta}x^2+\ldots\ldots$$

$$+(-1)^n\begin{bmatrix}n\\n\end{bmatrix}\frac{(q^{\gamma-\beta}-1)\ldots(q^{\gamma-\beta+n-1}-1)}{(q^{\gamma}-1)\ldots(q^{\gamma+n-1}-1)}\,q^{n\beta}x^n$$

where $x=1$.

---

* This is evidently the case of $F(\alpha,\beta,\gamma,1)=\dfrac{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)}$ due to Gauss.

† *i.e.* $1+\dfrac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}x\ldots$

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7. If all the A's after A_n and all the B's after B_n are zeros, then the quotient becomes

$$\frac{A_0 + A_1 x + \ldots + A_n x^n}{B_0 + B_1 x + \ldots + B_n x^n},$$

and the corresponding continued fraction will evidently terminate at the $(2n+1)^{\text{th}}$ convergent. The denominator and the numerator of the $(2n+1)^{\text{th}}$ convergent are respectively

$$C_0 + C_1 x + \ldots\ldots C_n x^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

and

$$C_{2n+1} + C_{2n} x + \ldots + C_{n+1} x^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

where $(-1)^m C_m$ is the determinant obtained by deleting the $(m+1)^{\text{th}}$ column of the array

$$\| \; (A_{2n})_{n+1} \, (B_{2n})_{n+1} \; \|$$

where $\quad A_{2n} = A_{2n-1} = \ldots = A_{n+1} = \ldots = B_{n+1} = \ldots = B_{2n} = 0.$

As the $2n^{\text{th}}$ convergent and the fraction are identically equal, then by comparing the various powers of x, we have the relations

$$C_0 : C_1 : \ldots\ldots : C_{n+1} = B_0 : B_1 : \ldots\ldots : B_n : A_n : \ldots\ldots A_0 ;$$

and also

$$\text{if} \quad B_m = 0, \text{ then } C_m = 0, \text{ where } m \not> n,$$

$$\text{and if} \quad A_{n-r+1} = 0, \text{ then } C_{n+r} = 0, \text{ where } r \not> n+1.$$

8. If $\Delta_{2r-1} \neq 0$, and if all* the bigradients

$$\begin{vmatrix} A_p & B_p \\ (A_{2r})_{r+1} & (B_{2r})_{r+1} \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

(where $p > 2r$ and $r < n$)

vanish, then the continued fraction will terminate at the $(2r+1)^{\text{th}}$ convergent, that is

$$\frac{A_0 + A_1 x + \ldots + A_n x^n}{B_0 + B_1 x + \ldots + B_n x^n} = \frac{b_0}{1+} \, \frac{b_1 x}{1} + \ldots + \frac{b_{2r} x}{1}$$

$$= \frac{C_{r+1} + C_{r+2} + \ldots\ldots + C_{2r+1} x^r}{C_0 + C_1 x + \ldots + C_r x^r} \ldots\ldots\ldots\ldots\ldots (15)$$

---

* Evidently $(n-r)$ in number, for in others all the elements of the first row are zero.

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where $(-1)^m C_m$ is the determinant obtained by deleting the $(m+1)^{\text{th}}$ column of the array

$$\left\| (A_{2r})_{r+1} (B_{2r})_{r+1} \right\|$$

As the denominator and the numerator of the convergent are both polynomials of the r^{th} degree, therefore

$$A_0 + A_1 x + \ldots + A_n x^n \ldots\ldots\ldots\ldots\ldots\ldots \text{(16)}$$

and $$B_0 + B_1 x + \ldots + B_n x^n \ldots\ldots\ldots\ldots\ldots\ldots \text{(17)}$$

must contain a polynomial of $(n-r)^{\text{th}}$ degree as a common factor.

Hence *if* $\Delta_{2r-1} \neq 0$, *then the vanishing of the* $(n-r)$ *bigradients* (14) *is the necessary and sufficient condition for the two equations*

$$A_0 + A_1 x + \ldots + A_n x^n = 0$$

and $$B_0 + B_1 x + \ldots + B_n x^n = 0,$$

having $(n-r)$ *and only* $(n-r)$ *common roots.*

This was first shown by Lemonnier.*

9. As (17) contains the denominator of (15) as a factor, therefore the continued fraction corresponding to

$$\frac{B_0 + B_1 x + \ldots + B_n x^n}{C_0 + C_1 x + \ldots + C_r x^r}$$

will terminate at the $(2n-2r+1)^{\text{th}}$ convergent. The denominator and the numerator of the $(2n-2r+1)^{\text{th}}$ convergent are respectively

$$d_0 + d_1 x + \ldots\ldots\ldots\ldots + d_{n-r} x^{n-r} \ldots\ \ldots\ldots\ldots \text{(18)}$$

and $$d_{2n-2r+1} + d_{2n-2r}\, x + \ldots\ldots + d_{n-r+1}\, x^{n-r} \ldots\ldots\ldots \text{(19)}$$

where $(-1)^m d_m$ is the determinant obtained by deleting the $(m+1)^{\text{th}}$ column of the array

$$\left\| (B_{2n-2r})_{n-r+1} (C_{2n-2r})_{n-r+1} \right\| .\dagger$$

As the denominator (18) is independent of x, therefore

$$d_1 = d_2 = d_3 = \ldots = d_{n-r} = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(20)}$$

Similarly from (16) and the numerator of (15) we obtain

$$e_0 + e_1 x + \ldots\ldots + e_{n-r}\, x^{n-r} \ldots\ldots\ldots\ldots\ldots\ldots \text{(21)}$$

$$e_{2n-2r+1} + \ldots\ldots + e_{n-r+1}\, x^{n-r} \ldots\ldots\ldots\ldots\ldots\ldots \text{(22)}$$

---

* *Memoire sur l'élimination. Annales de l'École Norm. Sup.* (2) 7 (1878), p. 151.

† The C's with suffixes higher than n are to be replaced by zeros.

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as the denominator and the numerator of the $(2n-2r+1)^{\text{th}}$ convergent to the corresponding continued fraction. Here $(-1)^m e_m$ is the determinant obtained by deleting the $(m+1)^{\text{th}}$ column of the array

$$\left\| \;(A_{2n-2r})_{n-r+1}\; (k_{2n-2r})_{n-r+1}\; \right\|$$

where $k_0 = c_{2n+1}$, $k_1 = c_{2n}$, and so on, $k_n = c_{n+1}$.

Hence we have also

$$e_1 = e_2 = e = \ldots = e_{n+r} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

As the numerator (19) and (22) divided respectively by d_0 and e_0 become the common factor, therefore we have

$$d_0 : d_{2n-2r+1} : d_{2n-2r} : \ldots : d_{n-r+1}$$
$$= e_0 : e_{2n-2r+1} : e_{2n-2r} : \ldots : e_{n-r+1} = \ldots\ldots\ldots\ldots\ldots (24)$$

We see that if the condition (14) is satisfied, then each of the conditions (20), (23), and (24) is satisfied. If all the conditions (20), (23), and (24) are satisfied, then the condition (14) is satisfied. But for certain values of r the fulfilment of the condition (20) will also imply the fulfilment of the condition (14).

10. Let f_n be the numerator of the n^{th} convergent of the continued fraction

$$\frac{a_1}{1+}\,\frac{a_2 x}{1+}\,\frac{a_3 x}{1+}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

It can be easily shown that *if* $n < m < 2n+1$, *and if* f_m *contains* f_n *as a factor, then the* m^{th} *convergent will be equal* to the* n^{th} *convergent.*

Hence if $2r > n-1$, then the vanishing of all the determinants (20) will also imply the vanishing of all the determinants (14).

But if $m > 2n$, then f_m may contain f_n as a factor, and still the two convergents may not be equal. As, for example, in the case of the continued fraction

$$\frac{1}{1}-\frac{x}{1}-\frac{x}{1}-\frac{x}{1}-\ldots,$$

* If n is odd, m is odd also; if n is odd and m even, then at least two of the convergents are equal, but they may not be the n^{th} and the m^{th} convergent.

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we have

$$f_{2n}=1-\binom{2n-1}{1}x+\binom{2n-2}{2}x^2-\ldots+(-1)^n\binom{n}{n}x^n$$

and $f_{2n+1}=1-\binom{2n}{1}x+\binom{2n-1}{2}x^2-\ldots\ldots+(-1)^n\binom{n+1}{n}x^n.$

It can be easily shown that

$$f_{2^n(r+1)-1}=f_r\,(f_{r+1}-xf_{r-1})\,(f_{2(r+1)}-xf_{2r})\ldots\ldots(f_{2^{n-1}(r+1)}-xf_{2^{n-1}(r+1)-2})$$

where $f_{2n+2}-xf_{2n}$

$$=1-\frac{2n+2}{1}x+\binom{2n-1}{1}\frac{2n+2}{2}x^2-\binom{2n-2}{2}\frac{2n+2}{3}x^3+\ldots\ldots$$

$$\ldots\ldots+(-1)^{n+1}\binom{n}{n}\frac{2n+2}{n+1}x^{n+1}$$

$$=1-(2n+2)\int_0^x f_{2n}\,dx.$$

As f_{2m-1} is the numerator of the $2m^{\text{th}}$ convergent, so we see that though f_{2n+1} contains f_n as a factor, still the two convergents are unequal. In this case the condition (14) does not necessarily depend on (20).

11. *If $n < m < 3n-1$, and if the m^{th} convergent contains the n^{th} convergent as a factor, then the m^{th} convergent shall be equal to the n^{th} convergent.* Thus, if $r > \frac{n}{3} - \frac{1}{6}$, then the fulfilment of the conditions (20) and (23) also implies the fulfilment of the condition (14).

But if $r < \frac{n}{3} - \frac{1}{6}$, then though both the conditions (20) and (23) may be satisfied, still (14) may not be satisfied.

The necessary* conditions for the 8^{th} convergent containing the 3^{rd} convergent of the continued fraction (25) as a factor, and at the same time the two convergents remaining unequal, are

$$a_6+a_7+a_8=\frac{(a_2+a_3)^2+a_6\,a_8}{a_2+a_3}$$

$$=\frac{a_3^{\,2}+a_6\,a_8}{a_3},$$

---

* The conditions are sufficient if none of the a's is zero.

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which are the same as the vanishing of the two continuants

$$\begin{vmatrix} \theta & 1 & & \\ a_6 & 1 & 1 & \\ & a_7 & \theta & 1 \\ & & a_8 & 1 \end{vmatrix}$$

where $\theta = a_2 + a_3$ and a_3.

The necessary conditions for the n^{th} convergent containing the 3^{rd} as a factor, and at the same time the two convergents remaining unequal, are the vanishing of the two continuants of order $n-4$,

$$\begin{vmatrix} \theta & 1 & & \\ a_6 & 1 & 1 & \\ & a_7 & \theta & 1 \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

In the case of the continued fraction

$$\frac{1}{1+}\,\frac{3x}{1+}\,\frac{x}{1+}\,\frac{ax}{1+}\,\frac{cx}{1+}\,\frac{2x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\ldots$$

the 3^{rd} convergent is $\dfrac{1+x}{1+4x}$, and the 8^{th} convergent is

$$\frac{1+x}{1+4x}\cdot\frac{1+(5+a+c)x+(4+4a+3c)x^2}{1+(5+a+c)x+(4+4a+3c)x^2+3ax^3}.$$

Though the 8^{th} convergent contains the 3^{rd} as a factor, still they are unequal.

The necessary conditions for the 11^{th} convergent containing the 4^{th} as a factor, and at the same time the two convergents remaining unequal, are

$$p_2 = a_7(a_9 + a_{10} + a_{11}) + a_8(a_{10} + a_{11}) + a_9 a_{11}$$

$$= (a_2 + a_3 + a_4)\frac{a_7 a_9 a_{11}}{a_2 + a_4} + a_2 a_4$$

$$p_1 = a_7 + a_8 + a_9 + a_{10} + a_{11}$$

$$= \frac{a_7 a_9 a_{11}}{a_2 a_4} + a_2 a_3 a_4$$

and

$$p_1 = \frac{p_2(a_2 + a_3) - a_7 a_9 a_{11}}{(a_3 + a_4)^2} + a_3 + a_4.$$

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This last condition is the same as

$$\begin{vmatrix} \theta & 1 & & & & \\ a_7 & 1 & 1 & & & \\ & a_8 & \theta & 1 & & \\ & & a_9 & 1 & 1 & \\ & & & a_{10} & \theta & 1 \\ & & & & a_{11} & 1 \end{vmatrix} = 0$$

where $\theta = a_3 + a_4$.

If instead of (11) we take $n\,(>11)$, one of the conditions shall always be the vanishing of a continuant of order $n-5$.

The conditions are derived from the recurrence-formulae connecting three successive convergents. These recurrence-formulae also show that if $n < m < 3n-1$, then the m^{th} convergent cannot contain the n^{th} convergent as a factor unless some of the convergents are equal. But if the convergents are considered in connection with the corresponding series, then it will be evident that the m^{th} convergent is equal to the n^{th} convergent.

It can be easily shown that if the $(3n-1)$ convergent contains the n^{th} convergent as a factor, and at the same time the two convergents remain unequal, then

$${}^*K(a_{n+3}\,a_{n+4}\dots a_{3n-1}) = K(a_2\,a_3\dots a_n)\,K(a_3\,a_4\dots a_n),$$

and if the $3n^{\text{th}}$ convergent contains the n^{th} convergent as a factor, then

$$K(a_{n+3}\,a_{n+4}\dots a_{3n-1}) = K(a_2\,a_3\dots a_n)\,K(a_3\,a_4\dots a_n).$$

12. The quotient of the two series

$$\frac{A_0 + \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dots}{B_0 + \dfrac{B_1}{x} + \dfrac{B_2}{x^2} + \dots} \qquad \dots\dots\dots\dots \quad (26)$$

can also be converted into a continued fraction of the form

$$\frac{a_1}{1} + \frac{a_2}{x+b_2} + \frac{a_3}{x+b_3} + \dots \qquad \dots\dots\dots\dots \quad (27)$$

---

* $K(a_2\,a_3\dots)$ denotes the continuant $\begin{vmatrix} 1 & -1 & & \\ a_2 & 1 & -1 & \\ & a_3 & 1 & -1 \\ \dots & \dots & \dots & \dots \end{vmatrix}$

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The elements of the continued fraction are

$$b_n = \frac{{}^1D_{2n-3}}{D_{2n-3}} - \frac{{}^1D_{2n-1}}{D_{2n-1}}$$

$$a_n = \frac{D_{2n-5}\,D_{2n-1}}{D^2_{2n-3}},$$

and $f_n(x)$ and $\phi_n(x)$, the denominator and the numerator of the $(n+1)^{\text{th}}$ convergent are given by

$$f_n(x) = x^n - \frac{{}^1D_{2n+1}}{D_{2n+1}}x^{n-1} + \frac{{}^2D_{2n+1}}{D_{2n+1}}x^{n-2} - \ldots + (-1)^n\frac{{}^nD_{2n+1}}{D_{2n+1}}$$

$$\phi_n(x) = \frac{{}^{2n+1}D_{2n+1}}{D_{2n+1}}x^n - \frac{{}^{2n}D_{2n+1}}{D_{2n+1}}x^{n-1} + \ldots + (-1)^n\frac{{}^{n+1}D_{2n+1}}{D_{2n+1}}$$

where $(-1)^m\,{}^mD_{2n+1}$ is the determinant obtained by deleting the $(m+1)^{\text{th}}$ column $(D_{2n+1} = {}^0D_{2n+1})$ of the array

$$\left\| \,(A_{2n})_{n+1}\,(B_{2n})_{n+1}\, \right\|.$$

The successive convergents to the continued fraction (27) have the property that if the n^{th} convergent, as well as the quotient (26), are expanded as a power-series in $\frac{1}{x}$, then the first $(2n-1)$ terms of these two expansions will be, term for term, the same.

This may be easily proved in the following way:—

If the quotient $(26) = c_0 + \frac{c_1}{x} + \frac{c_2}{x} + \ldots\ldots$,

then the 3^{rd} convergent $\dfrac{\beta_5 x^2 + \beta_4 x + \beta_3}{x^2 + \beta_1 x + \beta_2}$

$$= c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \frac{c_4}{x^4} + \frac{v}{x^5} + \ldots\ldots$$

Multiplying the two sides of this relation by the denominator of (26) we obtain

$$\frac{\beta_5 x^2 + \beta_4 x + \beta_3}{x^2 + \beta_1 x + \beta_2}\left(B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \ldots\right)$$

$$= \left(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{A_4}{x^4} + \frac{w}{x^5} + \ldots\right)$$

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Now, multiplying up and equating the coefficients of $\frac{1}{x^2}$, $\frac{1}{x}$, x^0, x, $\frac{1}{x^2}$ to zero, we obtain

$$\begin{aligned}
A_4+\beta_1 A_3+\beta_2 A_2 &= \beta_3 B_2+\beta_4 B_3+\beta_5 B_4 \\
A_3+\beta_1 A_2+\beta_2 A_1 &= \beta_3 B_2+\beta_4 B_2+\beta_5 B_3 \\
A_2+\beta_1 A_1+\beta_2 A_0 &= \beta_3 B_0+\beta_4 B_1+\beta_5 B_2 \\
A_1+\beta_0 A_0 &= \beta_4 B_0+\beta_5 B_1 \\
A_0 &= \beta_5 B_0
\end{aligned}$$

Solving these simultaneous equations for β's, we obtain

$$\frac{1}{D_5}=\frac{-\beta_1}{{}^1D_5}=\frac{\beta_2}{{}^2D_5}=\frac{\beta_3}{{}^3D_5}=\frac{-\beta_4}{{}^4D_5}=\frac{\beta_5}{{}^5D_5}.$$

From the recurrence-formulae connecting the three successive convergents we have

$$f_4=(x+b_5)\,f_3+a_5 f_2$$

and

$$\phi_4=(x+b_5)\,\phi_3+a_5\,\phi_2\,.$$

Equating to zero the coefficients of the various powers of x, we obtain

$$\begin{aligned}
1 &= 1 \\
-\frac{{}^1D_9}{D_9} &= -\frac{{}^1D_7}{D_7}+b_5 \\
\frac{{}^2D_9}{D_9} &= \frac{{}^2D_7}{D_7}-b_5\frac{{}^1D_7}{D_7}+a_5 \\
-\frac{{}^3D_9}{D_9} &= -\frac{{}^3D_7}{D_7}+b_5\frac{{}^2D_7}{D_7}-a_5\frac{{}^1D_5}{D_5} \\
\frac{{}^4D_9}{D_9} &= 0\ -b_5\frac{{}^3D_7}{D_7}-a_5\frac{{}^2D_5}{D_5} \\
-\frac{{}^5D_9}{D_9} &= 0\ +b_5\frac{{}^4D_7}{D_7}-a_5\frac{{}^3D_5}{D_5} \\
\frac{{}^6D_9}{D_9} &= \frac{{}^4D_7}{D_7}-b_5\frac{{}^5D_7}{D_7}+a_5\frac{{}^4D_5}{D_5} \\
-\frac{{}^7D_9}{D_9} &= -\frac{{}^5D_7}{D_7}+b_5\frac{{}^6D_7}{D_7}-a_5\frac{{}^5D_5}{D_5} \\
\frac{{}^8D_9}{D_9} &= \frac{{}^6D_7}{D_7}-b_5\frac{{}^7D_7}{D_7}+0 \\
\frac{{}^9D_9}{D_9} &= -\frac{{}^7D_7}{D_7}+0\qquad-0\,.
\end{aligned}$$

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Multiplying the first five equations in order by A_7, A_6, A_5, A_4, and A_3, and the last five equations by B_3, B_4, B_5, B_6, and B_7, and then adding up vertically, we find that

$$0=\frac{D_9}{D_7}+a_5\frac{D_7}{D_5}$$

$$\therefore \quad a_5=-\frac{D_5\,D_9}{D_7^2},$$

and from the second equation we have

$$b_5=\frac{{}^1D_7}{D_7}-\frac{{}^1D_9}{D_9}.$$

Here we can show that

If $D_{2n+1}\neq 0$ *and if all the bigradents*

$$\begin{vmatrix} A_p & B_p \\ (A_{2n})_{n+1} & (B_{2n})_{n+1} \end{vmatrix}$$

$$(\text{where } p=2n+1,\ 2n+2,\ \dots\ 2n+r)$$

vanish, then the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, ... $(n+r)^{\text{th}}$ *convergents are equal to each other, and the expansion of the quotient of the two series will agree with the expansion of the* $(n+1)^{\text{th}}$ *convergent as far as the* $(2n+r+1)^{\text{th}}$ *term (inclusive).* The proof is similar to that of Art. 4.

13. It has been stated in Art. 11 that if $n<m<3n-1$ and if the m^{th} convergent contains the n^{th} convergent as a factor, then the two convergents are equal. Here we shall show the nature of the remaining factor when $m>3n-1$. Both n and m are supposed to be odd integers, that is, the convergents are those of the continued fraction (27) of Art. 12.

If $r>n$ and if

$$\phi r=(x^{r-n}+q_{r-n-1}\,x^{r-n-1}+\ \dots\ +q_1x+q_0)\,\phi n$$
$$fr=(x^{r-n}+p_{r-n-1}\,x^{r-n-1}+\ \dots\ +p_1x+p_0)fn$$

then $\qquad p_{r-n-1},\ p_{r-n-2},\ \dots,\ p_{r-3n}$

$$=q_{r-n-1},\ q_{r-n-2},\ \dots,\ q_{r-3n} \qquad \dots\dots\dots\dots \qquad (y)$$

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And also if the first of the determinants

$$\begin{vmatrix} A_p & B_p \\ (A_{2n})_{n+1} & (B_{2n})_{n+1} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (x)$$

(where $p = 2n+1,\ 2n+2,\ \ldots$)

does not vanish, then, in general,

$$p_{r-3n-1},\ p_{r-3n-2},\ \cdots,\ p_0$$

$$\neq q_{r-3-n},\ \ldots\ldots\ldots\ldots,\ q_0 .$$

But if the first l determinant of (x) vanish, then, in addition to (y), we have also

$$p_{r-3n-1},\ \cdots,\ p_{r-3n-l}$$

$$= q_{r-3n-1},\ \cdots,\ q_{r-3n-l} .$$

The p's and q's can be expressed in terms of A's and B's.

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