# ON CERTAIN DISCRETE INEQUALITIES INVOLVING PARTIAL SUMS

PAUL R. BEESACK

**1. Introduction.** Our aim in this paper is to prove inequalities of the form

$$(1) \qquad \sum_{i=1}^{n} x_i^{\alpha}\left(\sum_{j=1}^{i} x_j\right)^{\beta} \leqq A_n(\alpha, \beta)\left(\sum_{i=1}^{n} x_i\right)^{\alpha+\beta},$$

or

$$(2) \qquad \sum_{i=1}^{n} x_i^{\alpha}\left(\sum_{j=1}^{i} x_j\right)^{\beta} \geqq a_n(\alpha, \beta)\left(\sum_{i=1}^{n} x_i\right)^{\alpha+\beta},$$

for all real values of the parameters $\alpha, \beta$ and all non-negative (in some cases all positive) $x_i$. Obviously, $a_n$ is finite in all cases, and we shall show that $A_n$ is finite if $\alpha$ and $\alpha + \beta$ are both non-negative. In all cases, we obtain sharp values of the constants $a_n, A_n$ (when finite), as well as bounds for these constants, and their behaviour as $n \to \infty$. In case $\alpha < 0$, we naturally consider only positive $x_i$, otherwise the $x_i$ may be non-negative. Although we always write $x_i \geqq 0$ in the following, this should be read as $x_i > 0$ in case $\alpha < 0$; similar remarks apply to the parameter $t$ introduced below.

We shall use the sequential optimization technique of dynamic programming to obtain our results. The analysis, as well as the results, depend on the region of the plane in which the point $(\alpha, \beta)$ lies. Figure 1 shows the nine cases which we consider in separate sections, although the values of $a_n$ and $A_n$ are not the same throughout each region or on each boundary line.

In the last section, we indicate how the inequalities can be applied to obtain discrete analogues of integral inequalities of Opial (**2**) and Yang (**4**). One such inequality was proved recently by Wong (**3**), and more extensive results have been obtained by Lee (**1**).

**2. The recursion relations.** For each integer $k \geqq 1$ and each real $y \geqq 0$ (or $y > 0$ if $\alpha + \beta < 0$), set

$$(3) \qquad F_k(y) = \sup_{\substack{\Sigma_1^k x_i = y; \\ x_i \geqq 0}} \sum_{i=1}^{k} x_i^{\alpha}\left(\sum_{j=1}^{i} x_j\right)^{\beta}.$$

Then $F_1(y) = y^{\alpha+\beta}$, and

$$(4) \qquad F_{k+1}(y) = \sup_{0 \leqq x \leqq y} \{x^{\alpha} y^{\beta} + F_k(y - x)\}, \qquad k = 1, 2, \ldots.$$

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Received September 15, 1967.

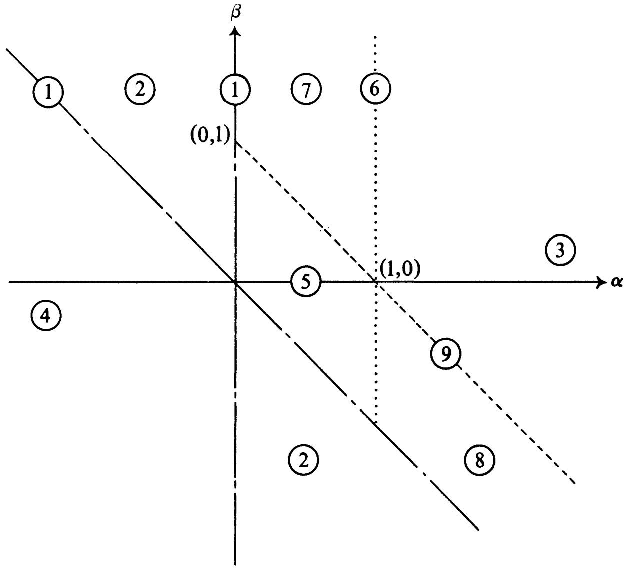


FIGURE 1

It is convenient to make the substitution $x = ty$, $0 \leqq t \leqq 1$, so that

$$(5) \qquad F_{k+1}(y) = \sup_{0 \leqq t \leqq 1} \{y^{\alpha+\beta} t^{\alpha} + F_k[(1-t)y]\} \equiv \sup_{0 \leqq t \leqq 1} G_{k+1}(t).$$

We note that if sup is replaced by inf throughout, then (4) and (5) remain valid. For clarity we shall write $f_k$, $g_k$ in place of $F_k$, $G_k$ when dealing with infima; $f_1(y) = y^{\alpha+\beta}$ is also valid.

We have that

$$(6) \qquad G_2(t) = y^{\alpha+\beta}\{t^{\alpha} + (1-t)^{\alpha+\beta}\} = g_2(t).$$

Dealing with suprema for now, note that if $\alpha \geqq 0$ and $\alpha + \beta \geqq 0$, then

$$(7) \qquad F_2(y) = h_2(t_2) y^{\alpha+\beta},$$

where

$$(8) \qquad h_2(t_2) = \sup\{t^{\alpha} + (1-t)^{\alpha+\beta}\} = \sup h_2(t),$$

is attained for some $t_2 \in [0, 1]$. It also follows that $A_2(\alpha, \beta) = h_2(t_2)$. In fact, it is clear that we have

$$(9) \qquad A_n(\alpha, \beta) = h_n(t_n) \quad \text{for } n \geqq 1,$$

https://doi.org/10.4153/CJM-1969-022-1 Published online by Cambridge University Press

where the functions $h_n$ are defined recursively by

$$\begin{cases} h_1(t) \equiv 1, \\ h_n(t) \equiv t^\alpha + h_{n-1}(t_{n-1})(1-t)^{\alpha+\beta}, \quad n \geqq 2, \end{cases} \tag{10}$$

and $t_n$ is any number $(0 \leqq t_n \leqq 1)$ such that

$$h_n(t_n) = \sup_{0 \leqq t \leqq 1} h_n(t).$$

In case $0 < t_n < 1$, we note that we necessarily have that $h_n'(t_n) = 0$, so that $t_n$ must satisfy the equation

$$k(t) \equiv t^{\alpha-1}(1-t)^{1-(\alpha+\beta)} = \frac{\alpha+\beta}{\alpha} h_{n-1}(t_{n-1}), \qquad n \geqq 2. \tag{11}$$

For convenience we define $t_1 = 1$, and list below certain relations based on (10) and (11) which we shall use repeatedly:

$$h_n'(t) = \alpha t^{\alpha-1} - (\alpha+\beta)h_{n-1}(t_{n-1})(1-t)^{\alpha+\beta-1}; \tag{12}$$

$$h_n''(t) = \alpha(\alpha-1)t^{\alpha-2} + (\alpha+\beta)(\alpha+\beta-1)h_{n-1}(t_{n-1})(1-t)^{\alpha+\beta-2}; \tag{13}$$

$$k'(t) = t^{\alpha-2}(1-t)^{-(\alpha+\beta)}(\alpha - 1 + \beta t); \tag{14}$$

$$h_n(t_n) = \frac{\alpha+\beta t_n}{\alpha+\beta} t_n^{\alpha-1} \quad \text{if } h_n'(t_n) = 0 \qquad (0 < t_n < 1); \tag{15}$$

$$h_n(t_n) = h_{n-1}(t_{n-1}) \frac{\alpha+\beta t_n}{\alpha}(1-t_n)^{\alpha+\beta-1} \quad \text{if } h_n'(t_n) = 0 \quad (0 < t_n < 1). \tag{16}$$

The same results apply with sup replaced by inf throughout, and we shall use the same notation (that is, $h_n$) for the successive functions in this case also. Here, of course, we have that $a_n(\alpha, \beta) = h_n(t_n)$, where the $h_n$ are defined by (10) and $h_n(t_n) = \inf h_n(t)$, for *all* $\alpha$ and $\beta$.

**3.** $\alpha(\alpha+\beta) = 0$**.** Suppose first that $\alpha = 0$, so that

$$G_2(t) = y^\beta\{1 + (1-t)^\beta\} = g_2(t).$$

If $\beta > 0$, then $\sup G_2(t) = 2y^\beta$ and $\inf g_2(t) = y^\beta$, and it is clear that $\sup G_n(t) = ny^\beta$, $\inf g_n(t) = y^\beta$ for each $n \geqq 1$. Hence,

$$a_n(0, \beta) = 1, \quad A_n(0, \beta) = n \qquad \textit{if } \beta > 0. \tag{17}$$

If $\beta < 0$, we obtain $\inf g_n(t) = ny^\beta$, $\sup G_n(t) = \infty$; the latter is easily seen directly from (1) by letting $x_1 \to 0+$. Hence,

$$a_n(0, \beta) = n, \quad A_n(0, \beta) = \infty \qquad \textit{if } \beta < 0. \tag{18}$$

Obviously, $a_n(0, 0) = A_n(0, 0) = n$.

Now suppose that $\alpha + \beta = 0$. From (10) we see at once that if $\alpha > 0$, $\sup h_n(t) = n$ and $\inf h_n(t) = 1$ for each $n \geqq 1$, so that

(19) $$a_n(\alpha, -\alpha) = 1, \quad A_n(\alpha, -\alpha) = n \qquad \text{if } \alpha > 0.$$

Similarly, we obtain

(20) $$a_n(\alpha, -\alpha) = n, \quad A_n(\alpha, -\alpha) = \infty \qquad \text{if } \alpha < 0;$$

again, the latter can be seen directly from (1) (for $n \geqq 2$) by taking $x_n > 0$ and letting $x_1 \to 0+$.

**4.** $\alpha(\alpha + \beta) < 0$. It is clear from (10) that $\sup h_n(t) = \infty$ in either of these cases (if $n \geqq 2$). On the other hand, by (12), $h_n'$ has the same sign on (0, 1) in either of the cases. It follows that

(21) $$a_n(\alpha, \beta) = 1, \quad A_n(\alpha, \beta) = \infty \qquad \text{if } \alpha(\alpha + \beta) < 0.$$

The latter result (for $n \geqq 2$) can be seen by fixing $x_1 > 0$ and letting $x_n \to \infty$ or $0+$ in (1) according as $\alpha > 0$ or $\alpha < 0$.

**5.** $\alpha > 1, \alpha + \beta > 1$. In this case, $h_n''(t) > 0$ for $0 < t < 1$ by (13), so that each $h_n$ is convex. It follows that $\sup h_n(t) = 1$ for all $n \geqq 1$. The same result holds even if $\alpha = 1$ $(\beta > 0)$; if, in addition, $\beta = 0$, then $h_n(t) \equiv 1$. Hence,

(22) $$\begin{cases} A_n(\alpha, \beta) = 1 & \text{if } \alpha \geqq 1, \alpha + \beta \geqq 1, \\ a_n(1, 0) = 1. \end{cases}$$

To deal with infima, we note first that $k(0) = 0$, $k(1-) = \infty$, and $k'(t) > 0$ for $0 < t < 1$. Hence, each of equations (11) has a unique root $t_n \in (0, 1)$, and $h_n'(t_n) = 0$. *Thus,*

(23) $$a_n(\alpha, \beta) = h_n(t_n) \quad \text{if } \alpha > 1, \alpha + \beta > 1,$$

*where the functions $h_n$ are defined by equations* (10), *and $t_n$ is the unique solution on* (0, 1) *of equation* (11). From (10) it is obvious that since $h_n(0) = h_{n-1}(t_{n-1})$, we have that

$$h_n(t_n) < h_{n-1}(t_{n-1}) < \ldots < h_2(t_2) \leqq 2^{-\alpha}(1 + 2^{-\beta}) < 1 = h_1(t_1).$$

It then follows from the increasing character of $k$ on (0, 1) that the sequence $\{t_n\}$ is a strictly decreasing sequence of positive numbers.

Although we have upper bounds for $a_n(\alpha, \beta)$, in the context of (2) we are more concerned with lower bounds. To this end, we note by (16) that

$$h_n(t_n) > h_{n-1}(t_{n-1})\left(1 - \frac{\alpha - 1}{\alpha} t_n\right)(1 - t_n)^{\alpha+\beta-1} > \alpha^{-1} h_{n-1}(t_{n-1})(1 - t_n)^{\alpha+\beta-1},$$

since $\beta > 1 - \alpha$ and $\alpha > 1$. On the other hand, $k$ is strictly increasing so that, if $t = \bar{t}$ is the unique solution of $k(t) = (\alpha + \beta)/\alpha$, we must have that

$0 < t_n < \bar{t}$, and hence $h_n(t_n) > \alpha^{-1}h_{n-1}(t_{n-1})(1 - \bar{t})^{\alpha+\beta-1}$. Consequently, we obtain the lower bound

(24) $$h_n(t_n) > \{\alpha^{-1}(1 - \bar{t})^{\alpha+\beta-1}\}^n \quad \text{if } n \geqq 2, \alpha > 1, \alpha + \beta > 1.$$

*We shall also show that* $\lim t_n = \lim h_n(t_n) = 0$. To prove this, we note first that $\lim t_n = a$ exists, where $1 > a \geqq 0$. Hence, by (15),

$$\lim h_n(t_n) = \frac{\alpha + \beta a}{\alpha + \beta} a^{\alpha-1},$$

and from (16), $\alpha = (\alpha + \beta a)(1 - a)^{\alpha+\beta-1}$, if $a \neq 0$. However, denoting the right side of this latter equation by $g(a)$, we have that $g(0) = \alpha$, $g(1) = 0$, and $g'(a) < 0$ for $0 \leqq a < 1$. It follows that $a = 0$, and hence that $\lim h_n(t_n) = 0$.

**6.** $\alpha < 0, \alpha + \beta < 0$. As in the preceding section, $h_n$ is convex. Now, of course, $h_n(0+) = h_n(1-) = \infty$ for $n > 1$, so that

(25) $$A_n(\alpha, \beta) = \infty \quad \text{if } \alpha < 0, \alpha + \beta < 0.$$

The analysis for infima proceeds precisely as in § 5 except that now $k$ *decreases* steadily on $(0, 1]$ from $\infty$ to 0. *Hence,*

(26) $$a_n(\alpha, \beta) = h_{n-1}(t_{n-1}) \quad \text{if } \alpha < 0, \alpha + \beta < 0,$$

*where the $h_n$ and $t_n$ are defined by equations* (10) *and* (11). From (10) we see that $h_n(t_n) > 1 + h_{n-1}(t_{n-1})$ for all $n \geqq 2$, so that

(27) $$a_n(\alpha, \beta) > n \quad \text{if } n \geqq 2, \alpha < 0, \alpha + \beta < 0.$$

Moreover, since $k$ is decreasing, it follows from (11) that $\{t_n\}$ is a strictly decreasing sequence; by using (15) and (27) we see that $\lim t_n = 0$.

**7.** $0 < \alpha < 1, 0 < \alpha + \beta < 1$. By (13), $h_n''(t) < 0$ for $0 < t < 1$, so that $-h_n$ is strictly convex, and $\inf h_n(t) = 1$ for all $n \geqq 1$. Clearly, the same result holds if either $\alpha = 1$ or $\alpha + \beta = 1$. Hence,

(28) $$a_n(\alpha, \beta) = 1 \quad \text{if } 0 < \alpha \leqq 1, 0 < \alpha + \beta \leqq 1.$$

In the present case, $k'(t) < 0$ for $0 < t < 1$ and $k$ decreases on $(0, 1]$ from $\infty$ to 0. Hence, each of equations (11) has a unique root $t_n \in (0, 1)$ with $h_n'(t_n) = 0$. It follows that

(29) $$A_n(\alpha, \beta) = h_n(t_n) \quad \text{if } 0 < \alpha < 1, 0 < \alpha + \beta < 1,$$

*where the functions $h_n$ are defined recursively by equations* (10), *and $t_n$ is the unique solution on* (0, 1) *of equation* (11). From (10) we see that

$$1 < h_2(t_2) < \ldots < h_{n-1}(t_{n-1}) = h_n(0) < h_n(t_n) < 1 + h_{n-1}(t_{n-1}).$$

Therefore,

(30) $$1 < h_n(t_n) < n \quad \text{if} \quad n \geqq 2, 0 < \alpha < 1, 0 < \alpha + \beta < 1.$$

Since $k$ is decreasing, it follows that the sequence $\{t_n\}$ is again a strictly decreasing sequence. If $a = \lim t_n$, then $1 > a \geqq 0$. If $a > 0$, then by (15),

$$\lim h_n(t_n) = \frac{\alpha + \beta a}{\alpha + \beta} a^{\alpha - 1},$$

and by (16), $\alpha(1 - a)^{1-(\alpha+\beta)} = \alpha + \beta a$ which is impossible since

$$(1 - a)^{1-(\alpha+\beta)} < \alpha + \beta a$$

for $0 < \alpha + \beta < 1$, and $0 < a < 1$. It follows that $\lim t_n = 0$, and hence $\lim h_n(t_n) = \infty$ by (15).

**8.** $\alpha = 1, \alpha + \beta > 0$. If $\beta > 0$, then the analysis of § 5 remains unchanged except that now $k$ increases from 1 to $\infty$ on $[0, 1)$, and we must verify that each of equations (11) *has* a root on $(0, 1)$. This will be the case if and only if

$$(1 + \beta) h_{n-1}(t_{n-1}) > 1 \quad \textit{for } n \geqq 2. \tag{31}$$

We shall prove (31) by induction, incidentally obtaining a better lower bound than would be obtained by setting $\alpha = 1$ in (24). Now, (31) is certainly true if $n = 2$. Moreover, if it is true for any $n = k \geqq 2$, then $t_k$ is well-defined, and in fact, from (11) with $\alpha = 1$,

$$t_k = 1 - m_{k-1}^{-1/\beta}, \quad \text{where } m_{k-1} = (1 + \beta) h_{k-1}(t_{k-1}).$$

Using (16), we obtain

$$h_k(t_k) = (1 + \beta t_k) h_{k-1}(t_{k-1}) \cdot m_{k-1}^{-1} = 1 + \beta(t_k - 1) h_{k-1}(t_{k-1}) m_{k-1}^{-1},$$

or

$$h_k(t_k) = 1 - \beta / \{(1 + \beta)^{1+\beta} h_{k-1}(t_{k-1})\}^{1/\beta}. \tag{32}$$

Thus, $(1 + \beta) h_k(t_k) > 1$ if and only if $\beta + m_k^{-1/\beta} > (1 + \beta) m_k^{-1/\beta}$, that is, if and only if

$$\beta m_k^{1/\beta} + 1 > \beta + 1;$$

the latter inequality is, however, true by our induction assumption since $\beta > 0$. Hence, *the result* (23) *of* § 5 *is also valid when* $\alpha = 1, \beta > 0$. In this case, the $a_n(\alpha, \beta) = h_n(t_n)$ may also be computed directly from the recursion relation (32), and satisfy the inequality (31). By proceeding as in § 5, it is easy to verify that $\lim t_n = 0$ and $\lim h_n(t_n) = (1 + \beta)^{-1}$.

If $\alpha = 1$ and $0 < \alpha + \beta < 1$, that is, $-1 < \beta < 0$, the analysis of § 7 remains unchanged except that now $k$ decreases from 1 to 0 on $[0, 1]$. Again, we must verify that each of equations (1) *has* a root on $(0, 1)$. This will be the case if and only if

$$0 < (1 + \beta) h_{n-1}(t_{n-1}) < 1 \quad \textit{for } n \geqq 2. \tag{33}$$

The proof of this by induction is precisely the same as before, *so that the result* (29) *of* § 7 *is also valid if* $\alpha = 1, -1 < \beta < 0$. The recursion relations (32) are also valid in this case, and the results $\lim t_n = 0$, $\lim h_n(t_n) = (1+\beta)^{-1}$ follow by setting $\alpha = 1$ in § 7.

**9.** $0 < \alpha < 1, \alpha + \beta > 1$. In this case, we note that $h_n(0) = h_{n-1}(t_{n-1}) > 0$ and $h_n(1) = 1$, by (10). Moreover, from (12), $h_n'(0+) = +\infty$ and $h_n'(1) = \alpha > 0$ so that $h_n'$ has at least two zeros on (0, 1) provided $h_{n-1}(t_{n-1}) \geqq 1$. The latter is clearly the case if $n = 2$, or for all $n$ when dealing with suprema. To see that $h_n'$ has precisely two zeros on (0, 1) in such circumstances, we note that $k(0+) = k(1-) = +\infty$ from (11) and $k'(t) = 0$ if and only if $t = a = (1-\alpha)/\beta$ by (14). Hence, $k$ is decreasing on $(0, a]$ and increasing on $[a, 1)$. It follows that $h_n'$ has at most two zeros, hence precisely two zeros on (0, 1) if either $n = 2$ or when dealing with suprema. We have also proved that

$$k(a) = \frac{\beta^\beta}{(1-\alpha)^{1-\alpha}(\alpha+\beta-1)^{\alpha+\beta-1}} < \frac{\alpha+\beta}{\alpha}. \tag{34}$$

Denoting the zeros of $h_n'$ by $t_n, t_n'$, where $0 < t_n < a < t_n' < 1$, we obviously have that $\sup h_n(t) = h_n(t_n)$. *Hence,*

$$A_n(\alpha, \beta) = h_n(t_n) \quad \textit{if } 0 < \alpha < 1, \alpha + \beta > 1, \tag{35}$$

*where $h_n(t_n)$ is defined by equations* (10) *and* (11). *In this case, however, equation* (11) *has two roots on* (0, 1), *and $t_n$ is the smaller of these two roots.* Moreover, it follows from (10) that

$$1 < h_2(t_2) < \ldots < h_n(t_n) < n. \tag{36}$$

Essentially, the same analysis as in § 7 shows that in the present case we must also have that $\lim t_n = 0$ and $\lim h_n(t_n) = \infty$.

It is somewhat more difficult to deal with the successive infima. The reason for this can already be seen when $n = 3$, where we have that $h_3(0) = h_2(t_2') < 1 = h_3(1)$, and $h_3'(0+) = +\infty$, $h_3'(1) = \alpha$. It is not obvious that $h_3'$ has *any* zeros on (0, 1), or even if it has, whether $\inf h_3(t)$ occurs at such a zero, or for $t = 0$. Nevertheless, we shall prove by induction that

$$f_n(y) = h_n(t_n')y^{\alpha+\beta},$$

where

$$\begin{cases} b < t_n' < 1, & h_n'(t_n') = 0, \\ k(a)(t_n')^{1-\alpha} < (\alpha + \beta t_n')/\alpha, \end{cases} \tag{37}$$

and $b$ is the unique root on (0, 1) of

$$s(t) \equiv (\alpha + \beta t)(1-t)^{\alpha+\beta-1} = \alpha. \tag{38}$$

Note first that

$$s(0) = \alpha, \quad s(1) = 0, \quad \text{and} \quad s'(t) = (\alpha+\beta)(1-t)^{\alpha+\beta-2}\{(1-\alpha) - \beta t\}.$$

Hence, $s$ is increasing on $[0, a]$ and decreasing on $[a, 1]$, so that (38) *has* a unique root $b \in (a, 1)$. Now, if $n = 1$, then the conditions (37) are satisfied, by (34), since we may take $t_1' = 1 - \epsilon$ for sufficiently small $\epsilon > 0$. If the result is valid for any $n \geqq 1$, then

$$f_{n+1}(y) = y^{\alpha+\beta} \inf h_{n+1}(t) = y^{\alpha+\beta} \inf\{t^\alpha + h_n(t_n')(1-t)^{\alpha+\beta}\}.$$

We shall prove that $h_{n+1}'(b) < 0$. Since

$$h_{n+1}(0) = h_n(t_n') \leqq h_n(0) = 1 = h_{n+1}(1),$$

and $h_{n+1}'(0+) = \infty$, $h_{n+1}'(1) = \alpha > 0$, it will follow from the character of $k$ on $(0, 1)$ that $h_{n+1}'$ has precisely two zeros on $(0, 1)$, say $t_{n+1}$ and $t_{n+1}'$, and that $0 < t_{n+1} < a < b < t_{n+1}' < 1$. We have that $h_{n+1}'(b) < 0$ if and only if

$$\alpha b^{\alpha-1} < (\alpha+\beta)h_n(t_n')(1-b)^{\alpha+\beta-1} \leftrightarrow \alpha < (\alpha+\beta)h_n(t_n')b^{1-\alpha}\alpha/(\alpha+\beta b),$$

since $b$ satisfies (38). Hence, $h_{n+1}'(b) < 0$ if and only if

$$(\alpha+\beta b)b^{\alpha-1} < (\alpha+\beta)h_n(t_n') = (\alpha+\beta t_n')(t_n')^{\alpha-1}$$

by (15). The latter inequality is valid by the first of the induction assumptions (30), since the function $r(t) \equiv (\alpha+\beta t)t^{\alpha-1}$ is strictly increasing on $[a, 1]$ and $a < b < t_n < 1$.

Denoting the two zeros of $h_{n+1}'$ by $t_{n+1}$ and $t_{n+1}'$ as above, it follows that $\inf h_{n+1}(t) = h_{n+1}(t_{n+1}')$ *provided* we can show that

$$h_{n+1}(t_{n+1}') < h_{n+1}(0) = h_n(t_n).$$

By (16), this is the case if and only if

$$(\alpha+\beta t_{n+1}')(1-t_{n+1}')^{\alpha+\beta-1} < \alpha,$$

and this follows from our remarks concerning the function $s$ since we have established that $b < t_{n+1}' < 1$.

In order to complete the induction, we shall show that if

$$y(t) = 1 + (\beta/\alpha)t - k(a)t^{1-\alpha},$$

then $y(t) > 0$ for $a \leqq t \leqq 1$, in particular for $t = t_{n+1}'$. To prove this, we note first that $y(1) = (\alpha+\beta)/\alpha - k(a) > 0$ by (34). Moreover,

$$y(a) = 1 + \frac{\beta}{\alpha}\frac{1-\alpha}{\beta} - k(a)\left(\frac{1-\alpha}{\beta}\right)^{1-\alpha}$$
$$= \frac{1}{\alpha} - \left(\frac{\beta}{\alpha+\beta-1}\right)^{\alpha+\beta-1},$$

which is positive if and only if $\alpha\beta^{\alpha+\beta-1} < (\alpha+\beta-1)^{\alpha+\beta-1}$. The latter inequality is easily proved by setting $\alpha+\beta = x$ and showing that $z(\alpha) \equiv \alpha(x-\alpha)^{x-1}$ is strictly increasing on $0 \leqq \alpha \leqq 1$ for each $x > 1$. Thus, $y(a) > 0$ and $y(1) > 0$. Moreover, $y''(t)$ is positive for all $t > 0$, and

$$y'(t) = 0 \leftrightarrow t^\alpha = \alpha(1-\alpha)k(a)/\beta.$$

Since $\alpha(1-\alpha)k(a)\beta^{-1} < a^{\alpha}$ is equivalent to $\alpha\beta^{\alpha+\beta-1} < (\alpha+\beta-1)^{\alpha+\beta-1}$, it follows that $y(t) > 0$ for $t \in [a, 1]$.

We have shown that

$$a_n(\alpha, \beta) = h_n(t_n') \quad \textit{if} \quad 0 < \alpha < 1, \alpha + \beta > 1. \tag{39}$$

*Here, the functions $h_n$ are defined by equations* (10), *and each of equations* (11) *has two roots on* (0, 1), *$t_n'$ being the larger of these two roots.*

Since $a = (1-\alpha)/\beta < b < t_n' < 1$ for all $n$, while $\{h_n(t_n')\}$ is strictly decreasing and the function $k$ is increasing on $[a, 1)$, it follows that the sequence $\{t_n'\}$ is also strictly decreasing. Moreover, since $r$ is strictly increasing on $[a, 1]$ we have that

$$h_n(t_n') > \frac{\alpha + \beta b}{\alpha + \beta} b^{\alpha - 1} \quad \text{for all } n \geqq 1. \tag{40}$$

Writing $\lim t_n = \bar{t}$, we have that $\bar{t} \geqq b$. Using (15), (16), and the decreasing character of the function $s$ on $[b, 1]$, it is easily seen that $\bar{t} = b$, and

$$\lim h_n(t_n') = \frac{\alpha + \beta b}{\alpha + \beta} b^{\alpha - 1}. \tag{41}$$

**10.** $\alpha > 1, 0 < \alpha + \beta < 1$. This case is similar to that of the preceding section, but roughly with the roles of $t$ and $1 - t$ interchanged. Hence, we shall deal with this case more briefly. We have that $h_n(0) = h_{n-1}(t_{n-1}) > 0$, $h_n(1) = 1$, $h_n'(0) = -(\alpha+\beta)h_{n-1}(t_{n-1}) < 0$, and $h_n'(1-) = -\infty$. On the other hand, $k$ is now increasing on $[0, a]$ and decreasing on $[a, 1]$ with $k(0) = k(1) = 0$, where $a = (1-\alpha)/\beta = (\alpha - 1)/(-\beta)$, and

$$k(a) = \frac{(-\beta)^{\beta}}{(\alpha-1)^{1-\alpha}(1-\alpha-\beta)^{\alpha+\beta-1}} > \frac{\alpha+\beta}{\alpha}, \tag{42}$$

since $h_2'$ has at least two, hence precisely two, zeros on (0, 1). Dealing with successive infima we obtain

$$a_n(\alpha, \beta) = h_n(t_n) \quad \textit{if } \alpha > 1, 0 < \alpha + \beta < 1, \tag{43}$$

*the numbers $h_n(t_n)$ again being defined by equations* (10) *and* (11). *In this case, equation* (11) *has two roots on* (0, 1) *and $t_n$ is the smaller of these roots.* The sequence $\{t_n\}$ is strictly decreasing and $t_n < a$ for $n > 1$. Moreover, it is clear that $1 > h_1(t_1) > \ldots > h_n(t_n) > \ldots$. The analysis of § 5 again shows that $\lim t_n = 0$ and $\lim h_n(t_n) = 0$.

Denoting the successive suprema by $h_n(t_n')$, one may prove by induction that, in this case, we have that

$$b < t_n' < 1, \quad h_n'(t_n') = 0, \quad k(a)(t_n')^{1-\alpha} > (\alpha + \beta t_n')/\alpha,$$

where $b$ is again the unique root on $(0, 1)$ of equation (38). The proof is essentially the same as before except that now the functions $s$, $r$, and $y$ introduced in § 9 satisfy the following conditions: $s$ is decreasing on $[0, a]$ and increasing on $[a, 1]$; $r$ is decreasing on $[a, 1]$; $y(t) < 0$ for $t \in [a, 1]$ (here the significant inequality is $(-\beta)^{1-(\alpha+\beta)} < \alpha(1-\alpha-\beta)^{1-\alpha-\beta}$, which is valid for $\alpha > 1, 0 < \alpha + \beta < 1$). In this case *we conclude that*

$$A_n(\alpha, \beta) = h_n(t_n') \quad \text{if } \alpha > 1, 0 < \alpha + \beta < 1. \tag{44}$$

*For each* $n \geqq 2$, $t_n'$ *is the larger of the two roots of equation* (11). The sequence $\{t_n\}$ is strictly decreasing with $\lim t_n = b$, $\{h_n(t_n')\}$ is strictly increasing, and

$$\lim h_n(t_n') = \frac{\alpha + \beta b}{\alpha + \beta} b^{\alpha - 1}. \tag{45}$$

**11.** $\alpha + \beta = 1, \alpha > 0$. We have already dealt with the case $\beta = 0$ in (22), so that only the cases $0 < \beta < 1$, $0 < \alpha < 1$, and $\beta < 0, \alpha > 1$ remain. We shall handle these cases simultaneously. For both, we note that $h_n(0) = h_{n-1}(t_{n-1})$ and $h_n(1) = 1$. However, $h_n$ is convex if $\alpha > 1$, while $-h_n$ is convex if $0 < \alpha < 1$. It follows at once that

$$A_n(\alpha, \beta) = 1 \quad \text{if } \alpha > 1, \alpha + \beta = 1, \tag{46}$$

$$a_n(\alpha, \beta) = 1 \quad \text{if } 0 < \alpha < 1, \alpha + \beta = 1. \tag{47}$$

Moreover, from (11) and (14), $k$ is increasing on $[0, 1]$ from 0 to 1 if $\alpha > 1$, while $k$ decreases from $+\infty$ to 1 on $(0, 1)$ if $0 < \alpha < 1$. In both these cases, we can solve (11) explicitly for $t_n$ to obtain

$$t_n = \{\alpha / h_{n-1}(t_{n-1})\}^{1/\beta} \quad \text{if } n \geqq 2, \tag{48}$$

on noting that $h_{n-1}(t_{n-1}) = \inf h_{n-1}(t) < 1$ if $\alpha > 1$, while $h_{n-1}(t_{n-1}) = \sup h_{n-1}(t) > 1$ if $0 < \alpha < 1$. *We thus have that*

$$a_n(\alpha, \beta) = h_n(t_n) \quad \text{if } \alpha > 1, \alpha + \beta = 1, \tag{49}$$

$$A_n(\alpha, \beta) = h_n(t_n) \quad \text{if } 0 < \alpha < 1, \alpha + \beta = 1, \tag{50}$$

where the $h_n(t_n)$ are defined by equations (10) and (48). From (10) we see that $\{h_n(t_n)\}$ is strictly decreasing if $\alpha > 1$, and strictly increasing if $0 < \alpha < 1$; hence, $\{t_n\}$ is strictly decreasing in either case, by (48). Using (16) we easily obtain the bounds

$$\begin{cases} h_n(t_n) > \alpha^{-n} & \text{if } \alpha > 1, \alpha + \beta = 1, \\ h_n(t_n) < \alpha^{-n} & \text{if } 0 < \alpha < 1, \alpha + \beta = 1, \end{cases} \tag{51}$$

and, by the usual argument, also obtain $\lim t_n = 0$, and

$$\begin{cases} \lim h_n(t_n) = 0 & \text{if } \alpha > 1, \alpha + \beta = 1, \\ \lim h_n(t_n) = \infty & \text{if } 0 < \alpha < 1, \alpha + \beta = 1. \end{cases} \tag{52}$$

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Using (48) and either (10) or (15), we see that the $h_n = h_n(t_n)$ are also given as the solution of the finite difference equation

$$(53)\qquad \begin{cases} h_1 = 1, \\ \Delta h_{n-1} = \beta\alpha^{\alpha/\beta} h_{n-1}^{-\alpha/\beta}, \quad n \geqq 2. \end{cases}$$

From (53) we can show that *if* $0 < \alpha < 1, \alpha + \beta = 1$, *then*

$$(54)\qquad h_n \leqq 1 + (n-1)\beta\alpha^{\alpha/\beta} \leqq n \quad \text{for } n \geqq 1,$$

which is a better estimate than (51) for large $n$ (or for all $n$ if $0 < \alpha < 3^{-1/3}$).

**12. Application to other discrete inequalities.** The discrete analogue of (an extension of) Opial's inequality which, as mentioned in the Introduction, was recently proved by Wong (**3**), may be stated in the following form. *If all* $x_j \geqq 0$, *and* $p \geqq 1$, *then*

$$(55)\qquad \sum_{i=1}^{n} x_i \Big( \sum_{j=1}^{i} x_j \Big)^p \leqq \frac{(n+1)^p}{p+1} \sum_{i=1}^{n} x_i^{p+1}.$$

Somewhat earlier, Yang (**4**, Lemma 7) proved the following generalization of Opial's inequality. *If $y$ is absolutely continuous on $[a, X]$ with $y(a) = 0$, then for $p, q \geqq 1$,*

$$(56)\qquad \int_a^X |y|^p |y'|^q\,dx \leqq \frac{q}{p+q}(X-a)^p \int_a^X |y'|^{p+q}\,dx.$$

Yang's proof of (56) is actually valid for all $p \geqq 0, q \geqq 1$. Opial's inequality is the special case of (56) obtained by setting $a = 0, p = q = 1$. Wong's result (55) is clearly the discrete analogue of (56) with $q = 1$. Recently, Lee (**1**) has obtained other discrete analogues of (56) involving both $p$ and $q$.

In order to obtain discrete analogues of (56) from the inequalities (1) and (2), we may make use of the following results:

$$(57)\quad n^{\gamma-1} \sum_{i=1}^{n} x_i^\gamma \leqq \Big( \sum_{i=1}^{n} x_i \Big)^\gamma \leqq \sum_{i=1}^{n} x_i^\gamma \qquad \textit{if all } x_i \geqq 0, 0 \leqq \gamma \leqq 1;$$

$$(58)\quad \sum_{i=1}^{n} x_i^\gamma \leqq \Big( \sum_{i=1}^{n} x_i \Big)^\gamma \leqq n^{\gamma-1} \sum_{i=1}^{n} x_i^\gamma \quad \textit{if all } x_i \geqq 0, \gamma \geqq 1;$$

$$(59)\quad \Big( \sum_{i=1}^{n} x_i \Big)^\gamma \leqq n^{\gamma-1} \sum_{i=1}^{n} x_i^\gamma \qquad \textit{if all } x_i > 0, \gamma < 0.$$

These results may be easily proved by the same methods used earlier, or by using Hölder's inequality, or the convexity of $x^\gamma$ if $\gamma > 1$, or $\gamma < 0$, or of $-x^\gamma$ if $0 < \gamma < 1$.

As a first example, if $p + q \geqq 1$ and $q \geqq 1$, then using (22), (46), and (58), we have that

$$(60)\qquad \sum_{i=1}^{n} x_i^q \Big( \sum_{j=1}^{i} x_j \Big)^p \leqq n^{p+q-1} \sum_{i=1}^{n} x_i^{p+q} \quad \textit{if all } x_i \geqq 0.$$

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When $q = 1$, the constant $n^p$ in (60) is larger than Wong's constant $(n+1)^p/(p+1)$ in (55) for all $p > 1$ and sufficiently large $n$, and in fact, for all $n \geqq 1$ if $p = 1$. However, (55) is false for $p < 1$, while (60) is still valid. Although Wong's result is sharp only for $p = 1$, (60) is *never* sharp since equality is attained in (1) under different conditions than in the right-hand part of (58).

As a second example, if $0 < q < 1$, $0 < p + q < 1$, then using (28), (29), (30), and (57) we have that

$$(61)\quad n^{p+q-1} \sum_{i=1}^{n} x_i^{p+q} \leqq \sum_{i=1}^{n} x_i^q \left( \sum_{j=1}^{i} x_j \right)^p \leqq h_n(t_n) \sum_{i=1}^{n} x_i^{p+q} < n \sum_{i=1}^{n} x_i^{p+q}.$$

Again, these inequalities are *not* sharp, for the same reason as before.

Another method of obtaining inequalities of the form (60) or (61) is to make use of the sharp special cases of such inequalities contained in (21), (46), (47), (49), and (50), which we rewrite as

$$(62)\quad \sum_{i=1}^{n} a_i \leqq \sum_{i=1}^{n} a_i^{\alpha} \left( \sum_{j=1}^{i} a_j \right)^{\beta} \quad \textit{if all } a_i > 0, \alpha < 0, \alpha + \beta = 1,$$

$$(63)\quad \sum_{i=1}^{n} a_i \leqq \sum_{i=1}^{n} a_i^{\alpha} \left( \sum_{j=1}^{i} a_j \right)^{\beta} \leqq h_n(t_n) \sum_{i=1}^{n} a_i \quad \textit{if } 0 < \alpha < 1, \alpha + \beta = 1,$$

$$(64)\quad h_n(t_n) \sum_{i=1}^{n} a_i \leqq \sum_{i=1}^{n} a_i^{\alpha} \left( \sum_{j=1}^{i} a_j \right)^{\beta} \leqq \sum_{i=1}^{n} a_i \quad \textit{if } \alpha > 1, \alpha + \beta = 1.$$

Now, considering the left side of (60), we let $a_i = x_i^{p+q}$; therefore $x_i = a_i^{1/(p+q)}$ and $x_i^q = a_i^{q/(p+q)}$, whence,

$$\sum_{i=1}^{n} x_i^q \left( \sum_{j=1}^{i} x_j \right)^p = \sum_{i=1}^{n} a_i^{q/(p+q)} \left( \sum_{j=1}^{i} a_i^{1/(p+q)} \right)^p.$$

Making use of (57)–(59) we have, for example, that

$$\sum_{i=1}^{n} x_i^q \left( \sum_{j=1}^{i} x_j \right)^p \leqq \sum_{i=1}^{n} a_i^{q/(p+q)} \left( \sum_{j=1}^{i} a_i^{p/(p+q)} \right) \qquad \text{if } 0 \leqq p \leqq 1 \text{ or } p < 0,$$

$$\sum_{i=1}^{n} x_i^q \left( \sum_{j=1}^{i} x_j \right)^p \leqq n^{p-1} \sum_{i=1}^{n} a_i^{q/(p+q)} \left( \sum_{j=1}^{i} a_i^{p/(p+q)} \right) \quad \text{if } p \geqq 1.$$

Setting $\alpha = q/(p+q)$, $\beta = p/(p+q)$, it now follows from (63) that

$$(65)\quad \sum_{i=1}^{n} x_i^q \left( \sum_{j=1}^{i} x_j \right)^p \leqq h_n(t_n) \sum_{i=1}^{n} x_i^{p+q}$$

$$\textit{if } p < 0, q < 0 \text{ or } 0 \leqq p \leqq 1, q < 0.$$

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Here, $h_n(t_n) \leqq 1 + (n - 1)pq^{q/p}(p + q)^{-(q+p)/p} \leqq n$ by (54). Similarly, it follows from (64) that

$$\sum_{i=1}^{n} x_i^q \left( \sum_{j=1}^{i} x_j \right)^p \leqq n^{p-1} \sum_{i=1}^{n} x_i^{p+q} \quad \textit{if } p \geqq 1, p + q < 0. \tag{66}$$

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*Carleton University,*
*Ottawa, Canada*