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ON THE UNITARIZED ADJOINT REPRESENTATION OF A SEMISIMPLE LIE GROUP II

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Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{A}$. Lebesgue measure on $\mathfrak{A}$ is invariant under the adjoint action of G; and so there is a natural unitary representation T_G of G on $L_2(\mathfrak{A})$ given by

$$T_G(g)f(x) = f(\mathrm{Ad}\, g^{-1}(x)), \quad g \in G, f \in L_2(\mathfrak{A}), x \in \mathfrak{A}.$$

In the paper [**3**], I considered the problem of decomposing T_G into its irreducible constituents (see [**3**, § 1, paragraph 2] for a brief historical comment on this problem). There I gave a complete description of the continuous spectrum of T_G. In this paper I shall prove a result that yields information on the discrete spectrum. I expect to complete the discussion in [**4**] (see the last paragraph of this paper for my reasons for dividing the consideration of the discrete spectrum into two separate articles).

We begin by quickly recalling the results of [**3**]. If Z_G is the center of G, it is obvious that $T_G|_{Z_G} \equiv 1$. Thus T_G is actually a representation of $G/Z_G \cong \mathrm{Ad}\,\mathfrak{A}$. We shall therefore assume, without loss of generality, that $Z_G = \{e\}$—that is, $G = \mathrm{Ad}\,\mathfrak{A}$. Now let K be a maximal compact subgroup of G, and choose a set of representatives $P_1, \ldots, P_r$ for the associativity classes of proper cuspidal parabolic subgroups of G. Write $P_i = M_i A_i N_i$ for a Langland's decomposition of P_i that is compatible with K, M_i^0 for the neutral component of M_i, and $\mathfrak{a}_i$ for the Lie algebra of A_i. Finally set $F_0 = Z_K$ and $F_i = Z_{(K \cap M_i^0)} \times \Gamma_i$, $i = 1, \ldots, r$, where $\Gamma_i = K \cap \exp \sqrt{-1}\, \mathfrak{a}_i$ (see [**3**, § 4]). The F_i are clearly compact abelian groups, and if P_1 is minimal then $F_1 = Z_{M_1}$. Theorems 5 and 9 of [**3**] assert that T_G is a subrepresentation of the (left) regular representation λ_G of G, has uniform infinite multiplicity, and

$$(*) \quad T_G \cong \begin{cases} \displaystyle\sum_{i=0}^{r}{}^{\oplus} \infty \ \mathrm{Ind}_{F_i}{}^G 1 & \mathrm{rank}\, G = \mathrm{rank}\, K \\ \displaystyle\sum_{i=1}^{r}{}^{\oplus} \infty \ \mathrm{Ind}_{F_i}^{G} 1 & \mathrm{rank}\, G > \mathrm{rank}\, K. \end{cases}$$

We also proved (in [**3**, Theorem 10]) that the principal series representations of G corresponding to P_i occur in the representation $\mathrm{Ind}_{F_i}{}^G 1$. Thus if rank $G >$ rank K, then $T_G \cong \lambda_G$ (see [**3**, Theorem 12]); and in general to complete the description of the constituents of T_G, we must look at its discrete spectrum to see "how much" of the discrete series occurs therein.

We first show that we may reduce to the case of simple groups.

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LEMMA 1. *Suppose* $\mathfrak{A} = \mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_n$ *is the decomposition of* $\mathfrak{A}$ *into simple Lie algebras. Then*

$$T_{\mathrm{Ad}\,\mathfrak{A}} \cong T_{\mathrm{Ad}\,\mathfrak{A}_1} \times \ldots \times T_{\mathrm{Ad}\,\mathfrak{A}_n}.$$

Proof. This is obvious, because $G \cong \mathrm{Ad}\,\mathfrak{A}_1 \times \ldots \times \mathrm{Ad}\,\mathfrak{A}_n$ and $L_2(\mathfrak{A}) \cong L_2(\mathfrak{A}_1) \otimes \ldots \otimes L_2(\mathfrak{A}_n)$.

Remark. Suppose $G = \mathrm{Ad}\,\mathfrak{A}$ is compact and B is a maximal torus. Then by [**3**, Theorem 1]

$$T_G \cong \infty \; \mathrm{Ind}_B{}^G\, 1.$$

Moreover by [**3**, Lemma 6]

$$\mathrm{Ind}_B{}^G\, 1 \approx \lambda_G,$$

where $\approx$ denotes quasi-equivalence. Thus

$$T_G \cong \infty\, \lambda_G.$$

Since G is compact $\infty\,\lambda_G \not\cong \lambda_G$. On the other hand, if G is non-compact then $\infty\,\lambda_G \cong \lambda_G$. Thus for the remainder of our deliberations we shall assume $G = \mathrm{Ad}\,\mathfrak{A}$ is non-compact and simple.

The key result of this paper is the following

THEOREM 2. *Suppose that some* $F_i \cong \mathbf{Z}_2{}^s$. *Then*

$$\mathrm{Ind}_{F_i}{}^G\, 1 \cong \lambda_G.$$

COROLLARY 3. *If one or more of the groups* F_i *is a product of* 2*-element groups, then* $T_G \cong \lambda_G$.

The corollary follows immediately from Theorem 2, formula (*), and the fact that λ_G has uniform infinite multiplicity. Now suppose some $F_i \cong \mathbf{Z}_2{}^s$. Let us write $F = F_i$. Then we have

PROPOSITION 4. *The representation* $\mathrm{Ind}_F{}^G\, \chi$ *is, up to unitary equivalence, independent of* $\chi \in \hat{F}$.

Theorem 2 is a consequence of Proposition 4 because

$$\lambda_G \cong \mathrm{Ind}_F{}^G\, \lambda_F \cong \sum_{\chi \in F}^{\oplus} \mathrm{Ind}_F{}^G\, \chi \cong \#(F)\, \mathrm{Ind}_F{}^G\, 1.$$

So we concentrate on the

Proof of Proposition 4. We shall first give the argument in case $1 \leqq i \leqq r$. Let $P = P_i$ be the corresponding cuspidal parabolic with $P = MAN$ the Langland's decomposition chosen earlier. Write $\bar{N}$ for the opposed nilradical, $\bar{\mathfrak{n}}$ for its Lie algebra. Given $\chi \in \hat{F}$, we set

$$\mathscr{W}_\chi = \{X \in \bar{\mathfrak{n}} : \mathrm{Ad}(f)X = \chi(f)X\}.$$

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Since $F \cong \mathbf{Z}_2{}^s$, $\chi(F) \in \{\pm 1\}$ and $\bar{\pi} = \sum_{\chi \in \hat{F}} \mathscr{W}_\chi$. Let $\mathscr{F} = \{\chi \in \hat{F} : \mathscr{W}_\chi \neq \{0\}\}$. We assert that $\bigcap_{\chi \in \mathscr{F}} \operatorname{Ker} \chi = \{e\}$. Indeed (since $F \subseteq K$), any element in the intersection must—by [**3**, Lemma 11]—be central in G. Since $Z_G = \{e\}$, the assertion is valid. This guarantees that $\mathscr{F}$ generates $\hat{F}$. For if the subgroup $\Lambda \subseteq \hat{F}$ generated by $\mathscr{F}$ is smaller than $\hat{F}$, then $\{e\} \neq \{f \in F : \lambda(f) = 1$ or all $\lambda \in \Lambda\} \subseteq \{f \in F : \chi(f) = 1$ for all $\chi \in \mathscr{F}\} = \{e\}$.

Now choose a minimal set $\{\chi_1, \ldots, \chi_s\}$ in $\mathscr{F}$ which generates $\hat{F}$. Then any $\chi \in \hat{F}$ can be written uniquely $\chi = \chi_1{}^{\epsilon_1} \ldots \chi_s{}^{\epsilon_s}$, where each ϵ_i is 0 or 1. We write

$$\chi^j = \chi_1{}^{\epsilon_1} \ldots \chi_j{}^{\epsilon_j}, \quad 1 \leqq j \leqq s$$

$$\chi^0 \equiv 1.$$

In order to prove the proposition, it suffices to show that for any $j = 1, \ldots, s$

$$\operatorname{Ind}_F{}^G \chi^j \cong \operatorname{Ind}_F{}^G \chi^{j-1}.$$

Fix an integer j between 1 and s. If $\epsilon_j = 0$ there is nothing to do—so assume $\epsilon_j = 1$. Write $\tau = \chi^j$, $\tau' = \chi^{j-1}$ so that $\tau = \chi_j \tau'$. Choose $W \in \mathscr{W}_{\chi_j}$, $W \neq 0$. Set $Q = \exp \mathbf{R}W$, a one-parameter subgroup of $\bar{N}$. Clearly F stabilizes Q and FQ is a closed subgroup of G. By induction in stages, it is therefore enough to prove

$$\operatorname{Ind}_F{}^{FQ} \tau \cong \operatorname{Ind}_F{}^{FQ} \tau'.$$

Let $E = \operatorname{Ker} \chi_j$. Since $\epsilon_j = 1$, the index $[F : E]$ is 2. Choose an element $z \in F - E$ and set $Z = \{z^\epsilon : \epsilon = 0, 1\} \cong \mathbf{Z}_2$. Then F is a direct product $F = E \times Z$. Also FQ is a direct product $FQ = E \times (ZQ)$. Next put

$$\tau = \tau_1 \times \tau_2, \quad \tau_1 = \tau|_E, \quad \tau_2 = \tau|_Z$$

$$\tau' = \tau_1{}' \times \tau_2{}', \quad \tau_1{}' = \tau'|_E, \quad \tau_2{}' = \tau'|_Z.$$

Clearly,

$$\operatorname{Ind}_F{}^{FQ} \tau = \operatorname{Ind}_{E \times Z}{}^{E \times (ZQ)} \tau_1 \times \tau_2 \cong \tau_1 \times \operatorname{Ind}_Z{}^{ZQ} \tau_2,$$

and similarly

$$\operatorname{Ind}_F{}^{FQ} \tau' \cong \tau_1{}' \times \operatorname{Ind}_Z{}^{ZQ} \tau_2{}'.$$

But since $\tau = \tau' \chi_j$, we have $\tau_1 = \tau_1{}'$; and so we are reduced to proving the following

LEMMA 5. *Let $Z \cong \mathbf{Z}_2$, $Q \cong \mathbf{R}$ and let ZQ be the natural semidirect product group. Then the representation $\operatorname{Ind}_Z{}^{ZQ} \tau$ is, up to unitary equivalence, independent of $\tau \in \hat{Z}$.*

Proof. Let τ_0 be the unique non-trivial element in $\hat{Z}$. One can prove this result by constructing an explicit intertwinning operator for $\pi_0 = \operatorname{Ind}_Z{}^{ZQ} \tau_0$ and $\pi_1 = \operatorname{Ind}_Z{}^{ZQ} 1$. In fact, we can realize π_0 and π_1 on $L_2(\mathbf{R})$ by

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$$\pi_0(z^\epsilon \exp tW)f(x) = (-1)^\epsilon f((-1)^\epsilon x + t), \quad f \in L_2(R)$$

$$\pi_1(z^\epsilon \exp tW)f(x) = f((-1)^\epsilon x + t), \quad \in L_2(R);$$

and then the intertwinning operator is exactly the Hilbert transform. We leave it to the reader to carry out the details; instead we give a short, alternate representation-theoretic proof.

It is a routine exercise in the use of the Mackey machine (see e.g. [**2**, IIIA]) to compute that the irreducible representations of ZQ are given by:

$$\pi_\tau(z^\epsilon \exp tW) = \tau(z^\epsilon), \quad \tau \in \hat{Z}$$

$$\pi_\rho = \operatorname{Ind}_Q{}^{ZQ}\, \chi_\rho, \quad \chi_\rho(\exp tW) = e^{i\rho t}, \quad \rho > 0.$$

Furthermore, the Plancherel measure on $(ZQ)^\wedge$ is just Lebesgue measure on the half-line $\rho > 0$. (The representations π_ρ occur twice in the regular representation and the representations π_τ are of measure 0.) But by the Subgroup Theorem [**2**, IIA1]

$$\pi_\rho|_Z = (\operatorname{Ind}_Q{}^{ZQ}\, \chi_\rho)|_Z \cong \operatorname{Ind}_{Q \cap Z}{}^Z\, \chi_\rho = \lambda_Z = 1 \oplus \tau_0.$$

Applying Anh reciprocity [**2**, IIA4], we conclude that

$$\operatorname{Ind}_Z{}^{ZQ}\, \tau_0 \cong \operatorname{Ind}_Z{}^{ZQ}\, 1 \cong \int_{\rho>0}^{\oplus} \pi_\rho d\rho,$$

thus proving Lemma 5.

This also completes the proof of Proposition 4 in the case $F = F_i$, $1 \leqq i \leqq r$. In case rank G = rank K and $F = F_0 = Z_K \cong \mathbf{Z}_2{}^s$, Proposition 4 is still valid. The method of proof is basically the same, although the preliminary set-up is somewhat different. Let $\mathfrak{A} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{A}$ corresponding to the maximal compact subgroup K. This time we diagonalize the action of $F = Z_K \subseteq K$ on $\mathfrak{p}$. Examining the details of the proof in the preceding case we see that the only fact needed is that no non-trivial element of F can have simultaneous eigenvalue 1. But that is true since such an element would necessarily be central in G. Having made that observation, we leave the precise details to the reader.

We shall now run down the list of non-compact real simple Lie groups G to see when one (or more) of the groups $F_i \cong \mathbf{Z}_2{}^s$. We use Helgason's listing [**1**, Ch. IX, § 4]. Some of the groups listed there are not centerless; thus it is to G/Z_G that our comments apply.

(i) G has a complex structure. Since rank $K <$ rank G, we have $T_G \cong \lambda_G$ by [**3**, Theorem 12].

(ii) $G = SL(n, \mathbf{R})$, $n \geqq 2$. G is $\mathbf{R}$-split. Therefore in a minimal parabolic subgroup $P = MAN$, we have $M = \Gamma \cong \mathbf{Z}_2{}^s$, $s \leqq$ rank G. Thus $T_G \cong \lambda_G$ by Corollary 3.

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(iii) $G = \mathrm{Sp}(n, \mathbf{R})$, $n \geqq 2$. G is $\mathbf{R}$-split and $T_G \cong \lambda_G$.

(iv) $G = SU^*(2n)$, $n \geqq 2$. Since rank $K <$ rank G, $T_G \cong \lambda_G$.

(v) $G = \mathrm{Sp}(m, n), m \geqq n \geqq 1$. Here $K = \mathrm{Sp}(m) \times \mathrm{Sp}(n)$ and $F = Z_K \cong \mathbf{Z}_2{}^2$.

(vi) $G = SO_e(m, n)$, $m \geqq n \geqq 1$, $m + n \neq 2, 4$. If m and n are both odd, then rank $K <$ rank G. Otherwise $K \cong SO(m) \times SO(n)$; and in a minimal parabolic $P = MAN$, $M \cong SO(m - n) \times \mathbf{Z}_2{}^{n-1}$. Thus at least one of Z_K or Z_M is of the form $Z_2{}^s$ unless $m = 4$ and $n = 2$. Note for future reference that $SO_e(4, 2)$ and $SU(2, 2)$ are locally isomorphic.

(vii) $G = SO^*(2n)$, $n \geqq 3$. In a minimal parabolic $P = MAN$ here, we have

$$M \cong \begin{cases} SU(2) \times \ldots \times SU(2), & n = 2k,\ k \text{ copies of } SU(2) \\ SU(2) \times \ldots \times SU(2) \times \mathbf{T}, & n = 2k + 1,\ k \text{ copies of } SU(2). \end{cases}$$

Thus, for n even, $Z_M \cong \mathbf{Z}_2{}^{n/2}$. Hence for $G = SO^*(4n), n \geqq 2$, we have $T_G \cong \lambda_G$.

(viii) G exceptional. We use the notation of [**1**, p. 354, Table II]:

(a) EI, EV, $EVIII$, FI and $G2$ are $\mathbf{R}$-split;

(b) EIV has rank $K <$ rank G;

(c) FII has $K \cong \mathrm{Spin}(9)$, so $Z_K \cong \mathbf{Z}_2$;

(d) EIX has $K \cong E_7 \times SU(2)$, so $Z_K \cong \mathbf{Z}_2{}^2$;

(e) EVI has $K \cong \mathrm{Spin}(12) \times SU(2)$, so $Z_K \cong \mathbf{Z}_2{}^3$;

(f) $EVII$ has in a minimal parabolic $M^0 \cong \mathrm{Spin}(8)$, so $Z_M \subseteq \mathbf{Z}_2{}^5$;

(g) EII has $K = SU(6) \times SU(2)$. But the universal covering group has center $\cong \mathbf{Z}_6$ and so Z_K corresponding to Ad $\mathfrak{A}$ is $\cong \mathbf{Z}_2$.

In the remaining groups—namely $SU(m, n)$, $SO^*(4n + 2)$, and $EIII$—the subgroups F always have a non-trivial toral part. In any event we can summarize our discussion in

THEOREM 6. *Let $G =$ Ad $\mathfrak{A}$ be non-compact semisimple with none of its simple factors locally isomorphic to $SU(m, n)$, $m \geqq n \geqq 1$, $m + n \geqq 3$, $SO^*(4n + 2)$, $n \geqq 1$, or EIII. Then $T_G \cong \lambda_G$.*

Regarding the original problem, it remains to conside only the three types of groups enumerated in Theorem 6. Each of these falls among the class of semisimple groups G for which G/K has a hermitian symmetric structure. In particular, in those cases the holomorphic discrete series is present. We shall address ourselves to these types in [**4**]. We decided to separate these out because the techniques we will employ there will be entirely different from those used here. Specifically, we shall rely heavily on recent work of Schmid on K-types of discrete series representations (holomorphic and otherwise) to obtain results on weights of these representations when restricted to smaller subgroups of K. Here we employed non-compact reciprocity and independence theorems more analogous to the methods of [**3**]. I close with the following remark. Work in progress on several other problems in semisimple groups has led to the investigation of representations of G induced from maximally compact Cartan subgroups. Both techniques discussed in this paragraph are proving useful in that investigation.

Added in proof. Due to the tardiness of Advances in Mathematics, this article, which is a sequel to [**3**], will appear first. [**3**] was submitted in Fall 1974 and accepted in Spring 1975. I write these lines in Fall 1977. I regret any possible inconvenience to the reader. Please write to me for preprints of [**3**] and see Notices AMS, vol. 22, p. 371.

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