1
Introduction

This book discusses an approach to the analysis of asymptotic and global properties of solutions to the equations of Einstein's theory of general relativity (the Einstein field equations) based on ideas arising in conformal geometry. This approach allows a geometric and rigorous formulation of problems and notions of great physical relevance in the context of general relativity. At the same time, it provides valuable insights into the properties of the Einstein field equations under optimal regularity conditions.

Before entering into the subject, it is useful to discuss the motivation behind this type of endeavour. Accordingly, a brief account of certain aspects of what can be called *mathematical general relativity* is necessary.

1.1 On the Einstein field equations

Einstein's theory of general relativity is the best theory of gravity we have. It is a relativistic theory of gravity which considers four-dimensional differentiable, orientable manifolds $\tilde{\mathcal{M}}$ endowed with a Lorentzian metric $\tilde{\boldsymbol{g}}$; a discussion of these differential geometric notions is provided in Chapter 2. The pair $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is called a ***spacetime***. Here, and in the rest of this book, quantities associated to the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ will be distinguished by a tilde ($\tilde{}$); the motivation behind this notation will become clear in the following. The gravitational field is described in general relativity as a manifestation of the curvature of spacetime.

The fundamental equations of general relativity, the ***Einstein field equations***, describe how matter produces the curvature of spacetime. They are given, in the abstract index notation discussed in Section 2.2.6, by

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda\tilde{g}_{ab} = \tilde{T}_{ab}, \tag{1.1}$$

where $\tilde{g}_{ab}$ is the abstract index version of $\tilde{\boldsymbol{g}}$, and where $\tilde{R}_{ab}$ and $\tilde{R}$ denote, respectively, the Ricci tensor and Ricci scalar of the metric $\tilde{\boldsymbol{g}}$. Moreover, λ is the so-called ***cosmological constant*** and $\tilde{T}_{ab}$ denotes the energy–momentum

https://doi.org/10.1017/9781009291347.002 Published online by Cambridge University Press

tensor of the matter in the spacetime. Precise definitions and conventions for the curvature tensors are provided in Chapter 2, while a discussion of the energy–momentum tensors for a range of matter models is provided in Chapter 9. The energy–momentum tensor satisfies the ***conservation equation***

$$\tilde{\nabla}^a \tilde{T}_{ab} = 0,$$

where $\tilde{\nabla}_a$ denotes the covariant derivative of the metric $\tilde{\boldsymbol{g}}$. The ***Bianchi identity*** satisfied by the Riemann curvature tensor $\tilde{R}^a{}_{bcd}$ of the metric $\tilde{\boldsymbol{g}}$ ensures the consistency between the conservation equation and the Einstein field equations. A ***solution to the Einstein field equations*** is a pair $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, together with a $\tilde{\boldsymbol{g}}$-divergence-free tensor $\tilde{T}_{ab}$ such that Equation (1.1) holds. In suitable open subsets of $\tilde{\mathcal{M}}$ the metric $\tilde{\boldsymbol{g}}$ is expressed, using some *local coordinates* (x^μ), in terms of its components $(\tilde{g}_{\mu\nu})$; here and in what follows, Greek indices are used as *coordinate indices*. In general, several coordinate charts will be needed to cover the spacetime manifold $\tilde{\mathcal{M}}$. Two metrics $\tilde{\boldsymbol{g}}$ and $\bar{\boldsymbol{g}}$ over $\tilde{\mathcal{M}}$ are said to be ***isometric*** if they are related, everywhere on $\tilde{\mathcal{M}}$, by some coordinate transformation.

In the cases where $\tilde{T}_{ab} = 0$, a direct computation shows that Equation (1.1) implies

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}. \tag{1.2}$$

In what follows, the latter will be known as the ***vacuum Einstein field equations*** and a solution thereof as an ***Einstein spacetime***. The full curvature of a four-dimensional manifold is described by the tensor $\tilde{R}^a{}_{bcd}$. This tensor has 20 independent components. By contrast, the Ricci tensor appearing in the Einstein field Equations (1.1) and (1.2) has only 10 independent components. Hence, even in the absence of a cosmological constant, where the vacuum field Equations (1.2) reduce to

$$\tilde{R}_{ab} = 0, \tag{1.3}$$

it is possible to have solutions with a non-vanishing Riemann tensor. As a consequence, solutions to the vacuum field equations play a special role in general relativity, as they describe *pure gravitational configurations*. Vacuum spacetimes are often deemed more fundamental, as they exclude potential pathologies which may arise from the choice of a particular matter model.

General relativity has two main domains of applicability: *cosmology* and *isolated systems*. To make use of the Einstein field Equations (1.1) within these two domains, one requires a number of idealisations. On the one hand, in cosmology it is usually assumed that the matter content of the universe can be described by a perfect fluid with an equation of state which depends on a particular cosmological era. It is a convention in mathematical relativity to refer to spacetimes with compact spacelike sections as ***cosmological spacetimes***. On the other hand, ***isolated systems*** are convenient idealisations of astrophysical objects for which it is assumed that the cosmological expansion has no influence. The transition between the regime of isolated systems and the cosmological one is a topic of fundamental relevance for the understanding of the physical content of the Einstein field equations; see, for example, Ellis (1984, 2002).

The validity of general relativity has been verified in a number of experiments covering a wide range of scenarios ranging from the dynamics of the solar system to cosmological scales; see, for example, Will (2014) for a discussion of the subject. Surveys of the physical content of general relativity and its various domains of applicability can be found, for example, in Poisson and Will (2015) and Shapiro (1999).

Note. *In the remainder of this chapter, in order to simplify the presentation, the discussion will be restricted to Einstein spaces, that is, solutions to the vacuum Equations* (1.2)*. The inclusion of matter very often requires a case-by-case analysis.*

1.2 Exact solutions

A natural first step to developing an understanding of the properties of solutions to the Einstein field equations is the construction of ***exact solutions***, that is, explicit solutions written in terms of *elementary functions* of some coordinates. The first non-trivial exact solution to the Einstein field equations ever obtained is the Schwarzschild solution. It describes a static spherically symmetric vacuum configuration; see Schwarzschild (1916), an English translation of which can be found in Schwarzschild (2003). Remarkably, despite the complexity of the field equations, the literature contains a vast number of exact solutions to the equations of general relativity; see, for example, Stephani et al. (2003) for a monograph on the subject. The number of solutions with a physical or geometric significance is, arguably, much smaller; see, for example, Bičák (2000) and Griffiths and Podolský (2009).

1.2.1 Construction of exact solutions

The construction of exact solutions to the Einstein field equations requires a number of assumptions concerning the nature of the solutions. The most natural assumptions involve the presence of continuous symmetries (*Killing vectors*) of some type in the solution, for example, spherical symmetry, axial symmetry, stationarity (including staticity) and homogeneity. Other types of assumptions involve the algebraic structure of the curvature tensors of the spacetime (e.g. the Petrov type of the Weyl tensor). These types of assumptions are harder to justify on a physical basis.

Exact solutions are usually constructed in a coordinate system adapted to the assumptions being made. Very often, these *natural coordinates* cover only a portion of the whole spacetime manifold. Thus, one needs to find new coordinate systems (charts) for the exact solution which allow one to uncover a full *maximal*

https://doi.org/10.1017/9781009291347.002 Published online by Cambridge University Press

analytic extension of the spacetime. This maximal extension usually paves the way to the interpretation of the exact solution and gives access to its global properties.

1.2.2 The limitations of exact solutions

Several of the well-known consequences of general relativity have been developed through the analysis of exact solutions, for example, the notion of a black hole. Thus, the study of exact solutions to the Einstein field equations helps to develop a physical and geometric intuition which, in turn, can lead to questions concerning more generic solutions. However, despite the valuable insights they provide, the construction of exact solutions is not a systematic approach to explore the *space of solutions of the theory*. In particular, this approach leaves open the question of whether certain properties of a solution are *generic*, that is, satisfied by a broader class of spacetimes. Moreover, exact solutions do not lend themselves to the analysis of dynamic situations such as, for example, the description of the gravitational radiation produced by an isolated system. Thus, it is not possible to address issues involving *stability* just by means of exact solutions. In order to analyse the above issues one has to consider whether it is possible to formulate an *initial value problem for the Einstein field equations* by means of which large classes of solutions can be constructed.

1.3 The Cauchy problem in general relativity

As in the case of many other physical theories, general relativity admits the formulation of an *initial value problem* (*Cauchy problem*). This aspect of the theory is obscured by both the *tensorial character of the Einstein field equations* and the *absence of a background geometry in the theory*; it is a priori not clear that the field equations give rise to a system of partial differential equations (PDEs) of a recognisable type.

Classical physical theories are expected to satisfy a ***causality principle****: the future of an event in spacetime cannot influence its past, and, moreover, signals must propagate at finite speed*. Among the three main types of PDEs (elliptic, hyperbolic and parabolic), *hyperbolic differential equations* are the only ones compatible with the causality principle. This observation suggests it should be possible to extract from the Einstein field equations a system of evolution equations with *hyperbolic properties*.

1.3.1 Hyperbolic reductions

The seminal work of Fourès-Bruhat (1952) has shown that the hyperbolic properties of the Einstein field equations can be made manifest by means of a suitable choice of coordinates. Following modern terminology, a choice of coordinates is a particular example of *gauge choice*. Indeed, by choosing the

https://doi.org/10.1017/9781009291347.002 Published online by Cambridge University Press

spacetime coordinates (x^μ) in such a way that they satisfy the wave equation associated with the metric $\tilde{\boldsymbol{g}}$, the Einstein field equations can be shown to imply a *system of quasilinear wave equations* for the components $(\tilde{g}_{\mu\nu})$ of the (a priori unknown) metric $\tilde{\boldsymbol{g}}$ with respect to the *wave coordinates*. For quasilinear wave differential equations there exists a developed theory which allows the formulation of a *well-posed Cauchy problem*. The use of wave coordinates is not the only way of bringing to the fore the hyperbolic aspects of the Einstein field equations. In this book, it will be shown that the Einstein field equations can be reformulated in such a way that after a suitable gauge choice they imply a so-called (first order) *symmetric hyperbolic evolution system* – a class of PDEs with properties similar to those of wave equations and for which a comparable theory is available. The procedure of extracting suitable hyperbolic *evolution equations* through a particular reformulation of the Einstein field equations and a suitable gauge choice is known as a ***hyperbolic reduction***; hyperbolic reductions are further discussed in Chapter 13. Besides its natural relevance in mathematical relativity, the construction of hyperbolic reductions for the Einstein field equations is of fundamental importance for numerical relativity; see, for example, Alcubierre (2008) and Baumgarte and Shapiro (2010).

In the same way that the Einstein field equations are geometric in nature, a proper formulation of the Cauchy problem in general relativity must also be done in a geometric way; see, for example, Choquet-Bruhat (2007). This idea is, in principle, in conflict with the discussion of hyperbolicity properties of the Einstein field equations, as the associated procedure of gauge fixing breaks the *spacetime covariance* of the field equations. As will be seen in the following, this tension can be resolved in a satisfactory manner.

1.3.2 Initial data and the constraint equations

The formulation of an initial value problem for the Einstein field equations requires the prescription of suitable initial data for the evolution equations on a three-dimensional manifold $\tilde{\mathcal{S}}$. This manifold will be later interpreted as a hypersurface of the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. An important feature of general relativity is that the initial data for the evolution equations implied by the Einstein field equations are constrained. The ***constraint equations of general relativity*** (*Einstein constraints*) can be formulated as a set of equations intrinsic to the initial hypersurface $\tilde{\mathcal{S}}$ for a pair of symmetric tensors $\tilde{\boldsymbol{h}}$ and $\tilde{\boldsymbol{K}}$ describing, respectively, the intrinsic geometry of the hypersurface (***intrinsic metric*** or ***first fundamental form***) and the way the initial hypersurface is curved within the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ – the so-called ***extrinsic curvature*** or ***second fundamental form***. A priori, it is not clear what the *freely specifiable data* for these constraint equations consist of, or whether, given a particular choice of free data, the equations can be solved. The systematic analysis of the constraint equations has shown that under suitable assumptions, they can be recast as a set of *elliptic partial differential equations*; see, for example, Bartnik and Isenberg

(2004). For this type of equation a theory is available to discuss the existence and uniqueness of solutions.

The constraint equations play a fundamental role in the theory and ensure that the solution of the evolution equations is, in fact, a solution to the Einstein field equations; this type of analysis is often called the *propagation of the constraints*. The constraint equations of general relativity will be discussed in Chapter 11.

1.3.3 The well-posedness of the Cauchy problem in general relativity

The formulation of the Cauchy problem in general relativity ensures, at least locally, the existence of a solution to the Einstein field equations which is consistent with the prescribed initial data. More precisely, one has the following result first proven in Fourès-Bruhat (1952).

Theorem 1.1 (***local existence of solutions to the initial value problem***) *Given a solution* $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ *to the Einstein constraint equations on a three-dimensional manifold* $\tilde{\mathcal{S}}$ *there exists a vacuum spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *such that* $\tilde{\mathcal{S}}$ *is a spacelike hypersurface of* $\tilde{\mathcal{M}}$*,* $\tilde{\boldsymbol{h}}$ *is the intrinsic metric induced by* $\tilde{\boldsymbol{g}}$ *on* $\tilde{\mathcal{S}}$ *and* $\tilde{\boldsymbol{K}}$ *is the associated extrinsic curvature.*

The spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ obtained as a result of Theorem 1.1 is called a ***development of the initial data set*** $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$. Not every spacetime can be *globally* constructed from an initial value problem. Those which can be constructed in this way are said to be ***globally hyperbolic***. There are important examples of spacetimes which do not possess this property – most noticeably, the *anti-de Sitter spacetime*. A general result concerning globally hyperbolic spacetimes states that their topology is that of $\mathbb{R} \times \tilde{\mathcal{S}}$ with each slice $\tilde{\mathcal{S}}_t \equiv \{t\} \times \tilde{\mathcal{S}}$ being intersected only once by each timelike curve in the spacetime. The slices $\tilde{\mathcal{S}}_t$ are known as ***Cauchy surfaces***. The above points will be further discussed in Chapter 14.

The Cauchy problem for the Einstein field equations provides an appropriate setting for the discussion of dynamics. In particular, it allows one to investigate whether a given solution of the Einstein field equations is *stable*, that is, whether its essential features are retained if the initial data set is perturbed. Moreover, it also allows one to analyse whether a given property of a solution is *generic*, that is, whether the property holds for all solutions in an open set in the *space of initial data*.

1.3.4 Geometric uniqueness and the maximal globally hyperbolic development

An important observation concerning Theorem 1.1 is that it does not ensure the uniqueness of the development $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ of the initial data set $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$: a different hyperbolic reduction procedure will, in general, give rise to an alternative development $(\tilde{\mathcal{M}}', \tilde{\boldsymbol{g}}')$. From the point of view of the Cauchy problem of general relativity, the solution manifold is not known a priori. Instead, it is obtained as a part of the evolution process.

Given that an initial data set for the Einstein field equations gives rise to an infinite number of developments (one for each *reasonable* gauge choice), it is natural to ask whether it is possible to combine these various developments to obtain a *maximal development*. This question is answered in the positive by the following fundamental result; see Choquet-Bruhat and Geroch (1969).

Theorem 1.2 (***existence of a maximal development***) *Given an initial data set for the Einstein field equations* $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$*, there exists a unique maximal development* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$*, that is, a development such that if* $(\tilde{\mathcal{M}}', \tilde{\boldsymbol{g}}')$ *is another development, then* $\tilde{\mathcal{M}}' \subseteq \tilde{\mathcal{M}}$ *and on* $\tilde{\mathcal{M}}'$ *the metrics* $\tilde{\boldsymbol{g}}$ *and* $\tilde{\boldsymbol{g}}'$ *are isometric.*

The *maximal development* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is also known as the *maximal globally hyperbolic development* of the data $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$. Theorem 1.2 clarifies the sense in which one can expect uniqueness from the Cauchy problem in general relativity; this idea is known as *geometric uniqueness*.

One can think of the maximal development of an initial data set as the largest spacetime that can be uniquely constructed out of an initial value problem. The boundary of this maximal development, if any at all, sets the limits of predictability of the data – accordingly, one has a close link with the notion of *classical determinism*. In certain spacetimes, it is possible to extend the maximal development of a hypersurface to obtain a *maximal extension*. Accordingly, in general, maximal developments and maximal extensions do not coincide. A further discussion of the Cauchy problem in general relativity is provided in Chapter 14.

1.3.5 Construction of maximal developments and global existence of solutions

Given some initial data set $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$, it is natural to ask, How can one construct its maximal development $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$? In general, this is a very difficult task, as it requires controlling the evolution dictated by the Einstein field equations under very general circumstances – something for which the required mathematical technology is not yet available. There are, nevertheless, some conjectures concerning the global behaviour of maximal developments. The origin of these conjectures goes back to Penrose (1969) – see Penrose (2002) for a reprint – and are usually known by the name *cosmic censorship*. In particular, the so-called *strong cosmic censorship* states that the maximal development of generic initial data for the Einstein field equations cannot be extended as a Lorentzian manifold.

Given an exact solution to the Einstein equations, if one knows its maximal extension, one can determine the maximal development $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ of one of its (Cauchy) hypersurfaces, say, $\tilde{\mathcal{S}}$. In what follows, let $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ denote the initial data implied on $\tilde{\mathcal{S}}$ by the spacetime metric $\tilde{\boldsymbol{g}}$. The explicit knowledge of the maximal development allows one to provide a physical interpretation of the solution and to analyse its global structure in some detail. One can now ask whether certain aspects of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ – say, its basic global structure – are shared by a wider class of solutions to the Einstein field equations. A strategy to address this question within the framework of the Cauchy problem in general relativity is to consider initial data sets $(\tilde{\mathcal{S}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{K}})$ which are, in some sense, close to the initial data for the exact solution. One can then try to show that the associated maximal globally hyperbolic development $(\bar{\mathcal{M}}, \bar{\boldsymbol{g}})$ has the desired global properties. If this is the case, one has obtained a statement about the *stability* of the solution and the *genericity* of the property one is interested in. The standard convention, to be used in this book, is to call $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$, respectively, the ***background spacetime*** and the ***background initial data set*** and $(\bar{\mathcal{M}}, \bar{\boldsymbol{g}})$ and $(\tilde{\mathcal{S}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{K}})$ the ***perturbed spacetime*** and ***perturbed initial data set***, respectively. In practice, the notion of closeness between initial data sets is dictated by the requirements of the PDE theory used to prove the existence of solutions to the evolution equations. In the previous discussion it has been assumed that the 3-manifolds on which the background and perturbed initial data are prescribed are the same. The stability analysis allows one to conclude that the spacetime manifolds $\tilde{\mathcal{M}}$ and $\bar{\mathcal{M}}$ are the same – they are, however, endowed with different metrics.

In analysing the stability of the background solution $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ one needs to show that the solutions to the evolution equations with perturbed initial data exist as long as the background solution. The expectation is that the assumption of having initial data close to data for an exact solution whose global structure is well understood will ease this task. In the following sections a strategy to exploit this assumption will be discussed.

1.4 Conformal geometry and general relativity

Special relativity provides a framework for the discussion of the notion of ***causality*** – that is, the relation between cause and effect – which is consistent with the *principle of relativity*. The *causal structure* of special relativity is determined by the light cones associated with the Minkowski metric $\tilde{\boldsymbol{\eta}}$. It allows the determination of whether a signal travelling not faster than the speed of light can be sent between two events – if this is the case, then the two events are said to be ***causally related***. More generally, one can talk of *Lorentzian causality*: any Lorentzian metric $\tilde{\boldsymbol{g}}$ gives rise to a causal structure determined by the light cones associated to $\tilde{\boldsymbol{g}}$. Thus, general relativity provides a natural generalisation of the notions of causality of special relativity – one in which the light cones vary from event to event in spacetime. Crucially, however, in general relativity the causal structure is a basic unknown of the theory.

The theory of hyperbolic differential equations provides notions of causality which, in principle, are independent from the notions of Lorentzian causality. It is, nevertheless, a remarkable feature of general relativity that locally, the propagation of fields dictated by the Einstein field equations is governed by the structure of the light cones of the solutions – the so-called *characteristic*

https://doi.org/10.1017/9781009291347.002 Published online by Cambridge University Press

surfaces of the evolution equations. Thus, the notions of Lorentzian and PDE causality coincide. This aspect of the Einstein field equations is further discussed in Chapter 14.

1.4.1 Conformal transformations and conformal geometry

Locally, a light cone can be described (away from its vertex) in terms of a condition of the form $\phi(x^\mu) = constant$ where $\phi : \tilde{\mathcal{M}} \to \mathbb{R}$ is such that

$$\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0. \qquad (1.4)$$

The structure of the light cones of a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is preserved by ***conformal rescalings***, that is, transformations of the spacetime metric of the form

$$\tilde{\boldsymbol{g}} \mapsto \boldsymbol{g} \equiv \Xi^2\tilde{\boldsymbol{g}}, \qquad \Xi > 0 \qquad (1.5)$$

where Ξ is a smooth function on $\tilde{\mathcal{M}}$ – the so-called ***conformal factor***. Throughout this book, the metrics $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ will be called the ***physical metric*** and the ***unphysical metric***, respectively. The rescaling (1.5) gives rise to a ***conformal transformation*** of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ to $(\tilde{\mathcal{M}}, \boldsymbol{g})$. Precise definitions and further discussion of these notions are provided in Chapter 5. In elementary geometry, conformal transformations are usually described as transformations preserving the angle between vectors. In Lorentzian geometry, they preserve the light cones; from (1.4) it follows that $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0$, so that the condition $\phi(x^\mu) = constant$ also describes the light cones of the metric $\boldsymbol{g}$.

One key aspect of conformal rescalings is that they allow one to introduce *conformal extensions* of the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$; see Figure 1.1. In a Riemannian setting, the most basic example of conformal extensions of manifolds is the so-called *conformal completion* of the Euclidean plane $\mathbb{R}^2$ into the 2-sphere $\mathbb{S}^2$ by

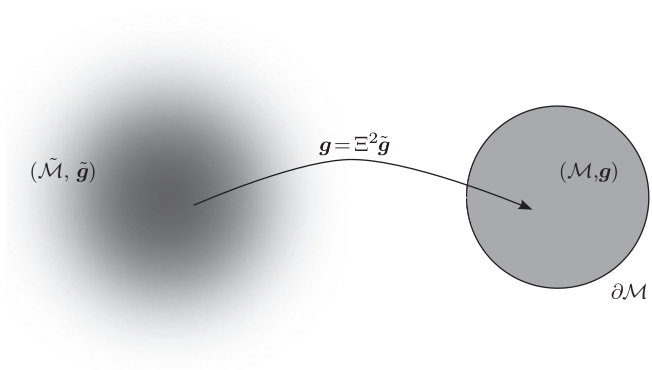

Figure 1.1 Schematic representation of the conformal extension of a manifold. The *physical* manifold $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ has infinite extension, while the *unphysical* (extended) manifold $(\mathcal{M}, \boldsymbol{g})$ is compact with boundary $\partial\mathcal{M}$. The boundary $\partial\mathcal{M}$ corresponds to the points for which $\Xi = 0$. Further details can be found in Chapter 5. Adapted from Penrose (1964).

https://doi.org/10.1017/9781009291347.002 Published online by Cambridge University Press

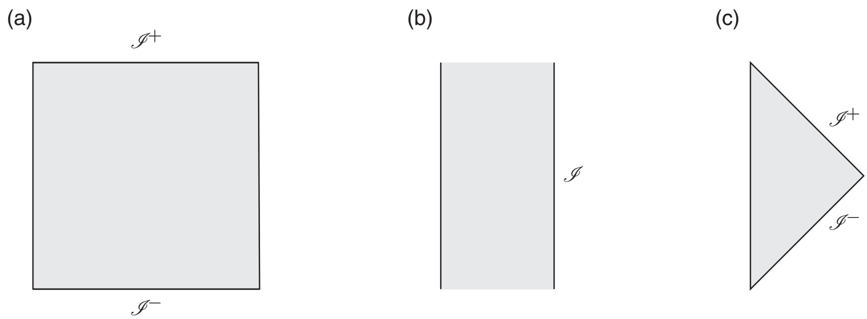

Figure 1.2 Penrose diagrams of the three spacetimes of constant curvature: (a) the de Sitter spacetime; (b) the anti-de Sitter spacetime; (c) the Minkowski spacetime. Details of these constructions can be found in Chapter 6.

means of stereographic coordinates. By suitably choosing the conformal factor Ξ, the metric $\boldsymbol{g}$ given by the rescaling (1.5) may be well defined even at the points where $\Xi = 0$. If this is the case, it can be verified that the set of points $\partial\mathcal{M}$ for which $\Xi = 0$ corresponds to *ideal points at infinity* for the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and is called the ***conformal boundary***. The pair $(\mathcal{M}, \boldsymbol{g})$ where $\mathcal{M}$ is the extended manifold obtained from attaching to $\tilde{\mathcal{M}}$ its conformal boundary is usually known as the ***unphysical spacetime***. Of particular interest are the portions of the conformal boundary which are hypersurfaces of the manifold $\mathcal{M}$ – these sets are characterised by the additional requirement of $\mathbf{d}\Xi \neq 0$, so that they have a well-defined normal. This part of the conformal boundary is denoted by $\mathscr{I}$.

Explicit calculations show that the three spacetimes of *constant curvature* – the Minkowski, de Sitter and anti-de Sitter spacetimes – can be conformally extended. The details of these constructions are described in Chapter 6. These conformal extensions are conveniently represented in terms of *Penrose diagrams*; see Figure 1.2. A discussion of the construction of Penrose diagrams can also be found in Chapter 6. The insights provided by the conformal extensions of these solutions are, in great measure, the fundamental justification for the use of conformal methods in general relativity.

1.4.2 Conformal geometry

The study of properties which are invariant under conformal transformations of a manifold is known as ***conformal geometry***. Associated to the metric $\boldsymbol{g}$ of the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ one has its covariant derivative (connection) ∇_a and its curvature tensors, say, $R^a{}_{bcd}$, R_{ab}, R. These objects can be related to the corresponding objects associated to the physical metric $\tilde{\boldsymbol{g}}$ ($\tilde{\nabla}_a$, $\tilde{R}^a{}_{bcd}$, $\tilde{R}_{ab}$ and $\tilde{R}$) and the conformal factor Ξ and its derivatives. Their transformation laws show, in particular, that the Riemann tensor, the Ricci tensor and the Ricci scalar are not conformal invariants. There is, however, another part of the curvature which

https://doi.org/10.1017/9781009291347.002 Published online by Cambridge University Press

is conformally invariant. It is described by the *Weyl tensor*, for which it holds that

$$\tilde{C}^a{}_{bcd} = C^a{}_{bcd}, \qquad \text{on } \tilde{\mathcal{M}}.$$

In view of the above, one can regard the Weyl tensor as a property of the collection of metrics conformally related to $\tilde{\boldsymbol{g}}$ – the ***conformal class*** $[\tilde{\boldsymbol{g}}]$. If the vacuum Einstein field Equations (1.3) hold, the Bianchi identities imply that

$$\tilde{\nabla}_a \tilde{C}^a{}_{bcd} = 0 \tag{1.6}$$

irrespectively of the value of the cosmological constant.

1.4.3 Conformal invariance of equations of physics

A number of equations in physics have nice conformal properties. The prototypical example is given by the source-free Maxwell equations

$$\tilde{\nabla}^a \tilde{F}_{ab} = 0, \qquad \tilde{\nabla}_{[a}\tilde{F}_{bc]} = 0, \tag{1.7}$$

where $\tilde{F}_{ab}$ denotes the ***Faraday tensor***. One can introduce an *unphysical Faraday tensor* F_{ab} by requiring it to coincide with $\tilde{F}_{ab}$ on $\tilde{\mathcal{M}}$. Using the transformation properties relating the covariant derivatives $\tilde{\nabla}_a$ and ∇_a, it follows that the Maxwell equations are ***conformally invariant***; that is, one has that

$$\nabla^a F_{ab} = 0, \qquad \nabla_{[a}F_{bc]} = 0.$$

The above equations are well defined everywhere on the unphysical spacetime manifold $\mathcal{M}$, in particular at the conformal boundary. These equations allow the extension of the definition of the unphysical field F_{ab} to the conformal boundary $\partial\mathcal{M}$.

In contrast to the Maxwell equations, the vacuum Einstein field Equations (1.2) are not conformally invariant. The transformation law for the Ricci tensor under the rescaling (1.5) implies the equation

$$R_{ab} = -\frac{2}{\Xi}\nabla_a\nabla_b\Xi - g_{ab}g^{cd}\left(\frac{1}{\Xi}\nabla_c\nabla_d\Xi - \frac{3}{\Xi^2}\nabla_c\Xi\nabla_d\Xi\right). \tag{1.8}$$

The above equation is, at least formally, singular at the points where $\Xi = 0$. Thus, it does not provide a good equation for the analysis of the evolution of the unphysical metric $\boldsymbol{g}$ on $\mathcal{M}$. Nevertheless, as pointed out by Penrose (1963) the Bianchi identity (1.6) has a nice *conformal covariance* property. More precisely, one has that

$$\tilde{\nabla}_a\left(\Xi^{-1}\tilde{C}^a{}_{bcd}\right) = 0.$$

The above equation suggests defining the ***rescaled Weyl tensor*** $d^a{}_{bcd} \equiv \Xi^{-1}\tilde{C}^a{}_{bcd}$. Under certain assumptions, the Weyl tensor can be shown to vanish at $\mathscr{I}$ so that the rescaled Weyl tensor is well defined at this portion of the conformal boundary – this important result is analysed in detail in Chapter 10. The rescaled

https://doi.org/10.1017/9781009291347.002 Published online by Cambridge University Press

Weyl tensor is not a conformal invariant; it transforms in a homogeneous fashion under the rescaling (1.5). The above discussion leads to the equation

$$\tilde{\nabla}_a d^a{}_{bcd} = 0, \tag{1.9}$$

the so-called ***Bianchi equation***. In addition, in view of the symmetries of the Weyl tensor it can be shown that

$$\tilde{\nabla}_{[e} d^a{}_{|b|cd]} = 0. \tag{1.10}$$

Note the similarity between Equations (1.9) and (1.10) and the Maxwell Equations (1.7). In particular, the equations are regular even at the conformal boundary. These equations are full of physical significance, as the Weyl tensor can be thought of as describing the *free gravitational field*, that is, a gravitational analogue of the Faraday tensor. Chapter 8 provides a detailed derivation and discussion of the equations presented in this section.

1.4.4 Asymptotics of the gravitational field and asymptotic simplicity

One of the basic predictions of general relativity is the existence of gravitational waves propagating at the speed of light across the fabric of spacetime. As a dynamical process governed by the Einstein field equations, gravitational radiation is closely related to the structure of the light cones of spacetime – thus, if one wants to analyse gravitational radiation one has to examine the propagation of the gravitational field along null directions. This analysis is complicated by the absence of a background geometry so that, a priori, it is not clear what the asymptotic behaviour of the gravitational field should be. This concern lies at the heart of the subject of the ***asymptotics of spacetime*** – that is, the study of the limit behaviour of fields at large distances and large times and the characterisation of spacetimes by data obtained by taking such limits.

In theories which describe fields on a given background, one can discuss limits *at infinity* in a meaningful way in terms of the background geometry. The situation is radically different in general relativity, where the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ – with respect to which the limits of fields derived from $\tilde{\boldsymbol{g}}$ are to be formulated – is the central objects of study. Accordingly, making sense of limiting procedures in general relativity is a delicate process and requires a careful analysis of the geometry and the way it is determined by the Einstein field equations. An approach to this analysis is provided by Penrose's suggestion that the close relation between the propagation of the gravitational field and the structure of null cones which holds locally is also preserved at large scales and that the asymptotic behaviour of the gravitational field can be conveniently analysed in terms of conformal extensions of the spacetime; see Penrose (1963, 1964) and Penrose (2011) for a reprint of the latter reference. With this idea in mind, Penrose introduced the notion of ***asymptotically simple spacetimes***, namely, spacetimes admitting a *smooth* conformal extension which is similar to that of one of the three constant curvature spacetimes. Proceeding in this manner, one attempts to single out a class of *sufficiently well-behaved spacetimes* for which it is possible to relate the structure of the light cones in spacetime to the structure of the field equations and the large-scale behaviour of their solutions. For asymptotically simple spacetimes the causal character of $\mathscr{I}$ is determined by the sign of the cosmological constant; moreover, as already seen, the Weyl tensor vanishes on the conformal boundary – the latter is the basic observation in a collection of results known generically as ***peeling***.

Minkowski-like spacetimes, that is, those asymptotically simple spacetimes for which $\lambda = 0$, are of particular relevance in the study of asymptotics with regard to their connection to the notion of isolated systems in general relativity; compare the discussion at the end of Section 1.1. For this type of spacetime $\mathscr{I}$ is a null hypersurface describing idealised *observers at infinity*. Penrose's original insight was to use the notion of asymptotic simplicity as a way of characterising isolated systems in general relativity – this idea has been called ***Penrose's proposal*** by Friedrich (2002). One of the appealing features of this approach to the study of isolated systems is that it provides a general framework in which notions of physical interest such as gravitational radiation and the associated mass/momentum-loss can be rigorously formulated and analysed. A substantial amount of work has been invested in pursuing these ideas, as attested by the sprawling literature on the subject. An exposition of the notion of asymptotic simplicity, some of its basic consequences and Penrose's proposal is given in Chapters 7 and 10.

1.4.5 The conformal Einstein field equations

In view of Penrose's ideas on the relation between general relativity and conformal geometry one can ask: *to what extent is it possible to draw conclusions about the global structure of spacetimes from an analysis of the behaviour, under conformal rescalings, of the Einstein field equations?* As will be seen in this book, by considering this question one is led to *analyse the behaviour of solutions to the Einstein field equations under optimal regularity conditions.* To address the above question one needs a suitable set of equations to work with. As already observed, the direct transcription of the Einstein field equations as an equation for the unphysical metric $\boldsymbol{g}$ does not provide a set of equations which are adequate from the point of view of PDE theory.

An alternative set of field equations, the so-called ***conformal Einstein field equations***, has been constructed in the seminal work by Friedrich (1981a,b, 1983). The construction of this conformal representation of the equations begins with a revised reading of the *singular Equation* (1.8) not as an equation for the unphysical metric (or alternatively, its Ricci tensor) but for the derivatives of the conformal factor Ξ. To complete this alternative point of view one upgrades the curvature tensors to the level of unknowns and, accordingly, provides equations for them. The required equations are supplied by the Bianchi identities in a way which is consistent with the Einstein field equations satisfied by the physical metric $\tilde{\boldsymbol{g}}$. The resulting system consists of equations for the conformal factor and its first- and second-order derivatives, the unphysical metric $\boldsymbol{g}$ (through the definition of its Ricci tensor), the unphysical Ricci tensor R_{ab} and the rescaled Weyl tensor $d^a{}_{bcd}$ – the equation for the latter field is Equation (1.9). The equations derived by Friedrich have two key properties: (i) they are *formally regular* even at the points where $\Xi = 0$ and (ii) whenever $\Xi \neq 0$, they imply a solution to the Einstein field equations. The considerations leading to the conformal Einstein field equations will be discussed in Chapter 8.

The equations described in the previous paragraph are usually known as the *metric conformal field equations*. One can extend the basic construction to incorporate more *gauge freedom* so as to obtain a more flexible set of equations. A natural first step in this direction consists of rewriting the field equations in a frame formalism. This leads, in turn, in an almost direct way to the spinorial version of the equations; see below. A more extreme generalisation consists of a reformulation of the field equations in terms of a covariant derivative $\hat{\nabla}_a$ which is not the Levi-Civita connection of a metric, but which nevertheless respects the structure of the conformal class $[\tilde{\boldsymbol{g}}]$, a so-called *Weyl connection*. The resulting equations are known as the *extended conformal Einstein field equations*. As will be seen below, this particular formulation of the equations allows the use of gauges with conformally privileged properties.

Friedrich's conformal Einstein equations are not the only possible type of conformal representation of the Einstein field equations; see, for example, Mason (1995) and Anderson (2005a). In any case, they are the ones which have been studied in a more systematic manner in the literature.

1.4.6 Gauge conditions and conformal geodesics

As already mentioned, the procedure of hyperbolic reduction requires the specification of a gauge in terms of which the evolution equations are to be expressed. Earlier in this chapter, the notion of a gauge choice had been restricted to a specification of coordinates. For the conformal field equations, the gauge specification involves three aspects: a coordinate, a frame and a conformal aspect. The precise choice of these three aspects of the gauge depends on the particulars of the problem at hand. A discussion of the gauge freedom contained in the conformal field equations is given in Chapter 13.

The presence of a *conformal gauge freedom* – that is, the freedom to specify the representative in the conformal class one wants to work with – is one of the most attractive aspects of the conformal field equations. Given the bewildering freedom one has in this respect, the use of conformal gauges related to conformal invariants is a natural choice. *Conformal geodesics* are a good example of the type of invariants one can consider. These curves are defined through a set of equations which are invariant under conformal rescalings. In general, the conformal class $[\tilde{\boldsymbol{g}}]$ does not contain a metric for which the conformal geodesics can be recast as *standard (metric) geodesics*. However, there is always a *Weyl connection* for which they are affine geodesics. Conformal geodesics can be used to construct *conformal Gaussian gauge systems* for which coordinates and an adapted frame are propagated off an initial hypersurface. Conformal geodesics allow one to specify a *privileged unphysical metric* $\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$ where Θ is a conformal factor determined through the conformal geodesic equations. Crucially, for solutions to the vacuum field equations (1.2), the conformal factor Θ can be determined explicitly from the initial data for a congruence of these curves – it turns out to be a quadratic polynomial of a suitable parameter of the curves in the congruence. To fully exploit the advantages provided by conformal Gaussian systems, it is necessary to express the conformal field equations in terms of Weyl connections – these considerations lead to the already mentioned extended conformal field equations. Conformal geodesics and their properties are analysed in Chapter 5.

1.4.7 Spinors

This book adopts an approach to the extraction of information from the conformal Einstein field equations which makes systematic use of a formalism based on the so-called *2-spinors*. The use of spinors to carry out this analysis is not essential to the purposes of the book, but it has the advantage of simplifying certain algebraic aspects of the discussion.

Spinors are the most basic objects subject to Lorentz transformations. To every tensor and tensorial operation there exists a spinorial counterpart. More precisely, to every tensor of rank k there corresponds a spinor of rank $2k$. In some particular cases – for example, null vectors or the Weyl tensor – by exploiting symmetries one can associate to the tensor a spinor of the same rank k.

Spinors are well adapted to the discussion of the geometry of null hypersurfaces. Thus, it is not surprising that they are a valuable tool in the discussion of the Einstein field equations. In this book, spinorial representations of the conformal field equations are systematically used as a part of the hyperbolic reduction procedure. In particular, a 2-spinor formalism usually known as the *space spinor formalism*, which can be regarded as a spinorial analogue of the *1+3 formalism for tensors*, provides an almost completely algorithmic approach to the decomposition of the field equations into (symmetric hyperbolic) evolution equations and constraint equations. The basic spinorial formalism used in this book is described in Chapter 3, while the space spinor formalism is dealt with in Chapter 4.

1.5 Existence of asymptotically simple spacetimes

The conformal field equations provide a powerful tool for the analysis and construction of asymptotically simple spacetimes. In broad terms, they allow the reformulation of problems involving unbounded domains in the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ as problems on bounded domains of the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$. From the point of view of PDE theory, problems involving a finite existence time are simpler to analyse than global existence questions. Under the appropriate conditions, the existence of solutions to hyperbolic differential equations on a fixed finite time interval can be shown by invoking the property of *Cauchy stability*; this and other basic notions of PDE theory are discussed in Chapter 12 where a brief account of basic existence results for symmetric hyperbolic systems is given. Prior to its use with the conformal Einstein field equations, the technique for the analysis of evolution equations based on a combination of conformal techniques and Cauchy stability had been used to show the existence of global solutions of the Yang-Mills equations on the Minkowski and de Sitter spacetimes; see Choquet-Bruhat and Christodoulou (1981).

The remainder of this section provides a brief survey of some of the existence results for asymptotically simple spacetimes which have been obtained using the conformal Einstein equations. These results will be elaborated in Part IV of this book.

1.5.1 Characteristic initial value problems

Characteristic problems are a particular type of initial value problem where data are prescribed on *null initial hypersurfaces*. Typically, these data are prescribed on two intersecting null hypersurfaces $\mathscr{N}_1$ and $\mathscr{N}_2$. The relevant PDE theory then allows one to conclude the existence and uniqueness of solutions on neighbourhoods of $\mathscr{N}_1 \cap \mathscr{N}_2$ which are either to the future or to the past of their intersection. In a different type of characteristic problem one prescribes initial data on a null cone $\mathscr{N}$, including its vertex, and one endeavours to obtain a solution inside the cone – at least in a neighbourhood of the vertex. Conformal methods allow the formulation of characteristic problems for which initial data are prescribed on a null conformal boundary – in this case one talks of an ***asymptotic characteristic initial value problem***; see Friedrich (1981a,b, 1982, 1986c). An attractive feature of characteristic initial value problems is that the field equations, expressed in an adapted gauge, have structural properties which simplify their analysis. In particular, the constraint equations on the initial null hypersurfaces reduce to ordinary differential equations.

Asymptotic characteristic problems allow the aspects of the theory of the asymptotics of isolated systems to be set on a rigorous footing. The basic theory of characteristic problems for hyperbolic equations is discussed in Chapter 12. Applications of this theory to the conformal field equations are given in Chapter 18.

1.5.2 De Sitter-like spacetimes

The simplest type of standard (i.e. non-characteristic) initial value problem for the conformal Einstein field equations involves the construction of de Sitter-like spacetimes. In this case one considers compact initial hypersurfaces $\mathcal{S}$ which are diffeomorphic to the 3-sphere $\mathbb{S}^3$. One has the following concise statement first proved in Friedrich (1986b).

Theorem 1.3 (***global existence and stability of de Sitter-like spacetimes***) *Solutions to the Einstein field Equations* (1.2) *with a de Sitter-like value of the cosmological constant arising from Cauchy initial data close to data for the de Sitter spacetime are asymptotically simple.*

The proof of this result relies on the fact that a conformal representation of the *exact* de Sitter spacetime can be recast as a solution of the conformal Einstein field equations which extends beyond the conformal boundary. It follows from the general theory of hyperbolic equations that the solution of the evolution equations for an initial data set which is close to initial data for the background solution will give rise, in its development, to a spacelike hypersurface on which the conformal factor vanishes. This hypersurface can then be interpreted as the conformal boundary of the perturbed spacetime. Thus, the resulting perturbed spacetime has the same global structure as the de Sitter spacetime, and one can say that, in this case, the notion of asymptotic simplicity is *stable*. Remarkably, a variation of Theorem 1.3 allows for the possibility of prescribing initial data on the conformal boundary.

Theorem 1.3 can be extended to include the coupling of the gravitational field with various types of *trace-free matter*. A detailed discussion of the proof of Theorem 1.3 is given in Chapter 15.

1.5.3 Anti-de Sitter-like spacetimes

As already mentioned, the anti-de Sitter spacetime provides one of the basic examples of non-globally hyperbolic spacetimes. This peculiarity of the spacetime can be attributed to the timelike nature of its conformal boundary; this is further discussed in Chapter 14. As a consequence of the above, spacetimes with a global structure which is similar to that of the anti-de Sitter spacetime cannot be constructed using a standard initial value problem, and the initial data have to be supplemented by suitable boundary data on the hypothetic conformal boundary. This type of setting was first analysed in Friedrich (1995) and requires the identification of initial data which can be described as *anti-de Sitter-like* and appropriate boundary data for the conformal Einstein field equations on a timelike hypersurface representing the conformal boundary. It turns out that initial data sets $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ for anti-de Sitter-like spacetimes are characterised by the fact that they admit a conformal extension $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ such that $\mathcal{S}$ has a boundary $\partial\mathcal{S}$ with the topology of the 2-sphere $\mathbb{S}^2$. Based on the example of the exact anti-de Sitter spacetime one expects the conformal boundary to intersect $\mathcal{S}$ on $\partial\mathcal{S}$ and be of the form $\mathscr{I}_c = (-c, c) \times \partial\mathcal{S}$ for some $c > 0$. A detailed analysis of the conformal evolution equations on $\mathscr{I}_c$ reveals that suitable boundary data for the conformal field equations consists of a three-dimensional Lorentzian metric $\boldsymbol{\ell}$. In order to ensure the smoothness of solutions, the underlying PDE theory requires certain compatibility conditions (*corner conditions*) between the initial and the boundary data which are implied by the conformal field equations. Taking into account the above observations one has the following.

Theorem 1.4 (***local existence of anti-de Sitter-like spacetimes***) *Consider an anti-de Sitter-like initial data set* $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ *for the Einstein field equations and a Lorentzian three-dimensional metric* $\boldsymbol{\ell}$ *on* $\mathscr{I}_c$. *Assume that the above data satisfy suitable corner conditions. Then, there exists a solution to the Einstein field equations* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *with anti-de Sitter-like cosmological constant and an associated conformal extension* $(\mathcal{M}, \boldsymbol{g})$ *such that* $\tilde{\mathcal{S}}$ *is a spacelike hypersurface of* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *and so that* $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ *coincides with the intrinsic metric and extrinsic curvature implied by* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *on* $\tilde{\mathcal{S}}$. *Furthermore,* $\mathscr{I}_c$ *is the conformal boundary of* $(\mathcal{M}, \boldsymbol{g})$ *and the intrinsic metric of* $\mathscr{I}_c$ *implied by* $\boldsymbol{g}$ *belongs to the conformal class of* $\boldsymbol{\ell}$.

The proof of the above theorem is described in Chapter 17. The above theorem ensures only local existence of anti-de Sitter-like spacetimes, that is, the existence of a solution close to $\tilde{\mathcal{S}}$. It says nothing about the global existence or stability of solutions. Accordingly, it does not require assumptions on the smallness of the data. At the time of writing, the question of the stability (or lack thereof) is an open problem.

1.5.4 Minkowski-like spacetimes

The analysis of Minkowski-like spacetimes gives rise to some of the most challenging open problems in the application of conformal methods in general relativity.

In principle, one would like to construct Minkowski-like spacetimes by prescribing suitable *asymptotically Euclidean* initial data on a three-dimensional manifold $\tilde{\mathcal{S}}$ which is a Cauchy hypersurface of the hypothetic spacetime. However, it turns out that a simpler problem consists of the specification of initial data on a 3-manifold $\tilde{\mathcal{H}}$ describing a hypersurface of $\tilde{\mathcal{M}}$ which in the conformal extension intersects $\mathscr{I}$ —a so-called ***hyperboloid***. Hyperboloidal initial data sets $(\tilde{\mathcal{H}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ admit conformal extensions $(\mathcal{H}, \boldsymbol{h}, \boldsymbol{K})$ for which $\mathcal{H}$ is a manifold with boundary $\partial\mathcal{H}$ which has the topology of the 2-sphere $\mathbb{S}^2$ – this boundary corresponds to the intersection of the hyperboloid with $\mathscr{I}$. Hyperboloidal initial data sets are similar in structure to anti-de Sitter-like initial data. There is, in fact, a correspondence between the two; this relation is explored in Chapter 11. An important feature of hyperboloids is that they are not Cauchy hypersurfaces; that is, they do not allow the reconstruction of a whole Minkowski-like spacetime. Despite this shortcoming, one has the following *semi-global* existence and stability result first proved in Friedrich (1986b).

Theorem 1.5 (***semi-global existence and stability of the hyperboloidal initial value problem***) *Solutions to the hyperboloidal initial value problem for the Einstein Equation* (1.3) *with initial data* $(\tilde{\mathcal{H}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ *which are suitable perturbations of Minkowski hyperboloidal data are asymptotically simple to the future of* $\tilde{\mathcal{H}}$ *and have a conformal boundary with the same global structure as the conformal boundary of Minkowski spacetime.*

A detailed account of this result is given in Chapter 16. Aside from some technical details, the key ideas of the proof of this result are similar to those of Theorem 1.3 for de Sitter-like spacetimes. Again, a conformal point of view allows one to provide a global existence result for the Einstein field equations in terms of a problem involving a finite existence time. A proof of the non-linear stability of the Minkowski spacetime making use of initial data prescribed on a Cauchy initial hypersurface has been given in the work by Christodoulou and Klainerman (1993). This proof relies on a detailed analysis of the decay of the gravitational field using carefully constructed estimates. Remarkably, the main result of this work does not provide enough regularity at infinity for us to conclude that the spacetime obtained is asymptotically simple.

Time-independent solutions

An important source of intuition on the behaviour of general Minkowski-like spacetimes is provided by the analysis of *time-independent spacetimes*, that is, spacetimes possessing a continuous symmetry which (at least) in the asymptotic region is timelike. If the Killing vector of a time-independent solution is hypersurface orthogonal, then one speaks of a *static spacetime*. Otherwise, one has a *stationary solution*. In the vacuum case, static and stationary solutions can be thought of as describing the exterior gravitational field of some compact matter configuration. In addition, the Schwarzschild and Kerr spacetimes describe time-independent black holes. From the point of view of conformal geometry, their relevance lies in that they allow a detailed analysis of *spatial infinity*, that is, the portion of the conformal boundary intersecting the conformal extension $\mathcal{S}$ of a Cauchy hypersurface $\tilde{\mathcal{S}}$. Vacuum time-independent spacetimes can be shown to admit conformal extensions which are as smooth as one can expect.

Time-independent spacetimes are described by equations which, in a suitable gauge, are elliptic. This feature of this class of solutions explains many of their rigidity and uniqueness properties – in particular, they are characterised through a sequence of *multipole moments*. The analysis of these expansions and other asymptotic properties of static and stationary solutions can be performed in a very convenient manner through conformal methods. In addition, and quite remarkably, static spacetimes can be shown to have a close relation to spacetimes constructed from an asymptotic characteristic initial value problem on a light cone. These and further aspects of static solutions are discussed in Chapter 19.

Spatial infinity

The *asymptotic region* of Cauchy hypersurfaces of Minkowski-like spacetimes can be conformally extended to include a further point – the *point at infinity*. In these conformal extensions, domains in the asymptotic region are transformed into suitable neighbourhoods of the point at infinity. This *point compactification* procedure is a generalisation of the compactification of $\mathbb{R}^2$ into $\mathbb{S}^2$. From a spacetime perspective, the point at infinity gives rise to *spatial infinity* i^0. In this picture, i^0 can be thought of as the vertex of the light cone of $\mathscr{I}$, and the Minkowski-like spacetime corresponds to the exterior of the cone; this construction is analysed in Chapters 19 and 20.

The construction of Minkowski-like asymptotically simple spacetimes from Cauchy initial data requires a precise understanding of the behaviour of the gravitational field in a neighbourhood of spatial infinity. It was first observed by Penrose (1965) that for spacetimes with non-vanishing mass the conformal structure becomes singular at spatial infinity. As a consequence, the initial data implied by the Bianchi Equation (1.9) – which, as already discussed, is one of the key constituents of the conformal field equations – blows up at spatial infinity. The resulting singularity makes the analysis of solutions to the conformal field equations in this region of spacetime particularly challenging. This observation explains, to some extent, why the first results on the existence of Minkowski-like spacetimes were restricted to the developments of hyperboloidal initial data. Early attempts to analyse this situation – see, for example, Beig and Schmidt (1982), Beig (1984) and Friedrich (1988) – reached an impasse due to the lack of a suitable representation of spatial infinity. A breakthrough in this direction was given in Friedrich (1998c) where a representation of spatial infinity based on the properties of conformal geodesics, the so-called *cylinder at spatial infinity*, allows one to formulate a *regular* finite initial value problem for the conformal field equations at spatial infinity. In recent years, a considerable amount of work has been devoted to exploring the implications of this construction. The picture that has progressively emerged is that the conditions required to ensure the existence of asymptotically simple developments out of asymptotically Euclidean initial data are much more restrictive than what one would first expect.

The analysis of the structure of spatial infinity has been informed by developments in the construction of solutions to the constraint equations of general relativity. The exterior asymptotic gluing constructions introduced in Corvino (2000) and Corvino and Schoen (2006) allow one to *glue* static and stationary asymptotic regions to otherwise completely general asymptotically Euclidean initial data sets, the basic ideas of the exterior asymptotic gluing construction are briefly discussed in Chapter 11. As already observed, time-independent solutions to the Einstein field equations are well behaved in a neighbourhood of spatial infinity. Chruściel and Delay (2002) have shown that it is possible to combine this observation with Theorem 1.5 to obtain complete Minkowski-like asymptotically simple spacetimes. The spacetimes obtained in this manner are very special, as they are exactly static, or, more generally, stationary in a neighbourhood of spatial infinity – nevertheless, radiation is registered at null infinity. It is natural to ask whether it is possible to relax this *rigid* behaviour so as to obtain more general types of asymptotically simple spacetimes. The analysis of the *problem of spatial infinity* remains a challenging

https://doi.org/10.1017/9781009291347.002 Published online by Cambridge University Press

open area of research; an introductory discussion to the problem of spatial infinity is provided in Chapter 20.

1.6 Perspectives

At the time of writing, the use of conformal methods to analyse the global existence and stability of solutions to the Einstein field equations has been mainly restricted to asymptotically simple spacetimes. One of the motivations behind this book is to encourage researchers interested in the open problems of mathematical relativity to further extend the available conformal methods so as to make them suitable for the analysis of more complicated spacetimes – for example, black holes. From the author's point of view, the realisation of this vision requires the development of not only analytic tools, but also a computational framework which allows one to perform *numerical relativity* using the conformal field equations. Some ideas in this direction are put forward in the concluding Chapter 21.

1.7 Structure of this book

This book is divided in four parts. Throughout, a combination of abstract index notation and index-free notation has been used. An index-free notation has been preferred whenever it simplifies the presentation and emphasises structural aspects of an equation, while abstract indices are used, mostly, in detailed calculations. The spinorial conventions follow those in the monograph of Penrose and Rindler (1984). In view of the systematic use of spinors, this book adopts a $(+---)$ convention for the signature of Lorentzian metrics. As a consequence of this convention the sign of the cosmological constant in the de Sitter spacetime is negative, while for the anti-de Sitter spacetime it is positive. In order to avoid confusion – inasmuch as it is possible – with other sources, a negative cosmological constant will be described as being *de Sitter-like* and a positive one as being *anti-de Sitter-like*. Further details on conventions can be found in Chapters 2, 3 and 4.

Throughout this book ***bold italics*** are systematically used to denote that a given concept is being defined, while *italics* are used to highlight an idea; the attentive reader will realise that sometimes the distinction between these two is blurry.

The content of the four parts of this book can be briefly described as follows.

Part I (Geometric tools) provides a self-contained discussion of the differential geometric and spinorial notions that will be used throughout the book. The presentation and selection of material is tailored to the needs of the discussion in Parts II and III and the applications in Part IV. Chapter 2 gives a brief account of the required notions of differential geometry. The purpose of the chapter is not only to serve as a quick reference in later parts of the book but also to elaborate certain ideas which are not readily available elsewhere in the literature. Chapter 3 provides an account of 2-spinors, while Chapter 4 develops the so-called space spinor formalism. Chapter 5 provides an introduction to conformal geometry which covers not only the transformation formulae for the connection and curvature but also not so well-known topics such as Weyl connections and conformal geodesics – two key notions which will be further developed in Parts II and III.

Part II (General relativity and conformal geometry) provides an introduction to the use of conformal methods in general relativity. It also develops a toolkit of other mathematical methods which will be used to extract information from the Einstein field equations. Chapter 6 provides a brief survey of the construction of conformal extensions of basic solutions to the Einstein field equations – the Minkowski, de Sitter, anti-de Sitter and Schwarzschild spacetimes – as well as a general framework for the construction of Penrose diagrams of spherically symmetric static spacetimes. Chapter 7 provides a discussion of one of the leading themes of this book, the concept of asymptotically simple spacetimes and a formulation of the so-called Penrose's proposal. Chapter 8 gives a derivation and detailed discussion of the main tool of this book, the conformal Einstein field equations. Several versions of the equations are considered – metric, frame, spinorial and in terms of Weyl connections. Chapter 9 complements Chapter 8 and describes matter models amenable to treatment by means of conformal methods. Several of the main results of this book for the vacuum case can be generalised by including these matter models. Chapter 10 provides a brief discussion of the *formal* theory of the asymptotics of spacetime – sometimes also called *asymptopia*. This is a vast topic with a sprawling literature. It is thus impossible to do full justice to the subject in a concise chapter. Accordingly, the decision has been made to restrict the material to aspects of the subject which motivate the later parts of the book.

Part III (Methods of PDE theory) provides an account of PDE and spinorial methods that will be used systematically in Part IV to obtain statements about the existence of various types of solutions to the Einstein field equations. Chapter 11 provides a discussion of the constraint equations implied by the conformal Einstein field equations on spacelike and timelike hypersurfaces – the so-called conformal constraint equations. The proper discussion of this material requires the introduction of certain notions of elliptic PDE theory. This is done at various places in the chapter. Chapter 12 provides a discussion of the methods of the theory of hyperbolic PDEs which will be used in the latter parts of the book. This chapter has been written with the applications in Part IV in mind and covers basic local existence and uniqueness results for initial value, boundary value and characteristic initial value problems. Chapter 13 discusses in detail various hyperbolic reduction procedures for the conformal Einstein field equations by means of spinorial methods. The analysis is not restricted to the evolution systems, but also considers the subsidiary evolution equations required to prove the *propagation of the constraints*. Part III of the book concludes with Chapter

14 where a brief discussion of Lorentzian causality and key aspects of the Cauchy problem in general relativity are given.

Part IV (Applications) is concerned with applications of the conformal Einstein field equations to the analysis of the existence of asymptotically simple spacetimes. Chapter 15 analyses the global existence and stability of de Sitter-like spacetimes; see Theorem 1.3. Two different proofs are provided: the first one makes use of the standard conformal field equations and gauge source functions, and the second one relies on the extended conformal field equations and conformal Gaussian systems. Chapter 16 provides a proof of the semiglobal existence and stability result for hyperboloidal initial data for the Minkowski spacetime and a detailed analysis of the structure of the conformal boundary of the resulting spacetimes; see Theorem 1.5. Chapter 17 provides a discussion of the construction of anti-de Sitter-like spacetimes by means of an initial boundary value problem; see Theorem 1.5. Chapter 18 discusses a different setting for the construction of solutions to the conformal field equations, that of asymptotic characteristic initial value problems either on intersecting null hypersurfaces (one of them representing null infinity) or on a cone (representing past null infinity). Chapter 19 analyses the properties of static solutions by means of conformal methods. The main purpose of this chapter is to pave the way for the discussion of the problem of spatial infinity, which is analysed in Chapter 20. In particular, a discussion of the construction of the so-called cylinder at spatial infinity is provided.

The book concludes with Chapter 21, which provides a subjective selection of open problems in mathematical general relativity where it is felt that the use of conformal methods can provide fresh insights.

Further reading sections. Each chapter provides a brief literature survey. The purpose of this is to provide the interested reader a convenient point of entry into the literature in case more details or an alternative perspective on the subject are required.