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Darmon's Points and Quaternionic Shimura Varieties

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Abstract. In this paper, we generalize a conjecture due to Darmon and Logan in an adelic setting. We study the relation between our construction and Kudla's works on cycles on orthogonal Shimura varieties. This relation allows us to conjecture a Gross–Kohnen–Zagier theorem for Darmon's points.

1 Introduction

The theory of complex multiplication gives a collection of *Heegner points* on elliptic curves over **Q**, which are defined over class fields of imaginary quadratic fields. These points led to the proof of the Birch and Swinnerton-Dyer conjecture over **Q** for analytic rank 1 curves, thanks to the work of Gross, Zagier, and Kolyvagin.

Let us briefly recall the construction of Heegner points. If E is an elliptic curve over **Q**, then we know that E is modular. Let N be the conductor of E. There exists a modular form $f \in S_2(N)$ such that $L(E,s) = L(f,s)$. Denote by $\Phi_N \colon \Gamma_0(N)\backslash\mathcal{H} \to E(\mathbf{C})$ the modular uniformization that is obtained by taking the composition of the map $z_0 \in \mathcal{H} \mapsto c\int_{i\infty}^{z_0} 2\pi i f(z)\mathrm{d}z$ (here c denotes the Manin constant) with the Weierstrass uniformization. Let $K/\mathbf{Q}$ be an imaginary quadratic field. A Heegner point is a point $\Phi_N(z_0)$, where $z_0 \in \mathcal{H} \cap K$. It is the Abel–Jacobi image of z_0 in $\mathbf{C}/\Lambda_E \simeq E(\mathbf{C})$. The theory of complex multiplication shows that these points are defined over class fields of K.

In [7], Darmon gives a conjectural construction of *Stark–Heegner points*, which is a generalization of classical Heegner points. These points should help us to understand the Birch and Swinnerton-Dyer conjecture on one hand, and Hilbert's twelfth problem on the other.

In more concrete terms, assume that F is a totally real number field of degree d over **Q** and narrow class number 1. Let τ_j be its archimedean places and K/F some quadratic "ATR" extension (*i.e.*, K has exactly one complex place). Darmon defines a collection of points on elliptic curves E/F that are expected to be defined over class fields of K. In this case, the (conjectural, but partially proved by Skinner and Wiles) modularity of E gives the existence of a Hilbert modular form f on $\mathcal{H}^d$ whose periods appear, under some conjecture due to Oda, as a tensor product of periods of $E_{\tau_j} = E \otimes_{F,\tau_j} \mathbf{C}$. The construction explained in [8] can be seen as an exotic Abel–Jacobi map.

In this paper, we generalize Darmon's construction by removing the hypothesis "ATR" on K (but we assume that K is not CM) and the technical hypothesis that

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F has narrow class number 1. We replace the Hilbert modular variety used in the "ATR" case by a general quaternionic Shimura variety and define a suitable Abel–Jacobi map. We are able to specify the invariants of the quaternion algebra using local epsilon factors and to give a conjectural Gross–Zagier formula for these points. We conclude the paper by establishing a relation to Kudla's study of cycles on orthogonal Shimura varieties, in order to give a Gross–Kohnen–Zagier type conjecture.

Let us summarize the main construction of this paper. Let F be a totally real field of degree d and let $\tau_1, \ldots, \tau_d$ be its archimedean places. Fix $r \in \{2, \ldots, d\}$, and a quadratic extension K/F such that the set of archimedean places of F that split completely in K is $\{\tau_2, \ldots, \tau_r\}$. Let B/F be a quaternion algebra that splits at $\tau_1, \ldots, \tau_r$ and ramifies at $\tau_{r+1}, \ldots, \tau_d$. Let $G = \operatorname{Res}_{F/\mathbf{Q}} B^\times$. We will denote by $\mathrm{Sh}_H(G)$ the quaternionic Shimura variety of level H (a compact open subgroup of $G(\mathbf{A}_f)$) whose complex points are given by

$$\mathrm{Sh}_H(G)(\mathbf{C}) = G(\mathbf{Q})\backslash(\mathbf{C} \smallsetminus \mathbf{R})^r \times G(\mathbf{A}_f)/H,$$

where $\mathbf{A}_f$ is the set of finite adeles over $\mathbf{Q}$.

Fix an F-embedding $q\colon K \hookrightarrow B$. There is an action of $(K \otimes \mathbf{R})_+^\times/(F \otimes \mathbf{R})^\times$ on $(\mathbf{C} \smallsetminus \mathbf{R})^r$. By considering a suitable orbit of this action, we obtain for any $b \in G(\mathbf{A}_f)$ a real cycle $\mathscr{T}_b$ of dimension $r - 1$ on $\mathrm{Sh}_H(G)(\mathbf{C})$. Using the theorem of Matsushima and Shimura, we deduce that there exists an r-chain Δ_b on $\mathrm{Sh}_H(G)(\mathbf{C})$ such that $\partial\Delta_b$ is an integral multiple of $\mathscr{T}_b$.

Let E/F be an elliptic curve, assumed modular, *i.e.*, there exists a Hilbert modular eigenform $\tilde{\varphi}$ satisfying $L(E, s) = L(\tilde{\varphi}, s)$. We will assume that this form corresponds to an automorphic form φ on B by the Jacquet–Langlands correspondence. There exists a holomorphic differential form ω_φ of degree r on $\mathrm{Sh}_H(G)(\mathbf{C})$ naturally attached to φ. In general, the set of periods of ω_φ is a dense subset of $\mathbf{C}$. Fix some character β of the set of connected components of $(K \otimes \mathbf{R})_+^\times/(F \otimes \mathbf{R})^\times$. Following Darmon we define a modified differential form ω_φ^β whose periods are, assuming Yoshida's period conjecture, a lattice, homothetic to some sublattice of the Neron lattice of E.

The image of (a suitable multiple of) the complex number $\int_{\Delta_b} \omega_\varphi^\beta$ in $\mathbf{C}/\Lambda_E$ is independent of the choice of Δ_b. Hence it defines by Weierstrass uniformization a point P_b^β in $E(\mathbf{C})$. More precisely, denote by $\Phi\colon \mathbf{C}/\Lambda \to E(\mathbf{C})$ the Weierstrass uniformization given by a fixed embedding $\tau_{1,K}\colon K \hookrightarrow \mathbf{C}$, which extends $\tau_1\colon F \hookrightarrow \mathbf{C}$. We have the following conjecture.

Conjecture (5.1 below) $\quad P_b^\beta = \Phi\left(\int_{\Delta_b} \omega_\varphi^\beta\right) \in E(\mathbf{C})$ *lies in* $E(K^{ab})$ *and*

$$\forall a \in (K \otimes \widehat{\mathbf{Z}})^\times \qquad \operatorname{rec}_K(a) P_b^\beta = \beta(a_\infty) P_{q_{\mathbf{A}}(a)b}^\beta.$$

Let us assume this conjecture is true and denote by K_b^+ the field of definition of P_b^β. Let $\pi = \pi(\varphi)$ be the automorphic representation generated by φ; fix a character $\chi\colon \operatorname{Gal}(K_b^+/K) \to \mathbf{C}^\times$. Denote by $\varepsilon(\pi \times \chi, \frac{1}{2})$ the sign in the functional equation of the Rankin-Selberg L-function $L(\pi \times \chi, s)$ and by $\eta_K\colon F_{\mathbf{A}}^\times/F^\times \mathrm{N}_{K/F}(K_{\mathbf{A}}^\times) \to \{\pm 1\}$ the quadratic character of K/F. The following proposition proves that B is uniquely determined by K and the isogeny class of E/F.

Proposition (5.7 below) *Let $b \in \widehat{B}^{\times}$ and assume Conjecture 5.1. If*

$$e_{\overline{\chi}}(P_b^{\beta}) = \sum_{\sigma \in \mathrm{Gal}(K_b^+/K)} \chi(\sigma) \otimes P_b^{\beta} \in E(K_b^+) \otimes \mathbf{Z}[\chi]$$

is not torsion, then

$$\forall v \nmid \infty \qquad \eta_{K,v}(-1)\varepsilon\Big(\pi_v \times \chi_v, \frac{1}{2}\Big) = \mathrm{inv}_v(B_v) \quad \textit{and} \quad \varepsilon\Big(\pi \times \chi, \frac{1}{2}\Big) = -1.$$

The last part of this paper is focused on a conjecture in the spirit of the Gross–Kohnen–Zagier theorem. Assume that $E(F)$ has rank 1. Denote by P_0 some generator modulo torsion. For each totally positive $t \in \mathcal{O}_F$ such that (t) is square free and prime to the relative discriminant $d_{K/F}$ of K, denote by $K[t]$ the quadratic extension $K[t] = F(\sqrt{-D_0 t})$, where $D_0 \in F$ satisfies $\tau_j(D_0) > 0$ if and only if $j \in \{1, r+1, \dots, d\}$. Let $P_{t,1}$ be Darmon's point obtained for $K[t]$, $b = 1$ and $\beta = 1$, and set

$$P_t = \mathrm{Tr}_{K[t]_1^+/F}\, P_{t,1}.$$

The point P_t is in $E(F)$ under Conjecture 5.1, and it is assumed that there exists some integer $[P_t] \in \mathbf{Z}$ such that $P_t = [P_t]P_0$. In the spirit of [9, Conjecture 5.3] we conjecture the following.

Conjecture (6.11 below) *There exists a Hilbert modular form g of level 3/2 such that the $[P_t]$s are proportional to some Fourier coefficients of g.*

In our attempt to adapt Yuan, Zhang, and Zhang's proof in the CM case [31] to prove this conjecture, we obtained a relation between Darmon's points and Kudla's program; see Proposition 6.8.

2 Quaternionic Shimura Varieties

In this section we recall some properties of Shimura varieties associated with quaternion algebras. The standard references are [21] and Reimann's book [25]. The content of this section is more or less the transcription to Shimura varieties of what is done for curves in [5, 22].

Let F be a totally real field of degree $d = [F : \mathbf{Q}]$ and let $\tau_1, \dots, \tau_d$ be its archimedean places. Denote by $\overline{\mathbf{Q}} \subset \mathbf{C}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ so $\tau_j \colon F \hookrightarrow \overline{\mathbf{Q}}$. Fix $r \in \{2, \dots, d\}$ and a finite set S_B of non-archimedean primes satisfying

$$|S_B| \equiv d - r \bmod 2.$$

Let B be the unique quaternion algebra over F ramified at the set

$$\mathrm{Ram}(B) = \{\tau_{r+1}, \dots, \tau_d\} \cup S_B.$$

For each $j \in \{1, \dots, d\}$ we put $B_{\tau_j} = B \otimes_{F,\tau_j} \mathbf{R}$. It is not necessary but more convenient to fix for each $j \in \{1, \dots, r\}$ an $\mathbf{R}$-algebra isomorphism $B_{\tau_j} \xrightarrow{\sim} M_2(\mathbf{R})$.

The constructions given in this paper are independent of the choice of these isomorphisms, as in the author's Ph.D. thesis [11].

Let G be the algebraic group over $\mathbf{Q}$ satisfying $G(A) = (B \otimes_{\mathbf{Q}} A)^{\times}$ for every commutative $\mathbf{Q}$-algebra A. We will denote by $\mathrm{nr}\colon G(A) \longrightarrow (F \otimes_{\mathbf{Q}} A)^{\times}$ the reduced norm and by Z the center of G. For $j \in \{1, \dots, d\}$ let G_j be the algebraic group over $\mathbf{R}$ given by $G_j = G \otimes_{F,\tau_j} \mathbf{R}$; thus, $G_{\mathbf{R}} = G \otimes_F \mathbf{R}$ decomposes as $G_1 \times \cdots \times G_d$. For any abelian group A, denote by $\widehat{A}$ the group $A \otimes \widehat{\mathbf{Z}}$.

Let X be the $G(\mathbf{R})$-conjugacy class of the morphism $h\colon \mathbf{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_{m,\mathbf{C}}) \to G(\mathbf{R}) = G_1(\mathbf{R}) \times \cdots \times G_d(\mathbf{R})$ defined by

$$x + iy \longmapsto \Big(\underbrace{\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \dots, \begin{pmatrix} x & y \\ -y & x \end{pmatrix}}_{r \text{ times}}, \underbrace{1, \dots, 1}_{d-r \text{ times}}\Big).$$

The set X has a natural complex structure [20], and the following map is an holomorphic isomorphism between X and $(\mathbf{C} \smallsetminus \mathbf{R})^r$:

$$ghg^{-1} \longmapsto g \cdot (i, \dots, i) = \Big(\frac{a_1 i + b_1}{c_1 i + d_1}, \dots, \frac{a_r i + b_r}{c_r i + d_r}\Big),$$

where $g = (g_1, \dots, g_d) \in G(\mathbf{R})$ and for $j \in \{1, \dots, r\}$, g_j is identified with $\left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$.

Quaternionic Shimura Varieties Let H be an open-compact subgroup of $\widehat{B}^{\times}$. The quaternionic Shimura varieties considered in this paper are algebraic varieties $\mathrm{Sh}_H(G, X)$, whose complex points are given by

$$\mathrm{Sh}_H(G, X)(\mathbf{C}) = B^{\times} \backslash (X \times \widehat{B}^{\times}/H),$$

where the left-action of $B^{\times}$ and the right-action of H are given by

$$\forall k \in B^{\times}\ \forall h \in H\ \forall (x, b) \in X \times \widehat{B}^{\times} \qquad k \cdot (x, b) \cdot h = (kx, kbh).$$

Such Shimura varieties are defined over some number field called the reflex field. In our case this number field is

$$F' = \mathbf{Q}\Big(\sum_{j=1}^{r} \tau_j(\alpha),\ \alpha \in F\Big) \subset \overline{\mathbf{Q}} \subset \mathbf{C}.$$

We will denote by $[x, b]_H$ the element of $\mathrm{Sh}_H(G, X)(\mathbf{C})$ represented by (x, b) and by $[x, b]_{H\widehat{F}^{\times}}$ the corresponding element of the modified variety $\mathrm{Sh}_H(G/Z, X)(\mathbf{C}) = B^{\times} \backslash (X \times \widehat{B}^{\times}/HZ)$.

Remark 2.1 All automorphic forms that appear in this article have trivial central character. Thus the choice of using the quotient variety $\mathrm{Sh}_H(G/Z, X)(\mathbf{C})$ rather than $\mathrm{Sh}_H(G, X)(\mathbf{C})$ is made to simplify computations.

Remark 2.2 The complex Shimura varieties are compact whenever $B \neq M_2(F)$. The Hilbert modular varieties used by Darmon in [7, Chapters 7–8] are the quotient varieties obtained when $B = M_2(F)$ and $r = d$.

The Shimura varieties form a projective system $\{\mathrm{Sh}_H(G,X)\}_H$ indexed by open compact subgroups in $\widehat{B}^\times$. The transition maps $\mathrm{pr}\colon \mathrm{Sh}_H(G,X) \to \mathrm{Sh}_{H'}(G,X)$ are defined on complex points by $[x,b]_H \to [x,b]_{H'}$.

There is an action of $\widehat{B}^\times$ on the projective system $\{\mathrm{Sh}_H(G,X)\}_H$. The right multiplication by $g \in \widehat{B}^\times$ induces an isomorphism

$$[\cdot g]\colon \{\mathrm{Sh}_H(G,X)\}_H \xrightarrow{\sim} \{\mathrm{Sh}_H(G,X)\}_{g^{-1}Hg},$$

defined on complex points by $[\cdot g][x,b]_H = [x,bg]_{g^{-1}Hg}$.

Complex conjugation Fix $j \in \{1,\dots,r\}$. Let $h_j\colon \mathbf{S} \to G_{j,\mathbf{R}}$ be the morphism obtained by composing h with the j-th projection $G_{\mathbf{R}} \to G_{j,\mathbf{R}}$ and X_j the $G_j(\mathbf{R})$-conjugacy class of h_j. For $x_j = g_j h_j g_j^{-1} \in X_j$, the set $\mathrm{Im}(g_j h_j g_j^{-1})$ is a maximal anisotropic $\mathbf{R}$-torus in $G_{j,\mathbf{R}}$. The map $\ell_j\colon x_j \mapsto \mathrm{Im}(x_j)$ satisfies $|\ell_j^{-1}(\ell_j(x_j))| = 2$, thus there exists a unique antiholomorphic and $G_{j,\mathbf{R}}$-equivariant involution $t_j\colon X_j \to X_j$ such that for all $x_j \in X_j$,

$$\ell_j^{-1}(\ell_j(x_j)) = \{x_j, t_j(x_j)\}.$$

More precisely, under the identification $X_j \xrightarrow{\sim} \mathbf{C} \smallsetminus \mathbf{R}$, the map ℓ_j satisfies

$$\ell_j(x+iy) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right\} \quad \text{and} \quad \ell_j^{-1}\big(\ell_j(x+iy)\big) = \{x+iy, x-iy\}.$$

Note that the map t_j can be extended to complex points of the Shimura varieties by $t_j([x,b]_H) = [t_j(x),b]_H$; t_j acts trivially on X_k for $k \neq j$.

Differential forms In this section we recall some facts concerning differential forms on Shimura varieties. We will denote by $\Omega_H = \Omega_{H/F'}$ the sheaf of differentials of degree r on $\mathrm{Sh}_H(G,X)$ and by Ω_H^{an} the sheaf of holomorphic r-differentials on $\mathrm{Sh}_H(G,X)(\mathbf{C})$, provided that $\mathrm{Sh}_H(G,X)$ is smooth. Recall that the GAGA principle gives us the following isomorphism between global sections:

$$\Gamma\big(\mathrm{Sh}_H(G,X),\Omega_H\big) \otimes_{F'} \mathbf{C} \xrightarrow{\sim} \Gamma\big(\mathrm{Sh}_H(G,X)(\mathbf{C}),\Omega_H^{\mathrm{an}}\big).$$

Notice that in general, $\mathrm{Sh}_H(G,X)$ is not smooth. In this last case we will fix some integer $n \geq 3$ such that for each $\mathfrak{p}$ in $\mathrm{Ram}(B)$ we have $\mathfrak{p} \nmid n\mathcal{O}_F$ and for each $v \mid n\mathcal{O}_F$,

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isomorphisms $\iota_v\colon B_v \xrightarrow{\sim} M_2(F_v)$. The group

$$H' = \left\{ (h_v) \in H, \text{ s.t. } \forall v \mid n\mathcal{O}_F \ \iota_v(h_v) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n\mathcal{O}_{F_v} \right\}$$

is of finite index in H, and $\mathrm{Sh}_{H'}(G,X)$ is smooth. The map $\mathrm{Sh}_{H'}(G,X) \to \mathrm{Sh}_H(G,X)$ is a finite covering. We define $\Omega_H = \frac{1}{[H:H']} \sum_{\sigma\in H/H'} \sigma\Omega_{H'} = (\Omega_{H'})^H$. By abuse of language, we shall call an element of

$$\Gamma(\Omega_H) = \Gamma(\mathrm{Sh}_H(G,X),\Omega_H) = \left(\sum_{\sigma\in H/H'} \sigma \right) \Gamma(\mathrm{Sh}_{H'}(G,X),\Omega_{H'})$$

a global r-form on $\mathrm{Sh}_H(G,X)$. Remark that the space of global holomorphic r-forms $\varinjlim_H \Gamma(\Omega_H^{\mathrm{an}})$ is equipped with a canonical action of $\widehat{B}^\times$ given by pull-backs $[\cdot g]^*$.

Let $\varepsilon \in \{\pm 1\}^r$ and denote by $\Gamma((\Omega_H^{\mathrm{an}})^\varepsilon)$ the space of r-forms on $\mathrm{Sh}_H(G,X)(\mathbf{C})$ that are holomorphic (resp. anti-holomorphic) in z_j if $\varepsilon_j = +1$ (resp. if $\varepsilon_j = -1$). The maps t_j pulled-back on $\Gamma((\Omega_H^{\mathrm{an}})^\varepsilon)$ satisfy

$$t_j^*\colon \Gamma\big((\Omega_H^{\mathrm{an}})^\varepsilon\big) \longrightarrow \Gamma\big((\Omega_H^{\mathrm{an}})^{\varepsilon'}\big),$$

where $\varepsilon'_k = \varepsilon_k$ for $k \neq j$ and $\varepsilon'_j = -\varepsilon_j$.

When $\sigma \in \prod_{j=2}^r \{\pm 1\}$ we will define $e_j \in \{0,1\}$ by $\sigma_j = (-1)^{e_j}$ and t_σ^* by $\prod_{j=2}^r (t_j^*)^{e_j}$. Let $\beta\colon \prod_{j=2}^r \{\pm1\} \to \{\pm 1\}$ be a character and $\omega \in \Gamma(\Omega_H^{\mathrm{an}})$. We shall denote by ω^β the element $\omega^\beta = \sum_{\sigma\in\{\pm1\}^{r-1}} \beta(\sigma) t_\sigma^*(\omega)$ of $\bigoplus_\varepsilon \Gamma((\Omega_H^{\mathrm{an}})^\varepsilon)$.

Automorphic forms Let S_2^H be the space $S_{2,\dots,2,0,\dots,0}^H(B_{\mathbf{A}}^\times)$ of functions

$$\varphi\colon B_{\mathbf{A}}^\times \simeq G(\mathbf{R}) \times \widehat{B}^\times \longrightarrow \mathbf{C}$$

satisfying the following properties:

(1) $\forall g \in B^\times \ \forall b \in B_{\mathbf{A}}^\times \qquad \varphi(gb) = \varphi(b)$;
(2) $\forall g \in (\mathbf{R}^\times)^r \times G_{r+1}(\mathbf{R}) \times \cdots \times G_d(\mathbf{R}) \subset G(\mathbf{R}) \ \forall b \in B_{\mathbf{A}}^\times \qquad \varphi(bg) = \varphi(b)$;
(3) $\forall h \in H \ \forall b \in B_{\mathbf{A}}^\times \qquad \varphi(bh) = \varphi(b)$;
(4) $\forall g \in B_{\mathbf{A}}^\times \ \forall(\theta_1,\dots,\theta_r) \in \mathbf{R}^r$

$$\varphi\left(g\left[\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \dots, \begin{pmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{pmatrix}, 1,\dots,1 \right] \right) = e^{-2i\theta_1} \times \cdots \times e^{-2i\theta_r} \varphi(g);$$

(5) For all $g \in B_{\mathbf{A}}^\times$, the map

$$(x_1 + iy_1,\dots,x_r + iy_r) \mapsto \frac{1}{y_1\dots y_r} \varphi\left(g\left[\begin{pmatrix} y_1 & x_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} y_r & x_r \\ 0 & 1 \end{pmatrix}, 1,\dots,1 \right] \right)$$

is holomorphic on $\mathcal{H}^r$, where $\mathcal{H}$ denotes the Poincaré upper-half plane.

Remark that we do not need any assumption to obtain cuspidal forms as B will be assumed to differ from $M_2(F)$.

There is an action of $\widehat{B}^\times$ on $S_2 = \bigcup_H S_2^H$ defined by

$$\forall g \in \widehat{B}^\times,\ \forall \varphi \in S_2,\ \forall x \in B_{\mathbf{A}}^\times, \qquad g \cdot \varphi(x) = \varphi(xg);$$

thus S_2^H is the space of H-invariant functions in S_2.

By modifying properties (4) and (5) above we obtain the following new definition.

Definition 2.3 Let $\varepsilon\colon \{\tau_1, \dots, \tau_r\} \to \{\pm 1\}$ and $\varepsilon_i = \varepsilon(\tau_i)$. The space $(S_2^\varepsilon)^H$ is the space of maps $\varphi\colon B_{\mathbf{A}}^\times \simeq +G(\mathbf{R}) \times \widehat{B}^\times \to \mathbf{C}$ satisfying 1-3 above and

(4′) for all $g \in B_{\mathbf{A}}^\times$ and $(\theta_1 \dots \theta_r) \in \mathbf{R}^r$

$$\varphi\left(g\left(\begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix}, \dots, \begin{pmatrix}\cos\theta_r & -\sin\theta_r\\ \sin\theta_r & \cos\theta_r\end{pmatrix}, 1, \dots, 1\right)\right) = \\ e^{-2i\varepsilon_1\theta_1} \times \cdots \times e^{-2i\varepsilon_r\theta_r}\varphi(g);$$

(5′) for all $g \in B_{\mathbf{A}}^\times$, the map

$$(x_1 + iy_1, \dots, x_r + iy_r) \mapsto \frac{1}{y_1 \dots y_r}\varphi\left(g\left(\begin{pmatrix}y_1 & x_1\\ 0 & 1\end{pmatrix}, \dots, \begin{pmatrix}y_r & x_r\\ 0 & 1\end{pmatrix}, 1, \dots, 1\right)\right)$$

is holomorphic (resp. anti-holomorphic) in $z_j = x_j + iy_j \in \mathcal{H}$ if $\varepsilon_j = 1$ (resp. $\varepsilon_j = -1$).

We will denote by $S_2^{\hat{F}^\times}$ (resp. $(S_2^\varepsilon)^{\hat{F}^\times}$) the space of elements in S_2 (resp. S_2^ε) that are $\hat{F}^\times$-invariant.

We are now able to affirm the existence of relations between automorphic forms and r-forms on $\mathrm{Sh}_H(G, X)(\mathbf{C})$.

Proposition 2.4 *There exist bijections compatible with the $\widehat{B}^\times$-action between the following spaces:*

$$\begin{array}{lll} \Gamma(\Omega_H^{\mathrm{an}}) & \textit{and} & S_2^H \\ \Gamma((\Omega_H^{\mathrm{an}})^\varepsilon) & \textit{and} & (S_2^\varepsilon)^H \\ \Gamma(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), (\Omega_H^{\mathrm{an}})^\varepsilon) & \textit{and} & (S_2^\varepsilon)^{H\widehat{F}^\times}. \end{array}$$

This statement is completely analogous to [5, Section 3.6]; see [11, Propositions 1.2.2.4 and 1.2.2.5] for more details.

Matsushima–Shimura Theorem The decomposition of the cohomology of quaternionic Shimura varieties given by Matsushima–Shimura theorem will be useful in the following sections. Let us recall this result when $B \neq M_2(F)$ [10, 19]. Denote by h_F^+ the narrow class number of F.

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Theorem 2.5 *Let $m \in \{0,\dots,2r\}$. We have the following decomposition:*

$$H^m(\mathrm{Sh}_H(G,X)(\mathbf{C}),\mathbf{C}) \simeq$$

$$\begin{cases} \Big(\mathrm{Vect} \bigwedge_{\substack{i\in a\subset\{1,\dots,r-1\}\\ |a|=m/2}} \dfrac{\mathrm{d}z_i\wedge \mathrm{d}\overline{z_i}}{y_i^2}\Big)^s & \text{if } m\neq r,\\ \Big(\mathrm{Vect} \bigwedge_{\substack{i\in a\subset\{1,\dots,r-1\}\\ |a|=m/2}} \dfrac{\mathrm{d}z_i\wedge \mathrm{d}\overline{z_i}}{y_i^2}\Big)^s \oplus \bigoplus_{\varepsilon\in\{\pm1\}^r}(S_2^\varepsilon)^H & \text{if } m=r, \end{cases}$$

and

$$H^m(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C}) \simeq$$

$$\begin{cases} \Big(\mathrm{Vect} \bigwedge_{\substack{i\in a\subset\{1,\dots,r-1\}\\ |a|=m/2}} \dfrac{\mathrm{d}z_i\wedge \mathrm{d}\overline{z_i}}{y_i^2}\Big)^{s'} & \text{if } m\neq r,\\ \Big(\mathrm{Vect} \bigwedge_{\substack{i\in a\subset\{1,\dots,r-1\}\\ |a|=m/2}} \dfrac{\mathrm{d}z_i\wedge \mathrm{d}\overline{z_i}}{y_i^2}\Big)^{s'} \oplus \bigoplus_{\varepsilon\in\{\pm1\}^r}(S_2^\varepsilon)^{H\hat{F}^\times} & \text{if } m=r, \end{cases}$$

where s (resp. s') is the number of connected components of $\mathrm{Sh}_H(G,X)(\mathbf{C})$ (resp. of $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$).

3 Periods

3.1 Yoshida's conjecture

Let E/F be an elliptic curve, assumed modular in the sense that there exists a cuspidal, parallel weight two Hilbert modular form $\tilde{\varphi} \in S_2(\mathrm{GL}_2(F_{\mathbf{A}}))$ satisfying $L(E,s) = L(\tilde{\varphi},s)$. We shall assume that the automorphic representation generated by $\tilde{\varphi}$ is obtained by the Jacquet–Langlands correspondence from $\varphi \in S_2^{H\widehat{F}^\times}(B_{\mathbf{A}}^\times)$.

Denote by $\pi = \pi_\infty \otimes \pi_f$ the automorphic representation of $B_{\mathbf{A}}^\times/F_{\mathbf{A}}^\times$ generated by φ. We shall assume until Section 3.3, only for simplicity, that $\dim \pi_f^H = 1$.

The motivic conjecture of Yoshida is the following.

Conjecture 3.1 (Yoshida [30]) *Let $M = h^1(E)$ be the motive over F with coefficients in $\mathbf{Q}$ associated with E. The motive $M' = \bigotimes_{\{\tau_1,\dots,\tau_r\}} \mathrm{Res}_{F/F'} M$ over F' is isomorphic to the motive associated with the part $H^*(\mathrm{Sh}_{H\widehat{F}^\times}(G,X))^{(E)}$ of the cohomology for which Hecke eigenvalues are the same as E.*

Remark 3.2 Is the isomorphism between M' and $H^*(\mathrm{Sh}_{H\widehat{F}^\times}(G,X))^{(E)}$ canonical? This is an excellent question. In general, if such an isomorphism exists, it need not be unique up to a multiplicative constant (*e.g,* if E is defined over a proper subfield of F). However, there should always exist a canonical isomorphism between M' and $H^*(\mathrm{Sh}_{H\widehat{F}^\times}(G,X))^{(E)}$, which can be characterized geometrically. This will be shown in a forthcoming paper by Cornut and Nekovář.

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While looking at the ℓ-adic realization, this conjecture is in fact the Langlands cohomological conjecture. This case is known, up to semi-simplification,[1] thanks to Brylinski and Labesse in the case $B = \mathrm{M}_2(F)$ [2], Langlands in the case $B \neq \mathrm{M}_2(F)$ for primes of good reduction, [18], and Reimann and Zink [25,26] for a more general case.

Recall the following decompositions given by Yoshida in [30, Section 5.1], when we focus on $\tau'\colon F' \hookrightarrow \mathbf{C}$ induced by $\tilde{\tau}'\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$.

Betti cohomology There exists an isomorphism of $\mathbf{Q}$-vector spaces

$$\mathscr{I}: M'_{\mathrm{B}} \xrightarrow{\sim} \bigotimes_{j=1}^{r} M_{B,\tau_j}.$$

de Rham cohomology The map

$$\mathscr{J}: M'_{\mathrm{dR}} \xrightarrow{\sim} \Big(\bigotimes_{j=1}^{r} \big(M_{\mathrm{dR}} \otimes_{F,\tau_j} \overline{\mathbf{Q}}\big)\Big)^{\mathrm{Gal}(\overline{\mathbf{Q}}/F')}$$

is an isomorphism of F'-vector-spaces. The right-hand side is a tensor product of $\overline{\mathbf{Q}}$-vector spaces, and the action of $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/F')$ is given by

$$\bigotimes_{s\in\{\tau_1,\dots,\tau_r\}} (x_s \otimes_{F,s} a_s) \mapsto \bigotimes_{s\in\{\tau_1,\dots,\tau_r\}} \big(x_s \otimes_{F,\sigma s} \sigma(a_s)\big).$$

Comparison isomorphisms Let $I = \bigotimes_{j=1}^{r} I_{\tau_j}$, where

$$I_{\tau_j}\colon M_{\mathrm{B},\tau_j} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_{F,\tau_j} \mathbf{C}$$

are isomorphisms of $\mathbf{C}$-vector spaces, and I' is the following isomorphism over $\mathbf{C}$:

$$I'\colon M'_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C}.$$

The maps $I \circ (\mathscr{I} \otimes_{\mathbf{Q}} \mathrm{id}_{\mathbf{C}})$ and $(\mathscr{J} \otimes_{F'} \mathrm{id}_{\mathbf{C}}) \circ I'$ are known to satisfy

$$(\star)\qquad I \circ (\mathscr{I} \otimes_{\mathbf{Q}} \mathrm{id}_{\mathbf{C}}) = (\mathscr{J} \otimes_{F'} \mathrm{id}_{\mathbf{C}}) \circ I'\colon M'_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} \bigotimes_{j=1}^{r} (M_{\mathrm{dR}} \otimes_{F,\tau_j} \mathbf{C}).$$

Yoshida's period conjecture consists of the existence of the isomorphisms $\mathscr{I}$, $\mathscr{J}$, I, and I'. It is the Hodge–de Rham realization of the motivic conjecture above.

Complex conjugation Let c_{τ_j} be the complex conjugation on M_{B,τ_j}. We will need the following hypothesis, which allows us to compare c_{τ_j} with t_j^* on $M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C}$.

Hypothesis 3.3 *The action of t_j^* on $M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C}$ corresponds via the isomorphism*

$$(\mathscr{I} \otimes_{\mathbf{Q}} \mathrm{id}_{\mathbf{C}}) \circ (I')^{-1}\colon M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C} \longrightarrow M'_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \longrightarrow \Big(\bigotimes_{k=1}^{r} M_{\mathrm{B},\tau_k}\Big) \otimes_{\mathbf{Q}} \mathbf{C},$$

to the action of c_{τ_j} on M_{B,τ_j}.

[1] Since the Galois action on $H^r_\ell(\mathrm{Sh}_{H\widehat{F}^\times}(G,X))^{(E)}$ is semi-simple, the phrase "up to semi-simplification" can be omitted. This fact will be proved in a forthcoming paper by Cornut and Nekovář.

3.2 Lattices and Periods

Fix some $\omega_\varphi \neq 0$ in $F^r M'_{\mathrm{dR}}$. By definition of M', there exists a finite set of places S of F such that for $v \notin S$, $T_v\omega_\varphi = a_v(E)\omega_\varphi$, where T_v is the Hecke operator at the place v (these operators are defined in [5, Section 3.4] for quaternionic Shimura curves; the general case is completely analogous).

Let $\Omega_{E/F}$ be the sheaf of differentials on E/F. Fix $\eta \neq 0 \in H^0(E, \Omega_{E/F}) = F^1 M_{dR}$. For $j \in \{1, \dots, n\}$, let

$$\eta_j = \eta \otimes_{F,\tau_j} 1 \in H^0\big(E \otimes_{F,\tau_j} \overline{\mathbf{Q}}, \Omega_{(E\otimes_{F,\tau_j}\overline{\mathbf{Q}})/\overline{\mathbf{Q}}}\big) = (F^1 M_{\mathrm{dR}}) \otimes_{F,\tau_j} \overline{\mathbf{Q}}.$$

Then

$$\bigotimes_{j=1}^r \eta_j \in \Big(\bigotimes_{j=1}^r \big(F^1 M_{\mathrm{dR}} \otimes_{F,\tau_j} \overline{\mathbf{Q}}\big)\Big)^{\mathrm{Gal}(\overline{\mathbf{Q}}/F')} = \mathscr{J}(F^r M'_{\mathrm{dR}})$$

and there exists $\alpha \in F'^\times$ such that $\mathscr{J}(\alpha\omega_\varphi) = \eta_1 \otimes \cdots \otimes \eta_r$.

Let $j \in \{1, \dots, r\}$ and $E_j = E \otimes_{F,\tau_j} \mathbf{C}$. We shall denote by $H_1(E_j, \mathbf{Z})^\pm$ the eigenspaces of the complex conjugation action on $H_1(E_j, \mathbf{Z})$. Then

$$\Big\{ \int_\Upsilon \eta_j,\ \Upsilon \in H_1(E_j, \mathbf{Z})^\pm \Big\} = \mathbf{Z}\Omega_j^\pm,$$

where $\Omega_j^+ \in \mathbf{R} \setminus \{0\}$ and $\Omega_j^- \in i\mathbf{R} \setminus \{0\}$ are determined up to a sign. We fix the signs by imposing, *e.g.*, $\operatorname{Re}\big(\Omega_j^+\big) > 0$ and $\operatorname{Im}\big(\Omega_j^-\big) > 0$.

Fix a character $\beta\colon \{1\} \times \prod_{j=2}^r \{\pm 1\} \to \{\pm 1\}$, and write $\beta = \prod_{j=2}^r \beta_j$. We set

$$\omega_\varphi^\beta = \Big(\sum_{\sigma \in \{1\} \times \prod_{j=2}^r \{\pm 1\}} \beta(\sigma) t_\sigma^*\Big)\omega_\varphi = \prod_{j=2}^r \big(1 + \beta_j(-1)t_j^*\big)\,\omega_\varphi$$

and

$$\Omega^\beta = \prod_{j=2}^r \Omega_j^{\beta_j(-1)}.$$

The following identities

$$\Big(\bigotimes_{j=1}^r M_{\mathrm{B},\tau_j}\Big) \otimes_{\mathbf{Q}} \mathbf{C} = \bigotimes_{j=1}^r \mathrm{Hom}_{\mathbf{Z}}(H_1(E_j, \mathbf{Z}), \mathbf{C}) = \mathrm{Hom}_{\mathbf{Z}}\Big(\bigotimes_{j=1}^r H_1(E_j, \mathbf{Z}), \mathbf{C}\Big)$$

and Yoshida's conjecture show that the image of $\alpha\omega_\varphi^\beta$ under the map

$$(\mathscr{I} \otimes_{\mathbf{Q}} \mathrm{id}_{\mathbf{C}}) \circ I'^{-1} = I^{-1} \circ (\mathscr{J} \otimes_{F'} \mathrm{id}_{\mathbf{C}})\colon M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C} \longrightarrow \Big(\bigotimes_{j=1}^r M_{B,\tau_j}\Big) \otimes_{\mathbf{Q}} \mathbf{C}$$

is identified with the linear form

$$\begin{cases} \bigotimes_{j=1}^r H_1(E_j, \mathbf{Z}) & \longrightarrow \quad \mathbf{C}, \\ \Upsilon_1 \otimes \cdots \otimes \Upsilon_r & \longmapsto \quad \int_{\Upsilon_1 \otimes \cdots \otimes \Upsilon_r} \bigotimes_{j=1}^r \big(1 + \beta_j(-1)t_j^*\big)\,\eta_j. \end{cases} \tag{3.1}$$

Hypothesis 3.3 allows us to be more explicit. Let

$$\Upsilon_1 \otimes \cdots \otimes \Upsilon_r \in \bigotimes_{j=1}^{r} H_1(E_j, \mathbf{Z}),$$

then

$$\begin{aligned}\int_{\Upsilon_1 \otimes \cdots \otimes \Upsilon_r} \bigotimes_{j=1}^{r} \bigl(1 + \beta_j(-1)t_j^*\bigr)\,\eta_j &= \Bigl(\int_{\Upsilon_1} \eta_1\Bigr) \prod_{j=2}^{r} \int_{\Upsilon_j} (1 + \beta_j(-1)t_j^*)\eta_j \\ &= \Bigl(\int_{\Upsilon_1} \eta_1\Bigr) \prod_{j=2}^{r} \int_{\Upsilon_j + \beta_j(-1)c_{\tau_j}\Upsilon_j} \eta_j.\end{aligned}$$

and the linear form (3.1) takes values in $\Lambda_1 \Omega^\beta = (\mathbf{Z}\Omega_1^+ + \mathbf{Z}\Omega_1^-)\Omega^\beta$.

Under the dual isomorphism $\mathscr{I}^*$ of $\mathscr{I}$, the lattices

$$\bigotimes_{j=1}^{r} {}_{\mathbf{Z}} H_1(E_j, \mathbf{Z}) \subset \bigotimes_{j=1}^{r} {}_{\mathbf{Q}} M_{B,\tau_j}^* \quad \text{and} \quad \operatorname{Im}\bigl(H_r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{Z}) \longrightarrow (M_B')^*\bigr)$$

are commensurable. Thus there exists $\xi \in \mathbf{Z} \smallsetminus \{0\}$ such that

$$\xi \operatorname{Im}\bigl(H_r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{Z}) \longrightarrow (M_B')^*\bigr) \subset \mathscr{I}^*\Bigl(\bigotimes_{j=1}^{r} {}_{\mathbf{Z}} H_1(E_j, \mathbf{Z})\Bigr).$$

This proves the following proposition.

Proposition 3.4 *Under the hypothesis made in this section (E is modular, the multiplicity one in Yoshida's motivic conjecture and Hypothesis* 3.3*), there exist $\alpha \in F'^{\times}$ and $\xi \in \mathbf{Z} \smallsetminus \{0\}$ such that*

$$\forall \gamma \in H_r\bigl(Sh_H(G, X)(\mathbf{C}), \mathbf{Z}\bigr), \quad \forall \beta\colon \prod_{j=2}^{r} \{\pm 1\} \to \{\pm 1\}, \qquad \xi \int_\gamma \alpha \omega_\varphi^\beta \in \Lambda_1 \Omega^\beta.$$

3.3 General Case

When $m_H(\pi) = \dim \pi_f^H(\varphi) > 1$ Yoshida's conjecture reads as follows.

Conjecture 3.5 *The motive $H^r(\mathrm{Sh}_H(G, X))^{(E)}$ is isomorphic to*

$$\Bigl(\bigotimes_{\{\tau_1, \dots, \tau_r\}} \mathrm{Res}_{F/F'} M\Bigr)^{m_H(\pi)}.$$

In general the motive $H^r(\mathrm{Sh}_H(G, X))^{(E)}$ has rank $\neq 2^r$. We shall provide Betti and de Rham realizations of a submotive $M' \subset H^r(\mathrm{Sh}_H(G, X))^{(E)}$ of rank 2^r and an isomorphism $M' \xrightarrow{\sim} \bigotimes_{\{\tau_1, \dots, \tau_r\}} \mathrm{Res}_{F/F'} M$.

We need $0 \neq \omega_\varphi \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z, X)/F')^{(E)}$ satisfying the following conditions:

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- **de Rham cohomology:** The F'-vector space

$$M'_{\mathrm{dR}} := \Big(\bigoplus_{\sigma\in\{\pm1\}^r} \mathbf{C}t^*_\sigma(\omega_\varphi\otimes 1)\Big) \cap H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)}$$

has dimension 2^r.
Thus,

$$F^r M'_{\mathrm{dR}} := M'_{\mathrm{dR}} \cap F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)} = F'\omega_\varphi.$$

- **Betti cohomology:** Fix an isomorphism

$$I'\colon H^r_{\mathrm{B}}\big(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Q}\big)^{(E)} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} H^r_{\mathrm{dR}}\big(\mathrm{Sh}_H(G/Z,X)/F'\big)^{(E)} \otimes_{F'} \mathbf{C}.$$

The $\mathbf{Q}$-vector space

$$M'_{\mathrm{B}} := I'^{-1}(M'_{\mathrm{dR}}\otimes_{F'}\mathbf{C}) \cap H^r_{\mathrm{B}}\big(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Q}\big)^{(E)}$$

has dimension 2^r.

Definition 3.6 An element $\omega_\varphi \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)}$ is said to be rational if it satisfies the conditions above.

- **Comparison isomorphisms:** There exist isomorphisms

$$\mathscr{I}\colon M'_{\mathrm{B}} \xrightarrow{\sim} \bigotimes_{j=1}^r M_{\mathrm{B},\tau_j},$$

$$\mathscr{J}\colon M'_{\mathrm{dR}} \xrightarrow{\sim} \Big(\bigotimes_{j=1}^r (M_{\mathrm{dR}}\otimes_{F,\tau_j}\overline{\mathbf{Q}})\Big)^{\mathrm{Gal}(\overline{\mathbf{Q}}/F')},$$

$$I_{\tau_j}\colon M_{\mathrm{B},\tau_j}\otimes_{\mathbf{Q}}\mathbf{C} \xrightarrow{\sim} M_{\mathrm{dR}}\otimes_{F,\tau_j}\mathbf{C}.$$

Set $I = \bigotimes_{j=1}^r I_{\tau_j}$. We have

$$(\star)\qquad I\circ\big(\mathscr{I}\otimes_{\mathbf{Q}}\mathrm{id}_{\mathbf{C}}\big) = \big(\mathscr{J}\otimes_{F'}\mathrm{id}_{\mathbf{C}}\big)\circ I'\colon M'_{\mathrm{B}}\otimes_{\mathbf{Q}}\mathbf{C} \xrightarrow{\sim} \bigotimes_{j=1}^r \big(M_{\mathrm{dR}}\otimes_{F,\tau_j}\mathbf{C}\big).$$

As in Proposition 3.4 we have the following proposition.

Proposition 3.7 *Let $\omega_\varphi \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)}$ be rational. If E is modular and if Yoshida's conjecture is true, then there exist $\alpha\in F'^\times$ and $\xi\in\mathbf{Z}\smallsetminus\{0\}$ such that*

$$\forall\gamma\in H_r(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z}),\quad \forall\beta\colon \prod_{j=2}^r\{\pm1\}\to\{\pm1\},\qquad \xi\int_\gamma \alpha\omega_\varphi^\beta \in \Lambda_1\Omega^\beta.$$

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Example Let $H_1, H_2 \subset \widehat{B}^\times$ be compact open subgroups such that there exists $g \in \widehat{B}^\times$ satisfying $g^{-1}H_1 g \subset H_2$. Let $\omega_{\varphi_2} \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_{H_2}(G/Z, X)/F')^{(E)}$ be rational. Let us explain a way to obtain $\omega_{\varphi_1} \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_{H_1}(G/Z, X)/F')^{(E)}$ rational.

Let

$$\mathrm{pr}\colon \mathrm{Sh}_{g^{-1}H_1 g}(G/Z, X) \longrightarrow \mathrm{Sh}_{H_2}(G/Z, X)$$

be the map given by $[x, b]_{g^{-1}H_1 g} \mapsto [x, b]_{H_2}$ and let

$$[\cdot g]\colon \mathrm{Sh}_{H_1}(G/Z, X) \to \mathrm{Sh}_{g^{-1}H_1 g}(G/Z, X)$$

be given by $[x, b]_{H_1} \mapsto [x, bg]_{g^{-1}H_1 g}$. Let $\mathrm{pr}_g\colon \mathrm{Sh}_{H_1}(G/Z, X) \to \mathrm{Sh}_{H_2}(G/Z, X)$ be the composition of pr with $[\cdot g]$.

Choose $\theta_g \in \mathbf{Q}$. Set

$$\omega_{\varphi_1} := \sum_{\substack{g \in \widehat{B}^\times \\ \text{s.t. } g^{-1}H_1 g \subset H_2}} \theta_g \ \mathrm{pr}_g^*(\omega_{\varphi_2}),$$

$$(M_1')_{\mathrm{dR}} = \Big(\sum_g \theta_g \ \mathrm{pr}_g^*\Big)(M_2')_{\mathrm{dR}},$$

$$(M_1')_{\mathrm{B}} = \Big(\sum_g \theta_g \ \mathrm{pr}_g^*\Big)(M_2')_{\mathrm{B}}.$$

Proposition 3.8 *If $\omega_{\varphi_1} \neq 0$, then the map*

$$\sum_{\substack{g \in \widehat{B}^\times \\ \text{s.t. } g^{-1}H_1 g \subset H_2}} \theta_g \, \mathrm{pr}_g^*$$

is injective on $\bigoplus_{\sigma \in \{\pm 1\}^r} \mathbf{C} t_\sigma^*(\omega_{\varphi_2} \otimes 1)$, and $\omega_{\varphi_1} \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_{H_1}(G/Z, X)/F')^{(E)}$ is rational.

Proof Assume that $\omega = \sum_{\sigma \in \{\pm 1\}^r} \lambda_\sigma t_\sigma^* \omega_{\varphi_2} \in \bigoplus_{\sigma \in \{\pm 1\}^r} \mathbf{C} t_\sigma^*(\omega_{\varphi_2} \otimes 1)$ (where $\lambda_\sigma \in \mathbf{C}$) is such that $\sum_g \theta_g \, \mathrm{pr}_g^*(\omega) = 0$. We have the following equalities:

$$\begin{aligned}\sum_g \theta_g \, \mathrm{pr}_g^* \, \omega &= \sum_g \theta_g \, \mathrm{pr}_g^* \sum_\sigma \lambda_\sigma t_\sigma^* \omega_{\varphi_2} = \sum_\sigma \lambda_\sigma t_\sigma^* \sum_g \theta_g \, \mathrm{pr}_g^* \, \omega_{\varphi_2} \\ &= \sum_\sigma \lambda_\sigma t_\sigma^* \omega_{\varphi_1}.\end{aligned}$$

Thus,

$$\sum_\sigma \lambda_\sigma t_\sigma^* \omega_{\varphi_1} = 0 \in \bigoplus_{\sigma \in \{\pm 1\}^r} \mathbf{C} t_\sigma^* \omega_{\varphi_1},$$

and $\forall \sigma \in \{\pm 1\}^r$, $\lambda_\sigma t_\sigma^* \omega_{\varphi_1} = 0$. Hence for all $\sigma \in \{\pm 1\}^r$, $\lambda_\sigma \in 0$. The map

$$\sum_{g \in \widehat{B}^\times \ \text{s.t. } g^{-1}H_1 g \subset H_2} \theta_g \, \mathrm{pr}_g^*$$

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commutes with T_v, $v \notin S$ and is an isomorphism $\bigoplus \mathbf{C}t_\sigma^* \omega_{\varphi_2} \to \bigoplus \mathbf{C}t_\sigma^* \omega_{\varphi_1}$. Hence,

$$\omega_{\varphi_1} \in \Big(\bigoplus_{\sigma \in \{\pm 1\}^r} \mathbf{C}t_\sigma^*(\omega_{\varphi_1} \otimes 1) \Big) \cap F^r H^r_{\mathrm{dR}}\big(\mathrm{Sh}_{H_1}(G/Z, X)/F'\big)^{(E)}$$

is rational. ■

4 Toric Orbits

Let K/F be a quadratic extension satisfying the following properties:

(1) the places $\tau_2, \dots, \tau_r$ of F are split in K;
(2) the places $\tau_1, \tau_{r+1}, \dots, \tau_d$ are ramified in K;
(3) the places $\mathfrak{p} \in S_B$ are inert in K.

Thanks to the Albert–Brauer–Hasse–Noether theorem, there exists an F-embedding $q\colon K \hookrightarrow B$, unique up to conjugacy. We will denote by q_j (resp. $\widehat{q}, q_{\mathbf{A}}$) the induced embedding $K \hookrightarrow B_{\tau_j}$ (resp. $\widehat{K} \hookrightarrow \widehat{B}$, $K_{\mathbf{A}} \hookrightarrow B_{\mathbf{A}}$). For each place v of F, set $K_v = K \otimes_F F_v$.

4.1 Cycles on X

Let $T = \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)/\,\mathrm{Res}_{F/\mathbf{Q}}(\mathbf{G}_m)$. Thanks to Hilbert's Theorem 90 we have

$$T(A) = (K \otimes_{\mathbf{Q}} A)^\times/(F \otimes_{\mathbf{Q}} A)^\times$$

for every $\mathbf{Q}$-algebra A.

By abuse of notation, let us denote by $q\colon T \hookrightarrow G/Z(G)$ the embedding induced by $q\colon K \hookrightarrow B$. The group $T(\mathbf{R})$ is identified with $\prod_{j=1}^d K_{\tau_j}^\times/F_{\tau_j}^\times$. We denote, by abuse of notation, $q_j\colon K_{\tau_j}^\times/F_{\tau_j}^\times \to G_{j,\mathbf{R}}$.

Let $\pi_0(T(\mathbf{R}))$ be the set of connected components of $T(\mathbf{R})$ and denote by $T(\mathbf{R})^\circ$ the component of the identity. Fix a multi-orientation on $T(\mathbf{R})^\circ = \prod_{j=1}^d (K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ$ (*i.e.*, an orientation of each factor $(K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ$) and remark that

$$\pi_0(T(\mathbf{R})) = T(\mathbf{R})/T(\mathbf{R})^\circ \simeq \prod_{j=2}^r \{\pm 1\}.$$

We will focus on the orbits in X under the action of $q(T(\mathbf{R})^\circ)$ by conjugation.

Proposition 4.1 *Let $\mathscr{T}^\circ$ be an orbit of $q(T(\mathbf{R})^\circ)$ in X. Then $\mathscr{T}^\circ$ decomposes into a product of orbits in X_j under $q_j(T(\mathbf{R})^\circ)$ and is multi-oriented.*

Proof The first part of this assertion follows from the natural decomposition $X = X_1 \times \cdots \times X_r$. The orbit $\mathscr{T}^\circ$ decomposes into orbits under $q_j((K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ)$. For $j = 1$, $q_j((K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ) \simeq \mathbf{S}^1$ or a point and the orientation does not change. For $j \in \{2, \dots, r\}$, $q_j((K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ) \simeq \mathbf{R}_+^\times$. The action of $\mathbf{R}_+^\times$ on itself by multiplication does not change the orientation. Hence the multi-orientation induced on $\mathscr{T}^\circ$ by $T(\mathbf{R})^\circ$ is well defined. ■

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In the following sections we shall fix some $q(T(\mathbf{R})^\circ)$-orbit $\mathscr{T}^\circ$, whose projection on X_1 is a point.

Proposition 4.2 *$\mathscr{T}^\circ$ is a connected multi-oriented submanifold of real dimension $r-1$.*

Proof Recall that $\mathscr{T}^\circ$ is decomposed as $\mathscr{T}^\circ = \{z_1\} \times \mathscr{T}_2 \times \cdots \times \mathscr{T}_r$. Fix $x \in X$ such that $\mathscr{T}^\circ = q(T(\mathbf{R})^\circ \cdot x$. Then for $j \in \{2, \ldots, r\}$ we have $\mathscr{T}_j = q_j((K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ) \cdot \mathrm{pr}_j(x)$. The group $q_j((K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ)$ is naturally identified with $\mathbf{R}_+^\times$ and $\mathscr{T}_j$ is a connected oriented manifold of real dimension one. ■

As a corollary, we have the following decomposition:

$$\mathscr{T}^\circ = \{z_1\} \times \gamma_2 \times \cdots \times \gamma_r,$$

where z_1 is one of the two fixed points in the action of $q_1(T(\mathbf{R})^\circ)$ on X_1 and γ_j is an oriented connected submanifold of real dimension one in X_j.

When we use the identification of X with $(\mathbf{C} \smallsetminus \mathbf{R})^r$, the action of $T(\mathbf{R})$ on X by conjugation is an action of $\mathrm{PGL}_2(\mathbf{R})$ on $(\mathbf{C} \smallsetminus \mathbf{R})^r$ by homography. Let $z \in K \smallsetminus F$. For $j \in \{2, \ldots, r\}$ the matrix $q_j(z)$ is hyperbolic with exactly two fixed points in $\mathbf{P}^1(\mathbf{R})$, z_j and z_j'. The manifold γ_j is then a circle arc in the Poincaré upper half-plane joining z_j to z_j' (or a line if $z_j' = \infty$). Figure 1 gives some examples of what could the γ_js be in the case of circle arcs.

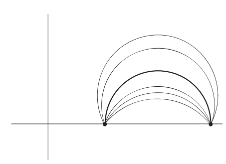

Figure 1: Case of circle arcs.

4.2 Tori on $\mathrm{Sh}_H(G/Z, X)(\mathbf{C})$

Let $b \in \widehat{B}^\times$. We will denote by $\mathscr{T}_b^\circ$ the following subset of $\mathrm{Sh}_H(G/Z, X)(\mathbf{C})$

$$\mathscr{T}_b^\circ = \left\{ [x, b]_{H\widehat{F}^\times}, x \in \mathscr{T}^\circ \right\}.$$

Proposition 4.3 *$\mathscr{T}_b^\circ$ is an oriented torus of real dimension $r-1$.*

Proof Let $x, x' \in \mathscr{T}^\circ$ and $b \in \widehat{B}^\times$; we know that

$$\begin{aligned} [x, b]_{H\widehat{F}^\times} = [x', b]_{H\widehat{F}^\times} &\iff \exists k \in B^\times \text{ and } h \in H\widehat{F}^\times \qquad (kx', kbh) = (x, b) \\ &\iff \exists k \in B^\times \cap bH\widehat{F}^\times b^{-1} \qquad kx' = x \end{aligned}$$

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Since the projection of $\mathscr{T}^\circ$ on X_1 is a point, we have $k \in B \cap q_1(K_{\tau_1}) = q_1(K)$ and

$$k \in q(K^\times) \cap bH\widehat{F}^\times b^{-1}.$$

Thus the stabilizer $\mathscr{W}$ of $\mathscr{T}_b^\circ$ under the action of $q(K^\times)$ is

$$\mathscr{W} = q(K^\times) \cap (bH\widehat{F}^\times b^{-1}),$$

which is commensurable with $\mathcal{O}_{K,+}^\times/\mathcal{O}_F^\times$. This quotient has rank $r-1$ over $\mathbf{Z}$ as a consequence of Dirichlet's units theorem

$$\mathcal{O}_{K,+}^\times/\mathcal{O}_F^\times \simeq \text{torsion} \times \mathbf{Z}^{r-1},$$

and the torsion is finite. The action of $T(\mathbf{R})^\circ$ on $\mathscr{T}^\circ$ is given by $\prod_{j=2}^r (K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ$, and there is an isomorphism

$$\prod_{j=2}^{r} (K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ \xrightarrow{\sim} \mathbf{R}^{r-1}.$$

The image $\widetilde{\mathcal{O}}$ of $\mathcal{O}_{K,+}^\times/\mathcal{O}_F^\times$ in $\mathbf{R}^{r-1}$ is isomorphic to $\mathbf{Z}^s$ with $s \leq r-1$. Denote by $\widetilde{\mathcal{O}}_K^\times$ the image of $\mathcal{O}_K^\times$ in $(K\otimes\mathbf{R})^{\times,\,\mathrm{N}_{K/\mathbf{Q}}=1}$. As

$$\prod_{j\notin\{2,\dots,r\}} K_{\tau_j}^\times/F_{\tau_j}^\times \qquad \text{and} \qquad \frac{(K\otimes\mathbf{R})^{\times,\,\mathrm{N}_{K/\mathbf{Q}}=1}}{\widetilde{\mathcal{O}}_K^\times}$$

are compact, $\mathbf{R}^{r-1}/\widetilde{\mathcal{O}}$ is compact. Thus, the image of $\mathcal{O}_{K,+}^\times/\mathcal{O}_F^\times$ in $\mathbf{R}^{r-1}$ is a lattice.

The set $\mathscr{T}_b^\circ$ is a principal homogeneous space under

$$q(K^\times)/\mathscr{W} \simeq (\mathbf{R}/\mathbf{Z})^{r-1}.$$

It is a real torus in $\mathrm{Sh}_H(G/Z,X)(C)$ of dimension $r-1$, which is oriented by the fixed multi-orientation on $\mathscr{T}^\circ$. ∎

For each $u \in \pi_0(T(\mathbf{R}))$ and $b \in \widehat{B}^\times$ let

$$\mathscr{T}_b^u = \left\{ [q(u)\cdot x, b]_{H\widehat{F}^\times},\ x \in \mathscr{T}^\circ \right\}.$$

It is a real oriented torus of dimension $r-1$.

Proposition 4.4 *The set*

$$\left\{ \mathscr{T}_b^u \mid b \in \widehat{B}^\times,\ u \in \pi_0(T(\mathbf{R})) \right\}$$

does not depend on the choice of the F-embedding $q\colon K \hookrightarrow B$.

Proof Let $\tilde{q}\colon K \hookrightarrow B$ be another F-embedding. Thanks to the Skolem–Noether theorem there exists $\alpha \in B^\times$ such that for all $k \in K$, $\widetilde{q}(k) = \alpha q(k)\alpha^{-1}$. Let $x_0 \in X$, and assume that $\mathscr{T}^\circ = q(T(\mathbf{R})^\circ)\cdot x_0$. We have $\widetilde{\mathscr{T}}^\circ := \tilde{q}(T(\mathbf{R})^\circ)\cdot\alpha(x_0) = \alpha\cdot\mathscr{T}^\circ$, and for each $u \in \pi_0(T(\mathbf{R}))$,

$$\alpha\cdot q(u)\cdot\mathscr{T}^\circ = \tilde{q}(uT(\mathbf{R})^\circ)\cdot\alpha\cdot x_0.$$

Let $b \in \widehat{B}^\times$. As $\alpha \in B^\times$, we have

$$\widetilde{\mathscr{T}}_b^u := \left[\tilde{q}(u)\widetilde{\mathscr{T}}^\circ, b\right]_{H\widehat{F}^\times} = \left[\alpha\cdot q(u)\cdot\mathscr{T}^\circ, b\right]_{H\widehat{F}^\times} = \left[q(u)\cdot\mathscr{T}^\circ, \alpha^{-1}\cdot b\right]_{H\widehat{F}^\times} = \mathscr{T}^u_{\alpha^{-1}b}.$$

The map $b \mapsto \alpha^{-1}b$ is a bijection. Thus,

$$\left\{\mathscr{T}_b^u,\ b \in \widehat{B}^\times,\ u \in \pi_0(T(\mathbf{R}))\right\} = \left\{\widetilde{\mathscr{T}}_b^u,\ b \in \widehat{B}^\times,\ u \in \pi_0(T(\mathbf{R}))\right\}. \qquad \blacksquare$$

Action of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$

Let us denote by K^{ab} the maximal abelian extension of K and by $\mathrm{rec}_K\colon K_{\mathbf{A}}^\times/K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ the reciprocity map normalized by letting uniformizers correspond to geometric Frobenius elements.

The group $K_{\mathbf{A}}^\times$ acts on $\{\mathscr{T}_b^u \mid b \in \widehat{B}^\times,\ u \in \pi_0(T(\mathbf{R}))\}$ by

$$\forall a = (a_\infty, a_f) \in K_{\mathbf{A}}^\times = K_\infty^\times \times \widehat{K}^\times\ \forall b \in \widehat{B}^\times \qquad a\cdot\mathscr{T}_b^u = \mathscr{T}^{q(a_\infty)u}_{\widehat{q}(a_f)b}.$$

The action of $k \in K^\times$ is trivial; as $q(k) \in B^\times$, the definition of $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$ gives

$$k\cdot\mathscr{T}_b^u = [q(k)q(u)\mathscr{T}^\circ,\quad \widehat{q}(k)b]_{H\widehat{F}^\times} = [q(u)\mathscr{T}^\circ, b]_{H\widehat{F}^\times} = \mathscr{T}_b^u.$$

The action of $F_{\mathbf{A}}^\times$ is trivial. For $a = (a_\infty, a_f) \in F_{\mathbf{A}}^\times$ and $b \in \widehat{B}^\times$, $\widehat{q}(a_f)b = b\widehat{q}(a_f)$ and $q(a_\infty)q(u)\mathscr{T}^\circ = q(u)\mathscr{T}^\circ$, hence

$$a\cdot\mathscr{T}_b^u = \left[q(a_\infty)q(u)\mathscr{T}^\circ, \widehat{q}(a_f)b\right]_{H\widehat{F}^\times} = \left[q(u)\mathscr{T}^\circ, b\right]_{H\widehat{F}^\times} = \mathscr{T}_b^u.$$

4.3 Special Cycles on $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$

In this section we construct some r-chain on $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$.

Proposition 4.5 *The homology class* $[\mathscr{T}_b^\circ] \in H_{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z})$ *of* $\mathscr{T}_b^\circ$ *is torsion.*

Proof Let us denote by pr the map

$$\mathrm{pr}\colon X \times \{b\} \longrightarrow \mathrm{Sh}_H(G/Z,X)(\mathbf{C}).$$

$\mathscr{T}_b^\circ$ is in the image of pr and

$$\mathrm{pr}^{-1}(\mathscr{T}_b^\circ) = (\{z_1\}\times\gamma_2\times\cdots\times\gamma_r)\times\{b\}.$$

Let $\omega \in H^{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C})$. As $r-1 \neq r$ we know thanks to the Matsushima–Shimura theorem that

$$\omega \in \Big(\mathrm{Vect} \bigwedge_{\substack{i\in a\subset\{1,\dots,r-1\}\\ |a|=m/2}} \frac{\mathrm{d}z_i \wedge \mathrm{d}\overline{z_i}}{y_i^2}\Big)^{s'}.$$

- If $r-1$ is odd, then $H^{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C}) = \{0\}$.
- If $r-1 = 2s$ is even, ω is the pull-back of $\bigwedge_{j=2}^{r} \omega^{(j)}$, where

$$\omega^{(j)} = 1 \qquad \text{or} \qquad \frac{\mathrm{d}x_j \wedge \mathrm{d}y_j}{y_j^2}.$$

With the notations of the proof of Proposition 4.3, $\mathscr{T}_b^{\circ}$ is a principal homogeneous space under $\mathscr{W}$. Fix a fundamental domain $\widetilde{\mathscr{W}}$ of $\mathscr{W}$ in $\gamma_2 \times \cdots \times \gamma_r$. The incompatibility of degrees gives

$$\int_{\mathscr{T}_b^{\circ}} \omega = \int_{\widetilde{\mathscr{W}}} \omega^{(2)} \wedge \cdots \wedge \omega^{(r)} = 0,$$

$$\forall \omega \in H^{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C}) \qquad \int_{\mathscr{T}_b^{\circ}} \omega = 0.$$

This proves that

$$[\mathscr{T}_b^{\circ}] = 0 \in H_r\big(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C}\big)$$

and that

$$[\mathscr{T}_b^{\circ}] \in H_r\big(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z}\big)$$

is torsion. ∎

Definition 4.6 Let $n \in \mathbf{Z}_{>0}$ be the exponent of $H_{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z})_{\mathrm{tors}}$. We will denote by Δ_b° any piece-wise differentiable r-chain verifying that $n[\mathscr{T}_b^{\circ}] = \partial\Delta_b^{\circ}$.

Proposition 3.7 proves that the value of

$$\Big(\frac{1}{\Omega^{\beta}}\xi\alpha \int_{\Delta_b^{\circ}} \omega_{\varphi}^{\beta}\Big) \in \mathbf{C}$$

modulo Λ_1 does not depend on the particular choice of Δ_b°. If $T(\mathbf{R})^{\circ}$ is fixed, then we have the following proposition.

Proposition 4.7 *Let $\mathscr{T}^{\circ}$ and $\mathscr{T}'^{\circ}$ be two special cycles such that $\mathrm{pr}_1(\mathscr{T}^{\circ}) = \mathrm{pr}_1(\mathscr{T}'^{\circ}) = \{z_1\}$. Assume that $\mathrm{pr}_j(\mathscr{T}^{\circ})$ and $\mathrm{pr}_j(\mathscr{T}'^{\circ})$ lie in the same connected component of X_j for each $j \in \{2,\dots,r\}$. Let n be the exponent of $H_{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z})_{\mathrm{tors}}$ and let Δ_b° and $\Delta_b'^{\circ}$ satisfy*

$$n[\mathscr{T}_b^{\circ}] = \partial\Delta_b^{\circ} \qquad \text{and} \qquad n[\mathscr{T}_b'^{\circ}] = \partial\Delta_b'^{\circ}.$$

Then we have

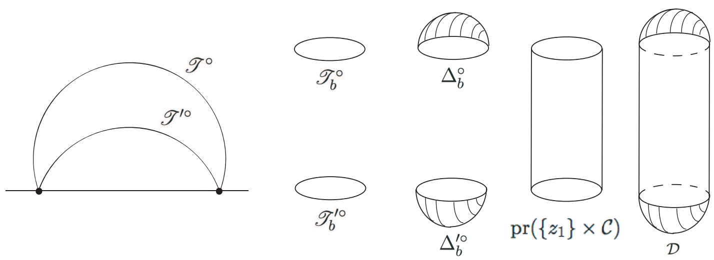
$$\int_{\Delta_b^{\circ}} \omega_{\varphi}^{\beta} = \int_{\Delta_b'^{\circ}} \omega_{\varphi}^{\beta} \pmod{\xi^{-1}\alpha^{-1}\Omega^{\beta}\Lambda_1}.$$

Proof Our hypothesis allows us to decompose $\Delta_b'^\circ - \Delta_b^\circ$ into

$$\Delta_b'^\circ - \Delta_b^\circ = \mathrm{pr}(\{z_1\} \times \mathcal{C}) + \mathcal{D},$$

where $\mathcal{D}$ is a cycle with $\partial\mathcal{D} = 0$ and pr is the map

$$\mathrm{pr}\colon \begin{cases} X & \longrightarrow \quad \mathrm{Sh}_H(G/Z, X)(\mathbf{C}) \\ x & \longmapsto \quad [x, b]_{H\widehat{F}^\times} \end{cases}$$



Let us show that $\int_{\Delta_b'^\circ - \Delta_b^\circ} \omega_\varphi^\beta \in \xi^{-1}\alpha^{-1}\Omega^\beta\Lambda_1$.

We have

$$\omega_\varphi^\beta = \sum_\varepsilon \omega_\varepsilon \in \bigoplus_{\varepsilon\colon \{\tau_1,\ldots,\tau_r\}\to\{\pm 1\}^r} \Gamma\big(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), (\Omega_H^{\mathrm{an}})^\varepsilon\big),$$

Each $\omega_\varepsilon \in \Gamma(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), (\Omega_H^{\mathrm{an}})^\varepsilon)$ satisfies $\mathrm{pr}^*(\omega_\varepsilon) = \mathrm{d}z_1 \wedge \omega_\varepsilon'$. We have

$$\int_{\mathrm{pr}(\{z_1\}\times\mathcal{C})} \omega_\varepsilon = \int_{\{z_1\}\times\mathcal{C}} \mathrm{d}z_1 \wedge \omega_\varepsilon' = 0,$$

thus $\int_{\{z_1\}\times\mathcal{C}} \omega_\varphi^\beta = 0$.

Thanks to Proposition 3.7 we have

$$\int_{\mathcal{D}} \omega_\varphi^\beta \in \xi^{-1}\alpha^{-1}\Omega^\beta\Lambda_1,$$

and the result follows. ■

Corollary 4.8 *The value modulo Λ_1 of*

$$\Big(\frac{1}{\Omega^\beta}\xi\alpha \int_{\Delta_b^\circ} \omega_\varphi^\beta\Big) \in \mathbf{C}$$

depends neither on the choice of $\mathcal{T}^\circ$ whose projection on X_1 is $\{z_1\}$ nor on Δ_b° satisfying $n[\mathcal{T}_b^\circ] = \partial\Delta_b^\circ$.

Remark 4.9 The value of $(1/\Omega^{\beta}\xi\alpha\int_{\Delta_b^{\circ}}\omega_{\varphi}^{\beta}) \in \mathbf{C}$ depends on the choice of the embedding q. We make no further mention of this dependence, nor of the dependence on z_1, as those objects are fixed in the whole paper.

Definition 4.10 We set

$$J_b^{\beta} = \frac{1}{\Omega^{\beta}}\xi\alpha\int_{\Delta_b^{\circ}}\omega_{\varphi}^{\beta} \pmod{\Lambda_1} \in \mathbf{C}/\Lambda_1,$$

the image of $\mathscr{T}_b^{\circ}$ by an exotic Abel–Jacobi map.

Properties of J_b^{β}

For each $u \in \pi_0(T(\mathbf{R}))$ let Δ_b^u be some piece-wise differentiable chain satisfying

$$n\big[[q(u)\cdot\mathscr{T}^{\circ}, b]_{H\widehat{F}^{\times}}\big] = \partial\Delta_b^u.$$

Proposition 4.11 *We have*

$$J_b^{\beta} = \frac{1}{\Omega^{\beta}}\xi\alpha\sum_{u\in\pi_0(T(\mathbf{R}))}\beta(u)\int_{\Delta_b^u}\omega_{\varphi} \pmod{\Lambda_1}.$$

Proof Let us identify $\pi_0(T(\mathbf{R}))$ with $\prod_{j=2}^{r}\{\pm 1\}$ and assume that the image of $T(\mathbf{R})^{\circ}$ is $(1,\dots,1)$. Then

$$\omega_{\varphi}^{\beta} = \sum_{u\in\pi_0(T(\mathbf{R}))}\beta(u)t_u^{*}(\omega_{\varphi}).$$

The chains $t_u\Delta_b^{\circ}$ and Δ_b^u are in the same connected component. Thus, using Proposition 4.7, we have

$$\int_{t_u\Delta_b^{\circ}}\omega_{\varphi} = \int_{\Delta_b^u}\omega_{\varphi}$$

and the result follows. ∎

Recall that $z_1 \in X_1$ is fixed by $q(K_{\tau_1}^{\times})$.

Proposition 4.12 *Let $\mathscr{T}^{\circ}$ and $\mathscr{T}'^{\circ}$ be two $q(T(\mathbf{R})^{\circ})$-orbits such that* $\mathrm{pr}_1(\mathscr{T}^{\circ}) = \mathrm{pr}_1(\mathscr{T}'^{\circ}) = \{z_1\}$. *There exists a unique $u \in \pi_0(T(\mathbf{R}))$ such that, for all $j \in \{2,\dots,r\}$,*

$$\mathrm{pr}_j(\mathscr{T}'^{\circ}) \quad \text{and} \quad \mathrm{pr}_j(q(u)\cdot\mathscr{T}^{\circ})$$

are in the same connected component of X_j.

If $J_b'^{\beta} \in \mathbf{C}/\Lambda_1$ denotes the value obtained from $\mathscr{T}'^{\circ}$, we have $J_b'^{\beta} = \beta(u)J_b^{\beta}$.

Proof Let $x, x' \in X$ be such that $\mathscr{T}^{\circ} = q(T(\mathbf{R})^{\circ})\cdot x$ (resp. $\mathscr{T}'^{\circ} = q(T(\mathbf{R})^{\circ})\cdot x'$). There exists $u \in \pi_0(T(\mathbf{R}))$ such that for all $j \in \{1,\dots,r\}$, $\mathrm{pr}_j(q(u)\cdot x)$ and $\mathrm{pr}_j(x')$

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are in the same connected component of X_j. As $\mathscr{T}'^{\circ} = q(u)\cdot\mathscr{T}^{\circ}$, the chain $\Delta_b'^{\circ}$, whose boundary up to torsion is $[\mathscr{T}'^{\circ}, b]_{H\hat{F}^\times}$, equals Δ_b^u. Thus,

$$\sum_{u'\in\pi_0(T(\mathbf{R}))}\beta(u')\int_{\Delta_b'^{u'}}\omega_\varphi = \sum_{u'\in\pi_0(T(\mathbf{R}))}\beta(u')\int_{\Delta_b^{uu'}}\omega_\varphi$$
$$= \beta(u)\sum_{u''\in\pi_0(T(\mathbf{R}))}\beta(u'')\int_{\Delta_b^{u''}}\omega_\varphi. \qquad\blacksquare$$

Let $q, q'\colon K\hookrightarrow B$ be two embeddings of F-algebras and $x\in X$, $\mathscr{T}^{\circ} = q(T(\mathbf{R})^{\circ})\cdot x$ (resp. $\mathscr{T}'^{\circ} = q'(T(\mathbf{R})^{\circ})\cdot x'$). There exists $a\in B^\times$ such that $q' = aqa^{-1}$ thanks to the Skolem–Noether theorem. For each $j\in\{1,\dots,r\}$, $\mathrm{pr}_j(\mathscr{T}^{\circ})$ and $\mathrm{pr}_j(\mathscr{T}'^{\circ})$ are in the same connected component of X_j if and only if $\tau_j(\mathrm{nr}(a)) > 0$.

Using Proposition 4.12 we obtain the following.

Proposition 4.13 *If* $\alpha = (\mathrm{sgn}\circ\tau_j(\mathrm{nr}(a)))_{j\in\{1,\dots,r\}}\in\{\pm1\}^{r-1}$, *then* $J_b'^{\beta} = \beta(\alpha)J_b^{\beta}$.

Let $\mathrm{N}_{B^\times}(K^\times)$ be the normalizer of $K^\times$ in $B^\times$. Let $a\in\mathrm{N}_{B^\times}(K^\times)\smallsetminus K^\times$. After multiplying a by an element in $K^\times$ we may assume for all $j\in\{2,\dots,r\}$, $\tau_j(\mathrm{nr}(a)) > 0$.

We have $\mathrm{pr}_1(q(a)\cdot\mathscr{T}^{\circ}) = t_1(z_1)$ and for all $j\in\{2,\dots,r\}$, $\mathrm{pr}_j(q(a)\cdot\mathscr{T}^{\circ}) = \mathrm{pr}_j(\mathscr{T}^{\circ})$, but the orientations of $\mathrm{pr}_j(q(a)\cdot\mathscr{T}^{\circ})$ and $\mathrm{pr}_j(\mathscr{T}^{\circ})$ are not the same.

Thus,

$$[t_1\mathscr{T}^{\circ}, b]_{H\widehat{F}^\times} = [q(a)\mathscr{T}^{\circ}, b]_{H\widehat{F}^\times} = [\mathscr{T}^{\circ}, \widehat{q}(a)^{-1}b]_{H\widehat{F}^\times},$$

but the orientations differ by $(-1)^{r-1}$. Hence we have the following proposition.

Proposition 4.14 *The tori* $\mathscr{T}_b^{\circ}$ *and* $t_1\mathscr{T}^{\circ}_{\widehat{q}(a)b}$ *are the same up to orientation.*

5 Generalized Darmon's Points

5.1 The Main Conjecture

Let $\Phi_1\colon \mathbf{C}/\Lambda_1 \xrightarrow{\sim} E_1(\mathbf{C})$ be the Weierstrass uniformization; *i.e.,* the inverse of Φ_1 is the Abel–Jacobi map for the differential η_1. For each $a_\infty\in K_\infty^\times$, fix some r-chain $q(a_\infty)\cdot\Delta_b^{\beta}$ satisfying $n[q(a_\infty)\cdot\mathscr{T}_b^{\beta}] = \partial q(a_\infty)\cdot\Delta_b^{\beta}$ and denote by $\beta(a_\infty)$ the sign

$$\beta(a_\infty) = \prod_{j=2}^{r}\beta\Big(\mathrm{sgn}\Big(\prod_{w|\tau_j}a_{\infty,w}\Big)\Big).$$

Conjecture 5.1 *The point*

$$P_b^{\beta} = \Phi_1\Big(\frac{1}{\Omega^{\beta}}\xi\alpha\int_{\Delta_b^{\beta}}\omega_\varphi\Big) = \Phi_1(J_b^{\beta})\in E_1(\mathbf{C})$$

lies in $E(K^{\mathrm{ab}})$ *and for all* $a = (a_\infty, a_f)\in K_{\mathbf{A}}^\times$,

$$\mathrm{rec}_K(a)P_b^{\beta} = \Phi_1\Big(\frac{\xi\alpha}{\Omega^{\beta}}\int_{q(a_\infty)\cdot\Delta^{\beta}_{\widehat{q}(a_f)b}}\omega_\varphi\Big) = \beta(a_\infty)P^{\beta}_{\widehat{q}(a_f)b}.$$

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Remark 5.2 The choice of $z_1 \in X_1^{q_1(K_{\tau_1}^\times)}$ fixes a morphism $h_1\colon \mathbf{S} \to G_{1,\mathbf{R}}$, hence a morphism $\mathbf{C}^\times = \mathbf{S}(\mathbf{R}) \to G_{1,\mathbf{R}}(\mathbf{R}) = B_{\tau_1}^\times = (B \otimes_{F,\tau_1} \mathbf{R})^\times$ satisfying $h_1(\mathbf{C}^\times) = q_1(K_{\tau_1}^\times)$. This fixes an embedding $\tau_{1,K}\colon K \hookrightarrow \mathbf{C}$ such that the diagram

$$\begin{tikzcd}
\mathbf{C}^\times \arrow[r, "h_1"] & (B \otimes_{F,\tau_1} \mathbf{R})^\times \\
 & (K \otimes_{F,\tau_1} \mathbf{R})^\times \arrow[u, "q_1"'] \arrow[ul, "\tau_{1,K}"]
\end{tikzcd}$$

commutes. We may fix $\tilde{\tau}_1\colon K^{\mathrm{ab}} \hookrightarrow \mathbf{C}$ above $\tau_{1,K}$ such that

$$\begin{tikzcd}
F \arrow[r, "\tau_1"] \arrow[d] & \mathbf{R} \arrow[r] & \mathbf{C} \\
K \arrow[rr] \arrow[urr, "\tau_{1,K}"] & & K^{\mathrm{ab}} \arrow[u, dotted, "\tilde{\tau}_1"']
\end{tikzcd}$$

commutes. Moreover the isomorphism

$$\begin{cases} \mathrm{Gal}(K^{\mathrm{ab}}/K) & \xrightarrow{\sim} \quad \mathrm{Gal}(\tilde{\tau}_1(K^{\mathrm{ab}})/\tau_{1,K}(K)) \\ \sigma & \longmapsto \quad \tilde{\tau}_1 \circ \sigma \circ \tilde{\tau}_1^{-1} \end{cases}$$

does not depend on the choice of $\tilde{\tau}_1$. If $\tilde{\tau}_1'$ is another embedding above $\tau_{1,K}$, then $\tilde{\tau}_1' = \tilde{\tau}_1 \circ \sigma'$ with $\sigma' \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$ and for all $\sigma \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$,

$$\tilde{\tau}_1' \circ \sigma \circ \tilde{\tau}_1'^{\,-1} = \tilde{\tau}_1 \circ \sigma' \sigma \sigma'^{\,-1} \circ \tilde{\tau}_1^{-1} = \tilde{\tau}_1 \circ \sigma \circ \tilde{\tau}_1^{-1},$$

because $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is commutative. Hence the Galois action of Conjecture 5.1 does not depend on the particular choice of $\tilde{\tau}_1$.

Remark 5.3 Using Conjecture 5.1, we obtain

$$\forall a_\infty \in K_\infty^\times, \qquad \mathrm{rec}_K(a_\infty) P_b^\beta = \beta(a_\infty) P_b^\beta.$$
$$\forall a \in F_{\mathbf{A}}^\times, \qquad \mathrm{rec}_K(a) P_b^\beta = P_b^\beta.$$

5.2 Field of Definition

Let $B_+^\times = \{b \in B^\times \mid \forall j \in \{2, \dots, r\},\ \tau_j(\mathrm{nr}(b)) > 0\}$. It is diagonally embedded in $(B \otimes \mathbf{R})^\times$. Set

$$K_b^+ = (K^{\mathrm{ab}})^{\mathrm{rec}_K(q_{\mathbf{A}}^{-1}(bH\hat{F}^\times b^{-1} B_+^\times))} \quad \text{and} \quad K_b := (K^{\mathrm{ab}})^{\mathrm{rec}_K(q_{\mathbf{A}}^{-1}(bH\hat{F}^\times b^{-1} B^\times))} \subset K_b^+.$$

Note that K_b and K_b^+ depend on the choice of the F-embedding $q\colon K \hookrightarrow B$.

Proposition 5.4 *Assuming Conjecture* 5.1*, the point* P_b^β *is defined over* K_b^+*:* $P_b^\beta \in E(K_b^+)$.

Proof Let $a = (1_\infty, bhfb^{-1})(a_\infty, 1_f) \in q_{\mathbf{A}}^{-1}(bH\hat{F}^\times b^{-1}B_+^\times)$ with $f \in \hat{F}^\times$ and $h \in H$. We have

$$\mathrm{rec}(a)P_b^\beta = \mathrm{rec}\big(q_{\mathbf{A}}^{-1}((1_\infty, bhfb^{-1})\big)\,P_b^\beta = P_{bhfb^{-1}b}^\beta = P_{bhf}^\beta = P_b^\beta. \qquad \blacksquare$$

Remark that rec_K induces a surjection

$$\mathcal{R}\colon \pi_0(T(\mathbf{R})) = \frac{(K \otimes_{\mathbf{Q}} \mathbf{R})^\times}{(F \otimes_{\mathbf{Q}} \mathbf{R})^\times (K \otimes_{\mathbf{Q}} \mathbf{R})_+^\times} \simeq \prod_{j=2}^{r}\{\pm 1\} \twoheadrightarrow \mathrm{Gal}(K_b^+/K_b).$$

Thus, we have the following proposition.

Proposition 5.5 *Assuming Conjecture* 5.1*, the points* P_b^β *lie in* $K_b^\beta = (K_b^+)^{\mathcal{R}(\mathrm{Ker}\,\beta)}$.

Remark 5.6 As $\mathrm{Ker}\beta$ has index 2 in $\prod_{j=2}^{r}\{\pm 1\}$, the field K_b^β has degree 1 or 2 over K_b.

Assume that the conductor N of E decomposes as $N = N_+N_-$ with $N_- = \mathfrak{p}_1 \dots \mathfrak{p}_t$, $\mathfrak{p}_i$ distinct prime ideals of $\mathcal{O}_F$ and $t \equiv d - r \bmod 2$. If

$$\mathrm{Ram}(B) = \{\tau_{r+1}, \dots, \tau_d\} \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\} \quad \text{and} \quad H = (R \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})^\times,$$

where $R \subset B$ is an Eichler order of level N_+, then K_b is a ring class field of conductor $\mathfrak{f}_b$ and K_b^+ a ring class field of conductor $\mathfrak{f}_b\mathfrak{f}_\infty$, where $\mathfrak{f}_\infty = \prod_{j=2}^{r}\tau_j$.

5.3 Local Invariants of B

Let π be the irreducible automorphic representation of $B_{\mathbf{A}}^\times$ generated by φ and

$$\eta_K = \eta_{K/F}\colon F_{\mathbf{A}}^\times/F^\times \mathrm{N}_{K/F}(K_{\mathbf{A}}^\times) \longrightarrow \{\pm 1\}$$

the quadratic character of K/F. For each place v of F let $\mathrm{inv}_v(B_v) \in \{\pm 1\}$ be the invariant of B: $\mathrm{inv}_v(B_v) = 1$ if and only if $B_v \simeq M_2(F_v)$.

Fix $b \in \widehat{B}^\times$ and a character $\chi\colon\ \mathrm{Gal}(K_b^+/K) \to \mathbf{C}^\times$, which will be identified with

$$K_{\mathbf{A}}^\times \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(K^{\mathrm{ab}}/K) \longrightarrow \mathrm{Gal}(K_b^+/K) \xrightarrow{\chi} \mathbf{C}^\times.$$

Let $L(\pi \times \chi, s)$ be the Rankin-Selberg L function, see [13, p. 132] and [14, Section 12]. This function admits, since π has trivial central character, a holomorphic extension to $\mathbf{C}$ satisfying

$$L(\pi \times \chi, s) = \varepsilon(\pi \times \chi, s)L(\pi \times \chi, 1 - s).$$

In this section, we prove the following proposition.

Proposition 5.7 *Let $b \in \widehat{B}^\times$ and assume Conjecture* 5.1. *If*

$$e_{\overline{\chi}}(P_b^\beta) = \sum_{\sigma \in \mathrm{Gal}(K_b^+/K)} \chi(\sigma) \otimes P_b^\beta \in E(K_b^+) \otimes \mathbf{Z}[\chi]$$

is not torsion, then $\beta = \chi_\infty$, for all $v \neq \tau_1$,

$$\eta_{K,v}(-1)\varepsilon\Big(\pi_v \times \chi_v, \frac{1}{2}\Big) = \mathrm{inv}_v(B_v) \quad \text{and} \quad \varepsilon\Big(\pi \times \chi, \frac{1}{2}\Big) = -1.$$

We shall use the following theorem ([27, 28]).

Theorem 5.8 *The equality $\eta_{K,v}(-1)\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \mathrm{inv}_v(B_v)$ holds if and only if there exists a non-zero invariant linear form $\ell_v \colon \pi_v \times \chi_v \to \mathbf{C}$ unique up to a scalar satisfying for all $a \in K_v^\times$ and for all $u \in \pi_v$,*

$$\ell_v(q_v(a)u) = \chi_v(a)^{-1}\ell_v(u)$$

i.e., *ℓ_v is $q(K_v^\times)$-invariant.*

Proof of Proposition 5.7 We follow the proof of [1, Proposition 2.6.2].

Let S' be a finite set of finite places of F containing the places where B, π, or K_b^+/F ramify, and such that the map $r = (r_v \colon K_v^\times \longrightarrow \mathrm{Gal}(K_b^+/K))_{v \in S'}$ obtained by composition

$$r \colon \prod_{v \in S'} K_v^\times \longrightarrow K_{\mathbf{A}}^\times \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(K^{\mathrm{ab}}/K) \longrightarrow \mathrm{Gal}(K_b^+/K)$$

is surjective.

For each $v \in S'$ let

$$j_v \colon \begin{cases} K_v & \hookrightarrow \quad B_v \\ k & \longmapsto \quad b_v^{-1} q_v(k) b_v, \end{cases}$$

and

$$j = (j_v)_{v \in S'} \colon \prod_{v \in S'} K_v \hookrightarrow \prod_{v \in S'} B_v.$$

As S' does not contain any archimedean place of F, for all $a \in \prod_{v \in S'} K_v^\times$,

$$\big[\mathcal{T}^\circ, \widehat{q}(a)b\big]_{H\hat{F}^\times} = \big[\mathcal{T}^\circ, bj(a)\big]_{H\hat{F}^\times}$$

and for all $a \in \prod_{v \in S'} K_v^\times$ and for all $b \in \widehat{B}^\times$, $\mathrm{rec}_K(a)P_b^\beta = P_{\widehat{q}(a)b}^\beta = P_{bj(a)}^\beta$.

Let $(K_v^\times)^\circ \subset K_v^\times$ be the inverse image of $(K_v^\times/\mathcal{O}_{K,v}^\times)^{\mathrm{Gal}(K/F)} \subset K_v^\times/\mathcal{O}_{K,v}^\times$.

We have

$$K_v^\times/\mathcal{O}_{K,v}^\times F_v^\times \xrightarrow{\sim} \begin{cases} 0 & \text{if } v \text{ is inert in } K/F, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } v \text{ ramifies in } K/F, \\ \mathbf{Z} & \text{if } v \text{ splits in } K/F, \end{cases}$$

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the quotient $(K_v^\times)^\circ/F_v^\times$ is compact and

$$D_v := K_v^\times/(K_v^\times)^\circ \xrightarrow{\sim} \begin{cases} \mathbf{Z} & \text{if } v \text{ splits in } K/F, \\ 0 & \text{otherwise,} \end{cases}$$

$$(K_v^\times)^\circ/\mathcal{O}_{K,v}^\times F_v^\times \xrightarrow{\sim} \begin{cases} \mathbf{Z}/2\mathbf{Z} & \text{if } v \text{ ramifies in } K/F, \\ 0 & \text{otherwise.} \end{cases}$$

For each $v \in S'$, $C_v = \mathcal{O}_{K,v}^\times \cap \operatorname{Ker}(r_v)$ is an open subgroup of $\mathcal{O}_{K,v}^\times$ and $V_v^\circ = (K_v^\times)^\circ/F_v^\times C_v$ is finite.

Let V_v be the following subset of $K_v^\times/F_v^\times C_v$:

- if v does not split in K/F, $V_v^\circ = K_v^\times/F_v^\times C_v$ and $V_v := V_v^\circ$;
- If v splits in K/F, we fix some section of $K_v^\times \twoheadrightarrow K_v^\times/(K_v^\times)^\circ \xrightarrow{\sim} \mathbf{Z}$.

Hence $K_v^\times = (K_v^\times)^\circ \times D_v$ and there exists $n_v \geq 1$ such that $\operatorname{Ker}(r_v|_{D_v}) = n_v D_v$.

Fix a set of representatives $D_v' \subset D_v$ of $D_v/n_v D_v$ and set $V_v = V_v^\circ D_v' \subset K_v^\times/F_v^\times C_v$.

Let $V = \prod_{v\in S'} V_v \subset \prod_{v\in S'} K_v^\times/F_v^\times C_v$, which is stable under multiplication by the abelian group $V^\circ = \prod_{v\in S'} V_v^\circ$ and such that $V \hookrightarrow \prod_{v\in S'} K_v^\times/F_v^\times C_v \xrightarrow{r} \operatorname{Gal}(K_b^+/K)$ is surjective with fibers of cardinality $\frac{|V|}{|\operatorname{Gal}(K_b^+/K)|}$. We have

$$\begin{aligned} \frac{|V|}{|\operatorname{Gal}(K_b^+/K)|} e_{\overline{\chi}}(P_b^\beta) &= \frac{|V|}{|\operatorname{Gal}(K_b^+/K)|} \sum_{\sigma\in\operatorname{Gal}(K_b^+/K)} \chi(\sigma) \otimes \sigma \cdot P_b^\beta \\ &= \sum_{a\in V} \chi(a) \otimes P_{bj(a)}^\beta. \end{aligned}$$

Fix some open-compact subgroup $H_1 \subset \bigcap_{a\in V} j(a) H j(a)^{-1}$. Using the maps

$$\operatorname{Sh}_{H_1}(G/Z, X) \xrightarrow{[\cdot j(a)]} \operatorname{Sh}_{j(a)^{-1}H_1 j(a)}(G/Z, X) \xrightarrow{\mathrm{pr}} \operatorname{Sh}_H(G/Z, X),$$

we have

$$\begin{aligned} \sum_{a\in V} \chi(a) \int_{\Delta_{bj(a)}^\circ} \omega_\varphi^\beta &= \sum_{a\in V} \chi(a) \int_{\Delta_b^\circ} [\cdot j(a)]^* \omega_\varphi^\beta \\ &= \int_{\Delta_b^\circ} \sum_{a\in V} \chi(a) [\cdot j(a)]^* \omega_\varphi^\beta = \int_{\Delta_b^\circ} \omega_1^\beta, \end{aligned}$$

where

$$\omega_1^\beta := \sum_{a\in V} \chi(a) [\cdot j(a)]^* \omega_\varphi^\beta.$$

Whenever

$$\frac{|V|}{|\operatorname{Gal}(K_b^+/K)|} e_{\overline{\chi}}(P_b^\beta) = \sum_{a\in V} \chi(a) \otimes P_{bj(a)}^\beta \in \mathbf{Z}[\chi] \otimes_{\mathbf{Z}} E(K_b^+) \subset \mathbf{Z}[\chi] \otimes_{\mathbf{Z}} \mathbf{C}/\Lambda_1$$

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is not torsion, there exists $\sigma\colon \mathbf{Z}[\chi] \hookrightarrow \mathbf{C}$ such that

$$\frac{\xi\alpha}{\Omega^\beta}\int_{\Delta_b^\circ}\sum_{a\in V}{}^{\sigma}\chi(a)[\cdot j(a)]^*\omega_\varphi^\beta \notin \mathbf{Q}[{}^{\sigma}\chi]\cdot\Lambda_1,$$

where ${}^\sigma\chi = \sigma\circ\chi$. The vector

$${}^\sigma\omega_1 = \sum_{a\in V}{}^\sigma\chi(a)[\cdot j(a)]^*\omega_\varphi \in \pi^{H_1}\cap\Gamma\big(\mathrm{Sh}_{H_1}(G/Z,X),\Omega_{H_1}\big)$$

is non-zero and invariant under $j(\prod_{v\in S'}(K_v^\times)^\circ)$. Moreover, for all $a\in\prod_{v\in S'}(K_v^\times)^\circ$, $j(a)\omega_1 = {}^\sigma\chi^{-1}(a)\omega_1$.

Let

$${}^\sigma\ell_{S'}\colon \bigotimes_{v\in S'}{}^\sigma\pi_v = \bigotimes_{v\in S'}\pi_v \longrightarrow \mathbf{C}({}^\sigma\chi^{-1})$$

be the $j(\prod_{v\in S'}(K_v^\times)^\circ)$-invariant projection on $\mathbf{C}\omega_1$.

Assume that $v\in S'$ does not split in K. In this case $(K_v^\times)^\circ = K_v^\times$ and ${}^\sigma\ell_{S'}$ induces a $q_v(K_v^\times)$-invariant linear form ${}^\sigma\ell_v\colon \pi_v\to\mathbf{C}({}^\sigma\chi_v^{-1})$. We have ${}^\sigma\ell_v(\omega_{1,v})\neq 0$, where

$$\omega_{1,v} = \sum_{a_v\in V_v}{}^\sigma\chi\circ r_v(a_v)[\cdot j_v(a_v)]^*\omega_\varphi.$$

As $\varepsilon_v(\pi_v\times{}^\sigma\chi_v,\frac12)$ is independent of $\sigma\colon\mathbf{Z}[\chi]\hookrightarrow\mathbf{C}$, Theorem 5.8 shows that

$$\eta_{K,v}(-1)\varepsilon\Big(\pi_v\times\chi_v,\frac12\Big) = \mathrm{inv}_v(B_v).$$

When $v\in S'$ splits in K or $v\notin S'\cup S_\infty$, the equality

$$\eta_{K,v}(-1)\varepsilon\Big(\pi_v\times\chi_v,\frac12\Big) = 1 = \mathrm{inv}_v(B_v)$$

follows from calculations that can be found, for example, in [23, Prop. 12.6.2.4].

Global sign If $v=\tau_j$ is an archimedean place, then $\varepsilon(\pi_v\times\chi_v,\frac12)=1$. Moreover $\eta_{K,v}(-1)=1$ if and only if $j\in\{2,\dots,r\}$ and $\mathrm{inv}_v(B_v)=1$ if and only if $j\in\{1,\dots,r\}$. Thus,

$$\eta_{K,v}(-1)\,\mathrm{inv}_v(B_v) = \begin{cases}-1\times 1 & \text{if } j=1,\\ 1\times 1 & \text{if } j\in\{2,\dots,r\},\\ -1\times -1 & \end{cases}$$

and for all $j\in\{1,\dots,d\}$,

$$\varepsilon_v\Big(\pi_v\times\chi_v,\frac12\Big) = \eta_{K,v}(-1)\,\mathrm{inv}_v(B_v)\times\begin{cases}-1 & \text{if } j=1,\\ 1 & \text{if } j>1.\end{cases}$$

Hence,

$$\varepsilon\Big(\pi\times\chi,\frac12\Big) = -\prod_v\eta_{K,v}(-1)\,\mathrm{inv}_v(B_v) = -1. \qquad\blacksquare$$

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5.4 Global Invariant Linear Form and a Conjectural Gross–Zagier Formula

For any open subgroup $H' \subset H$, $b \in \widehat{B}^\times$ and $u \in \pi_0(T(\mathbf{R}))$ fix

$$\Delta^u_{H',b} \in C^r\big(\mathrm{Sh}_{H'}(G/Z, X)(\mathbf{C}), \mathbf{Q}\big)$$

such that $\partial \Delta^u_{H',b} = [\mathscr{T}^u_{H',b}]$, where $\mathscr{T}^u_{H',b} = \{[q(u)x, b]_{H'\widehat{F}^\times},\ x \in \mathscr{T}^\circ\}$.

Recall that for all $u' \in \pi_0(T(\mathbf{R}))$, $t_{u'}\Delta^u_{H,b} = \Delta^{uu'}_b$.

Let π_∞ be the archimedean part of π. Fix $\varphi_\infty \in \pi_\infty$ a lowest weight vector of weight $(\underbrace{2, \dots, 2}_{r}, 0, \dots, 0)$ of π_∞ and ω_φ such that

$$\omega_\varphi = \varphi_\infty \otimes \varphi_f \in \pi_\infty \otimes \pi_f \subset S_2(B^\times_{\mathbf{A}}).$$

Let us denote by ${}_{\mathbf{Q}}\pi_f$ the sub $\mathbf{Q}[\widehat{B}^\times]$-module of π_f generated by φ_f.

Proposition 5.9 *The space ${}_{\mathbf{Q}}\pi_f$ is a $\mathbf{Q}$-vector space and ${}_{\mathbf{Q}}\pi_f \otimes_{\mathbf{Q}} \mathbf{C} \to \pi_f$ is surjective.*

Proof The space $\mathrm{Im}({}_{\mathbf{Q}}\pi_f \otimes_{\mathbf{Q}} \mathbf{C} \to \pi_f)$ is a zero subvector space of π_f invariant under $B^\times_{\mathbf{A}}$. As π_f is irreducible, we have $\mathrm{Im}({}_{\mathbf{Q}}\pi_f \otimes_{\mathbf{Q}} \mathbf{C} \to \pi_f) = \pi_f$, and ${}_{\mathbf{Q}}\pi_f \otimes_{\mathbf{Q}} \mathbf{C} \to \pi_f$ is surjective. ■

Fix $\eta \neq 0 \in H^0(E, \Omega_{E/F})$. There exists $\alpha \in F'^\times$ such that $\mathscr{J}(\alpha\omega_\varphi) = \eta$. Fix a continuous character of finite order $\chi\colon K^\times_{\mathbf{A}}/K^\times F^\times_{\mathbf{A}} \to \mathbf{Z}[\chi]^\times$. Let $H' \subset H$ be any open compact subgroup of $\widehat{B}^\times$ satisfying $\chi(q^{-1}_{\mathbf{A}}(H'F^\times_{\mathbf{A}})) = 1$. Assume that there exists $b_0 \in \widehat{B}^\times$ such that $b_0^{-1}H'b_0 \subset H$. Let pr_{b_0} be the map $\mathrm{Sh}_{H'}(G/Z, X) \to \mathrm{Sh}_H(G/Z, X)$ defined on complex points by

$$[x, b]_{H'\widehat{F}^\times} \mapsto [x, bb_0]_{H\widehat{F}^\times}.$$

Proposition 5.10 *If $b_0^{-1}H'b_0 \subset H$ for some $b_0 \in \widehat{B}^\times$, then for all $Z' \in C^r(\mathrm{Sh}_{H'}(G/Z, X)(\mathbf{C}), \mathbf{Z})$,*

$$\int_{Z'} \mathrm{pr}^*_{b_0}(\omega^{\chi_\infty}_\varphi) \in \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1.$$

Proof Let $Z = \mathrm{pr}_{b_0}(Z') \in C^r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{Z})$. We have

$$\int_{Z'} \mathrm{pr}^*_{b_0}\, \omega^{\chi_\infty}_\varphi = \deg(\mathrm{pr}_{b_0}\colon Z' \to Z) \int_Z \omega^{\chi_\infty}_\varphi.$$

Thanks to Proposition 3.7, we have $\int_Z \omega^{\chi_\infty}_\varphi \in \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1$, hence

$$\int_{Z'} \mathrm{pr}^*_{b_0}\, \omega^{\chi_\infty}_\varphi \in \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1.$$ ■

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Denote by $\mathrm{pr}\colon \mathrm{Sh}_{H'}(G/Z,X) \to \mathrm{Sh}_H(G/Z,X)$ the natural projection and by $(K\otimes\mathbf{R})_+^\times$ the set of elements in $(K\otimes\mathbf{R})^\times$ whose norm to F is positive at each place of F. We have $\pi_0(T(\mathbf{R})) = \frac{(K\otimes\mathbf{R})^\times}{(F\otimes\mathbf{R})^\times(K\otimes\mathbf{R})_+^\times}$.

The formula

$$\ell_\chi(\omega') = \frac{1}{[H:H']\deg(\mathscr{T}_{H',b} \xrightarrow{\mathrm{pr}} \mathscr{T}_{H,b})} \sum_{a\in \frac{K_{\mathbf{A}}^\times}{q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times}} \chi(a)\otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}} \omega' \pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}}\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1},$$

where

$$\partial\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)} = \left[\mathscr{T}^{q(a_\infty)}_{H',\widehat{q}(a_f)}\right],$$

is independent of the specific choice of $\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}$: we can assume that $\omega' = \mathrm{pr}_{b_0}^*(\omega_\varphi)$ for some $b_0 \in \widehat{B}^\times$; decompose each

$$a \in K_{\mathbf{A}}^\times/q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times$$

as $a = (a_f, 1_\infty)(1_f, a_\infty)$. Remark that

$$K_{\mathbf{A}}^\times/q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times = \widehat{K}^\times/\widehat{q}^{-1}(H'\widehat{F}^\times)\times(K\otimes\mathbf{R})^\times/(K\otimes\mathbf{R})_+^\times,$$

hence $a_f \in \widehat{K}^\times/\widehat{q}^{-1}(H'\widehat{F}^\times)$ and $a_\infty \in (K\otimes\mathbf{R})^\times/(K\otimes\mathbf{R})_+^\times$.

Thanks to Proposition 5.10, the formula

$$\begin{aligned}\sum_{a_\infty\in K_\infty^\times} \chi_\infty(a_\infty)\int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}} \omega' &= \sum_{a_\infty\in K_\infty^\times}\chi_\infty(a_\infty)\int_{\Delta_{H',\widehat{q}(a_f)}} t_{q(a_\infty)}\,\mathrm{pr}_{b_0}^*\,\omega_\varphi \\ &= \int_{\Delta_{H,\widehat{q}(a_f)}} \omega_\varphi^{\chi_\infty} \pmod{\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\end{aligned}$$

does not depend on the specific choice of $\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}$.

Thus, the expression of $\ell_\chi(\omega')$ above defines a linear form

$$\ell_\chi\colon S_2^{H'}\cap\mathbf{Q}[\widehat{B}^\times]\omega_\varphi \longrightarrow \mathbf{Q}(\chi)\otimes_{\mathbf{Q}}(\mathbf{C}/\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1).$$

To simplify the notations, let

$$\delta_{H',H} = \deg(\mathscr{T}_{H',b}\xrightarrow{\mathrm{pr}}\mathscr{T}_{H,b}) \qquad \text{and} \qquad W_{H'} = K_{\mathbf{A}}^\times/q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times.$$

Thus,

$$\ell_\chi(\omega') = \frac{1}{[H:H']\delta_{H',H}}\sum_{a\in W_{H'}}\chi(a)\otimes\int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}}\omega'.$$

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Proposition 5.11 (i) *Let $H'' \subset H' \subset H$ be open compact subgroups such that $\chi(q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^{\times})) = 1$ and pr^* the map $\mathrm{pr}^*\colon S_2^{H'}(B_{\mathbf{A}}^{\times}) \to S_2^{H''}(B_{\mathbf{A}}^{\times})$.*
If $\omega' \in S_2^{H'}(B_{\mathbf{A}}^{\times}) \cap \mathbf{Q}[\widehat{B}^{\times}]\omega_{\varphi}$, then $\ell_{\chi}(\omega') = \ell_{\chi}(\mathrm{pr}^*(\omega'))$ and ℓ_{χ} defines a linear form on $\mathbf{Q}[\widehat{B}^{\times}]\omega_{\varphi}$.

(ii) *We have for all $a \in \widehat{K}^{\times}$ $\forall \omega \in \mathbf{Q}[\widehat{B}^{\times}]\omega_{\varphi}$,*

$$\ell_{\chi}\big([\cdot\widehat{q}(a_f)]^*\omega\big) = \chi_f(a)^{-1}\ell_{\chi}(\omega).$$

(iii) *If χ factors through $\mathrm{Gal}(K_b^+/K)$ and if $P_b^{\beta} = \Phi_1\big(\int_{\Delta_{H,b}}\omega_{\varphi}^{\beta}\big) \otimes 1 \in \mathbf{C}/\mathbf{Q}\Lambda_1$, then*

$$\begin{aligned}
e_{\overline{\chi}}(P_b^{\chi_\infty}) &= \sum_{\mathrm{Gal}(K_b^+/K)} \chi(\sigma) \otimes \sigma(P_b^{\chi_\infty}) \in \mathbf{Q}(\chi) \otimes_{\mathbf{Q}} E(K_b^+) \subset \mathbf{Q}(\chi) \otimes_{\mathbf{Q}} \big(\mathbf{C}/\mathbf{Q}\Lambda_1\big)\\
&= \Phi_1(\ell_{\chi}([\cdot b]^*\omega_{\varphi})),
\end{aligned}$$

up to a non-zero rational factor.

Proof (i) Let $a \in \widehat{K}^{\times}$. We have $\mathrm{pr}(\Delta_{H'',\widehat{q}(a_f)}) = \Delta_{H',\widehat{q}(a_f)}$ and

$$\int_{\Delta_{H'',b}} \mathrm{pr}^*\,\omega' = \deg(\mathscr{T}_{H'',b} \longrightarrow \mathscr{T}_{H',b}) \int_{\Delta_{H',b}} \omega' = \delta_{H'',H'} \int_{\Delta_{H',b}} \omega'.$$

As $\chi(q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^{\times})) = 1$, we have (thanks to Proposition 5.10)

$$\begin{aligned}
\ell_{\chi}(\mathrm{pr}^*\,\omega') &= \frac{1}{[H:H'']\delta_{H'',H}} \sum_{a \in W_{H''}} \chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H'',\widehat{q}(a_f)}} \mathrm{pr}^*\,\omega' \\
&\qquad (\mathrm{mod}\ \mathbf{Q}(\chi) \otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1)\\
&= \frac{\delta_{H'',H'}}{\delta_{H'',H}} \sum_{a \in W_{H''}} \chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}} \omega' \\
&\qquad (\mathrm{mod}\ \mathbf{Q}(\chi) \otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1)\\
&= \frac{\delta_{H'',H'}}{[H:H'']\delta_{H'',H}} \sum_{a \in W_{H'}} [H':H'']\chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}} \omega' \\
&\qquad (\mathrm{mod}\ \mathbf{Q}(\chi) \otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1)\\
&= \frac{[H':H'']}{[H:H'']\delta_{H',H}} \sum_{a \in W_{H'}} \chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}} \omega' \\
&\qquad (\mathrm{mod}\ \mathbf{Q}(\chi) \otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1)\\
&= \ell_{\chi}(\omega').
\end{aligned}$$

(ii) Assume H'' is sufficiently small such that $[\cdot\widehat{q}(a_f)]^* \operatorname{pr}^* \omega \in S_2^{H''}$. We have

$$\ell_\chi([\cdot\widehat{q}(a_f)]^*\omega) = \ell_\chi([\cdot\widehat{q}(a_f)]^* \operatorname{pr}^* \omega)$$

$$= \frac{1}{[H:H'']\delta_{H'',H}} \sum_{a'\in W_{H''}} \chi(a') \otimes \int_{\Delta^{q(a'_\infty)}_{H'',\widehat{q}(a')}} [\cdot\widehat{q}(a_f)]^* \operatorname{pr}^* \omega \pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}$$

$$= \frac{1}{[H:H'']\delta_{H'',H}} \sum_{a'\in W_{H''}} \chi(a') \otimes \int_{\Delta^{q(a'_\infty)}_{H'',\widehat{q}(aa')}} \operatorname{pr}^* \omega \pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}$$

$$= \frac{1}{[H:H'']\delta_{H'',H}} \sum_{a''\in W_{H''}} \chi(a''a^{-1}) \otimes \int_{\Delta^{q(a''_\infty)}_{H'',\widehat{q}(a'')}} \operatorname{pr}^* \omega \pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}$$

$$= \chi_f(a)^{-1}\frac{1}{[H:H'']\delta_{H'',H}} \sum_{a''\in W_{H''}} \chi(a'') \otimes \int_{\Delta^{q(a''_\infty)}_{H'',\widehat{q}(a'')}} \operatorname{pr}^* \omega \pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}$$

$$= \chi_f(a)^{-1}\ell_\chi(\operatorname{pr}^*\omega) = \chi_f(a)^{-1}\ell_\chi(\omega)$$

(iii) As $\omega_\varphi \in S_2(B_{\mathbf{A}}^\times) = \bigcup_H S_2^H(B_{\mathbf{A}}^\times)$, there exists H' sufficiently small such that $\omega_\varphi \in S_2^{H'}$ and $[\cdot b]^*\omega_\varphi \in S_2^{H'}$. Let

$$m = [K_{\mathbf{A}}^\times / q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times : \operatorname{Gal}(K_b^+/K)]$$

and $\nu = 1/[H:H']\deg(\mathscr{T}_{H'} \to \mathscr{T}_H)$. We have

$$\ell_\chi(\circ[\cdot b]^*\omega_\varphi) = \nu \sum_{a\in \frac{K_{\mathbf{A}}^\times}{q_{\mathbf{A}}^{-1}(HF_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times}} \chi_f(a_f)\chi_\infty(a_\infty) \otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}} [\cdot b]^*\omega_\varphi \pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}$$

$$= \nu \sum_{a_f} \chi_f(a_f) \otimes \sum_{a_\infty} \chi_\infty(a_\infty)\operatorname{rec}_K(a_f)\cdot \int_{\Delta_{H',b}} t_{\operatorname{rec}_K(a_\infty)}\omega_\varphi \pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}$$

$$= \nu m \sum_{\sigma\in\operatorname{Gal}(K_b^+/K)} \chi(\sigma) \otimes \int_{\Delta_{H',b}} \sum_{a_\infty} \chi_\infty(a_\infty) t_{\operatorname{rec}_K(a_\infty)}\omega_\varphi \pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}$$

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$$= \nu m \sum_{\sigma \in \mathrm{Gal}(K_b^+/K)} \chi(\sigma) \otimes \int_{\Delta_{H',b}} \omega_\varphi^{\chi_\infty}$$

$$(\mathrm{mod}\ \mathbf{Q}(\chi) \otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1),$$

hence

$$e_{\overline{\chi}}(P_b^{\chi_\infty}) = \Phi_1(\ell_\chi([\cdot b]^* \omega_\varphi)).$$ ∎

Let us consider the Néron-Tate height $h_{\mathrm{NT}} \colon E(K^{\mathrm{ab}}) \times E(K^{\mathrm{ab}}) \to \mathbf{R}$ extended to an hermitian form

$$h_{\mathrm{NT}} \colon E(K^{\mathrm{ab}}) \otimes \mathbf{C} \times E(K^{\mathrm{ab}}) \otimes \mathbf{C} \longrightarrow \mathbf{C}.$$

Recall the condition for all $v \neq \tau_1$,

$$\varepsilon\Big(\pi_v \times \chi_v, \frac{1}{2}\Big)\eta_{K,v}(-1) = \mathrm{inv}_v(B)$$

from Proposition 5.8: if 5.4 fails, then $P_b^{\chi_\infty} \in E(K^{\mathrm{ab}})$ is torsion.

In general, there should be some $k(b, \omega_\varphi) \in \mathbf{C}$ such that for all $\sigma \colon \mathbf{Q}(\chi) \hookrightarrow \mathbf{C}$,

$$h_{\mathrm{NT}}(e_{\sigma\overline{\chi}}(P_b^{\chi_\infty})) = k(b, \omega_\varphi) L'\Big(\pi \times {}^\sigma\chi, \frac{1}{2}\Big),$$

as in Gross–Zagier, Zhang, and Yuan–Zhang–Zhang [12, 31, 33].

This formula explains the following conjecture.

Conjecture 5.12 *Let $K_\chi = (K^{\mathrm{ab}})^{\mathrm{Ker}(\chi)}$ be the extension of K trivializing χ. If for all $v \neq \tau_1$,*

$$\varepsilon\Big(\pi_v \times \chi_v, \frac{1}{2}\Big)\eta_{K,v}(-1) = \mathrm{inv}_v(B),$$

then there exists $b \in \widehat{B}^\times$ such that $k(b, \omega_\varphi) \neq 0$, and we have the following equivalences:

$$\begin{aligned}
&\ell_\chi \neq 0 \\
&\iff \exists b \in B_{\mathbf{A}}^\times \text{ such that } K_\chi \subset K_b^+ \text{ and } e_{\overline{\chi}}(P_b^{\chi_\infty}) \in \mathbf{Z}[\chi] \otimes E(K_b^+) \text{ is not torsion} \\
&\iff \exists \sigma \colon \mathbf{Q}(\chi) \hookrightarrow \mathbf{C} \qquad L'\Big(\pi \times {}^\sigma\chi, \frac{1}{2}\Big) \neq 0 \\
&\iff \forall \sigma \colon \mathbf{Q}(\chi) \hookrightarrow \mathbf{C} \qquad L'\Big(\pi \times {}^\sigma\chi, \frac{1}{2}\Big) \neq 0.
\end{aligned}$$

6 A Relation to Kudla's Program

The theorem of Gross–Kohnen–Zagier asserts that the positions of the traces to $\mathbf{Q}$ of classical Heegner points are given by the Fourier coefficients of some Jacobi form. The geometric proof of Zagier explained, for example, in [32] has been recently generalized by Yuan, Zhang, and Zhang in [31] using a result of Kudla and Millson [17]. In this section we establish a relation between Darmon's construction and Kudla's program. This is a first step in an attempt to apply the arguments of Zagier [32] and Yuan, Zhang, and Zhang's [31] to Darmon's points.

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6.1 Some Computations

Let us fix a modular elliptic curve E/F of conductor $N = N_+N_-$. Assume that $\mathrm{Ram}(B) = \{\tau_{r+1}, \dots, \tau_d\} \cup \{v \mid N_-\}$ and that the quadratic extension K/F satisfies the following hypothesis:

$$\forall v \mid N_+ \text{ splits in } K \qquad \forall v \mid N_- \text{ is inert in } K.$$

In particular, the relative discriminant $d_{K/F}$ is prime to N. Let R be an Eichler order of B of level N_+. Identify K with its image in B by q and assume that $K \cap R = \mathcal{O}_K$, $H = \hat{R}^\times$ (which implies that $\dim \pi_f^H = 1$).

Recall that h_1 defines an embedding $\tau_{1,K} \colon K \hookrightarrow \mathbf{C}$ and denote by c the non-trivial element of $\mathrm{Gal}(K/F)$. Assume that Conjecture 5.1 is true for $\beta = 1$ and let $P = \mathrm{Tr}_{K_1^+/K} P_1^1 \in E(K)$.

Proposition 6.1 *If ε is the global sign of E/F,* i.e., *$\Lambda(E/F, s) = \varepsilon\Lambda(E/F, 2-s)$, where Λ is the completed L-function of E/F, then $c(P) = -\varepsilon P$.*

Proof Assume that $K = F(i)$ and $B = K(j)$, with $i^2 = \mathfrak{a} \in F^\times$, $j^2 = \mathfrak{b} \in F^\times$ and $ij = -ji$. Recall that $\mathcal{T}_1^\circ = [\mathcal{T}^\circ, 1]_{H\hat{F}^\times}$ with $\mathcal{T}^\circ = \{z_1\} \times \gamma_2 \times \cdots \times \gamma_r$. Thus,

$$\begin{aligned} c(\mathcal{T}_1^\circ) &= [\{t_1 z_1\} \times \gamma_2 \times \cdots \times \gamma_r, 1]_{H\hat{F}^\times} = (-1)^{r-1}[j^{-1}(\mathcal{T}^\circ), 1]_{H\hat{F}^\times}, \\ c(\mathcal{T}_1^\circ) &= (-1)^{r-1}[\mathcal{T}^\circ, j]_{H\hat{F}^\times}, \end{aligned}$$

since $j \in B^\times$. This shows that $c(P_1) = (-1)^{r-1}P_j$. We will write P_j using only P_1. We will make the following abuse of language. For each place v of F, j_v shall denote the element $(1, \dots, 1, \underbrace{j_v}_{v}, 1 \dots) \in B_{\mathbf{A}}^\times$, and we will use the following lemma.

Lemma 6.2 *Let $b \in \widehat{B}^\times$ and v a place of F. When $v \mid N_+$, set $k_v \in K_v^\times$ corresponding to*

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi_v^{\mathrm{ord}_v(N_+)} \end{pmatrix},$$

where ϖ_v is an uniformizer of K_v. If $b_v = 1$, then

$$P_{bj_v} = \begin{cases} -\varepsilon_v P_b & \text{if } v \mid N_- \\ \varepsilon_v \,\mathrm{rec}_K(k_v^{-1}) P_b & \text{if } v \mid N_+ \\ P_b & \text{if } v \nmid N. \end{cases}$$

Proof of the lemma For each v inert in K/F we have

$$\begin{aligned} \mathrm{inv}_v(B) = 1 &\iff B_v \simeq M_2(F_v) \\ &\iff \mathfrak{b} \in \mathrm{N}_{K_v/F_v}(K_v^\times) = \mathcal{O}_{F_v}^\times F_v^{\times 2} \\ &\iff 2 \mid \mathrm{ord}_v(\mathfrak{b}) \end{aligned}$$

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As $\overline{j} = -j$, we have $\mathrm{nr}(j) = -j^2 = -\mathfrak{b}$ and

$$\mathrm{inv}_v(B) = 1 \iff 2 \mid \mathrm{ord}_v(\mathrm{nr}(j_v)).$$

If $v \mid N_-$, then $H_v = \mathcal{O}_{B_v}^\times$, where $\mathcal{O}_{B_v}$ is the unique maximal order in B_v, hence $H_v \lhd B_v^\times$ and $B_v^\times/H_v^\times \simeq \mathbf{Z}$ by choosing some uniformizer. As H_v is normal in $B_v^\times$, the map

$$[\cdot j_v]\colon\ \mathrm{Sh}_H(G/Z,X)(\mathbf{C}) \longrightarrow \mathrm{Sh}_{j_v^{-1}Hj_v}(G/Z,X)(\mathbf{C})$$

is well defined on $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$. Thus $[\mathscr{T}^\circ, bj_v]_{H\hat{F}^\times} = [\cdot j_v][\mathscr{T}^\circ, b]_{H\hat{F}^\times}$ and

$$\int_{\Delta^\circ_{bj_v}} \omega_\varphi = \int_{\Delta^\circ_b} [\cdot j_v]^*\omega_\varphi = \int_{\Delta^\circ_b} \pi_v(j_v)\omega_\varphi.$$

Decompose $\pi = \pi(\varphi) = \otimes'_v \pi_v$. We have

$$\pi_v\colon B_v^\times \xrightarrow{\mathrm{nr}} F_v^\times \xrightarrow{\mathrm{ord}_v} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \xrightarrow{\sim} \{\pm 1\}.$$

Let us denote by α the following unramified character

$$\alpha\colon F_v^\times \xrightarrow{\mathrm{ord}_v} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \xrightarrow{\sim} \{\pm 1\}$$

satisfying $\pi_v = \alpha \circ \mathrm{nr}$.

As $v \mid N_-$, E has multiplicative reduction in v. The character α is trivial if and only if E has split multiplicative reduction in v, *i.e.*, $\varepsilon_v = -1$.

Hence,

$$[\cdot j_v]^*\omega_\varphi = \alpha(\mathrm{nr}(j_v))\omega_\varphi = \begin{cases} \omega_\varphi & \text{if } \alpha = 1, \\ (-1)^{\mathrm{ord}_v(\mathrm{nr}(j))}\omega_\varphi & \text{otherwise.} \end{cases}$$

As $v \mid N_-$, $v \in \mathrm{Ram}(B)$ is inert in K/F and $\mathrm{inv}_v(B) = -1$, thus $2 \nmid \mathrm{ord}_v(\mathrm{nr}(j))$. Hence,

$$[\cdot j_v]^*\omega_\varphi = \alpha(\mathrm{nr}(j_v))\omega_\varphi = \begin{cases} \omega_\varphi = -\varepsilon_v\omega_\varphi & \text{if } \alpha = 1, \\ -\omega_\varphi = -\varepsilon_v\omega_\varphi & \text{otherwise} \end{cases}$$

and $P_{bj_v} = -\varepsilon_v P_b$.

If $v \mid N_+$, then we fix some uniformizer ϖ_v of F_v and an isomorphism $B_v \simeq \mathrm{M}_2(F_v)$ that identifies K_v with the set of diagonal matrices and R_v with

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_{F,v}) \;\Big|\; \varpi_v^{\mathrm{ord}_v(N_+)} \Big| c \right\}.$$

As $\mathrm{inv}_v(B_v) = 1$, j_v is a local norm. There exists $k_v \in K_v$ such that $j_v = \mathrm{N}_{K_v/F_v}(k_v)$. We may assume that $j_v^2 = 1$. Moreover, since j_v is in the normalizer of $K_v^\times$ in $B_v^\times$, we thus identify j_v to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Set

$$W_v = \begin{pmatrix} 0 & 1 \\ \varpi_v^{\mathrm{ord}_v(N_+)} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v^{\mathrm{ord}_v(N_+)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = k_v j_v.$$

This matrix is in the normalizer of R_v in B_v. As W_v normalize H_v,

$$[\mathscr{T}^\circ, bj_v]_{H\hat{F}^\times} = [\mathscr{T}^\circ, bk_v^{-1}W_v]_{H\hat{F}^\times} = [\cdot W_v][\mathscr{T}^\circ, bk_v^{-1}]_{H\hat{F}^\times}.$$

Decompose $\omega_\varphi = \bigotimes_{v|N_+} \omega_v \otimes \omega'$, where ω_v satisfies$[\cdot W_v]^*\omega_v = \varepsilon_v\omega_v$; then

$$\int_{\Delta^\circ_{bj_v}} \omega_\varphi = \varepsilon_v \int_{\Delta^\circ_{bk_v^{-1}}} \omega_\varphi.$$

As $b_v = 1$, $P_{bj_v} = \varepsilon_v \operatorname{rec}_K(k_v^{-1})P_b$.

If $v \nmid N$, then by a similar calculation we obtain $P_{bj_v} = \operatorname{rec}_K(k_v^{-1})P_b$. ■

End of the proof of Proposition 6.1 Lemma 6.2 implies that

$$c(P_1) = (-1)^{r-1} \prod_{v|N_-} (-\varepsilon_v) \prod_{v|N_+} \varepsilon_v \operatorname{rec}_K(k_v^{-1})P_1$$

and for all $a \in K_{\mathbf{A}}^\times$,

$$c(\operatorname{rec}_K(a)P_1) = (-1)^{r-1} \prod_{v|N_-} (-\varepsilon_v) \prod_{v|N_+} \varepsilon_v \operatorname{rec}_K(k_v^{-1}) \operatorname{rec}_K(a)P_1.$$

As $P \in E(K)$, we know that $\operatorname{rec}_K(k^{-1})P = P$. Thus

$$c(P) = (-1)^{r-1} \prod_{v|N_-} (-\varepsilon_v) \prod_{v|N_+} \varepsilon_v P = (-1)^{r-1}(-1)^{|\{v|N_-\}|} \prod_{v\nmid\infty} \varepsilon_v P. \tag{6.1}$$

We have to show that $(-1)^{r-1} \prod_{v|N_-}(-\varepsilon_v) \prod_{v|N_+} \varepsilon_v = -\varepsilon$. For each $v \mid \infty$ we have $\varepsilon_v = -1$. Since $\prod_{v|\infty} = (-1)^d$, the sign in equation (6.1) is

$$(-1)^d \underbrace{\prod_v \varepsilon_v}_{=\varepsilon} (-1)^{r-1}(-1)^{|\{v|N_-\}|}.$$

Recall that $\{v \mid N_-\} = \operatorname{Ram}(B) \cap S_f$. As $|\operatorname{Ram}(B)|$ is even, we have

$$(-1)^{|\{v|N_-\}|} = (-1)^{|\operatorname{Ram}(B)\cap S_\infty|} = (-1)^{d-r}.$$

Hence

$$c(P) = (-1)^d \varepsilon (-1)^{r-1}(-1)^{|\{v|N_-\}|}P = -\varepsilon P.$$ ■

Remark 6.3 The above computations are a particular case of a result of Prasad, [24, Theorem 4], which asserts that if $\operatorname{Hom}_{K_v^\times}(\pi_v, \mathbf{1}) \neq \{0\}$, then the nontrivial element in $\mathrm{N}_{B_v^\times}(K_v^\times)\backslash K_v^\times$ acts on $\operatorname{Hom}_{K_v^\times}(\pi_v, \mathbf{1})$ by multiplication by $\operatorname{inv}_v(B)\varepsilon_v = \operatorname{inv}_v(B)\varepsilon(\pi_v, \frac{1}{2}) \in \{\pm 1\}$.

6.2 Orthogonal Shimura Manifolds

Until the end of this paper we shall assume that $h_F^+ = 1$.

Let us recall some definitions used by Kudla [15] in the particular case $r = 1$. Let $n \in \mathbf{Z}_{\geq 0}$ and let (V, Q) be a quadratic space over F of dimension $n + 2$. We assume that the signature of $V \otimes_{\mathbf{Q}} \mathbf{R}$ is

$$(n, 2) \times (n + 1, 1)^{r-1} \times (n + 2, 0)^{d-r}.$$

Denote by D the symmetric space of $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GSpin}(V)$. D is the product of the oriented symmetric spaces of $V_j = V \otimes_{\tau_j, F} \mathbf{R}$. Thus $D = D_1 \times \dots D_d$, where D_j is the set of oriented positive subspaces in V_j of maximal dimension. For each $x \in V$ let x_j be the image of x in V_j. Assume that $Q(x)$ is totally positive. Set $V_x = x^\perp$, $G_x = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GSpin}(V_x)$, and for each $j \in \{1, \dots, d\}$,

$$D_{x_j} = \{z \in D_j \, z \perp x_j\}.$$

We shall focus on the following real cycle on the Shimura manifold $G(\mathbf{Q})\backslash D \times G(\widehat{\mathbf{Q}})/H$.

Definition 6.4 Let H be an open compact subgroup in $G(\widehat{\mathbf{Q}})$ and $g \in G(\widehat{\mathbf{Q}})$. The cycle $Z(x, g; H)$ is defined to be the image of the map

$$Z(x, g; H)\colon \begin{cases} G_x(\mathbf{Q})\backslash D_x \times G_x(\widehat{\mathbf{Q}})/H_x^g & \longrightarrow \quad G(\mathbf{Q})\backslash D \times G(\widehat{\mathbf{Q}})/H \\ G_x(\mathbf{Q})(y, u)H_x^g & \longmapsto \quad G(\mathbf{Q})(y, ug)H\widehat{F}^\times, \end{cases}$$

where H_x^g denotes $G_x(\widehat{\mathbf{Q}}) \cap gHg^{-1}$.

Example (including Proposition 6.5) Fix $D_0 \in F$ satisfying

$$\begin{cases} \tau_j(D_0) > 0 & \text{if } j \in \{1, r + 1, \dots, d\}, \\ \tau_j(D_0) < 0 & \text{if } j \in \{2, \dots, r\}. \end{cases}$$

Set $(V, Q) = (B^{\mathrm{Tr}=0}, D_0 \cdot \mathrm{nr})$. Then $(V \otimes_{F, \tau_j} \mathbf{R}, \tau_j \circ D_0 \cdot \mathrm{nr})$ has signature

$$\begin{cases} (1, 2) & \text{if } j = 1, \\ (2, 1) & \text{if } j \in \{2, \dots, r\}, \\ (3, 0) & \text{if } j \in \{r + 1, \dots, d\}. \end{cases}$$

Let $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GSpin}(V)$. The action of $B^\times$ on V by conjugation induces an isomorphism

$$\begin{array}{ccc} B^\times & \xrightarrow{\sim} & \mathrm{GSpin}(V) \\ b & \longmapsto & (v \mapsto bvb^{-1}), \end{array}$$

thus $G \simeq \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$.

Let $x \in V$ such that $Q(x) \gg 0$, and denote by x_j its image in $V \otimes_{F, \tau_j} \mathbf{R}$. Denote by K the quadratic extension $F + Fx$ and $T = \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m) \mathrm{Res}_{F/\mathbf{Q}}(\mathbf{G}_m)$ as above. Let q be the inclusion $K \hookrightarrow V \to B$.

Proposition 6.5 *The set* $D_x = D_{x_1} \times \cdots \times D_{x_r}$ *is a* $q(T(\mathbf{R}))^\circ$*-orbit in* D *whose projection on* D_1 *is a point.*

Proof As $x \in V$, $\mathrm{Tr}(x) = 0$ and $x^2 = -\mathrm{nr}(x) = -\frac{Q(x)}{D_0} \in F^\times$. Let $j \in \{1, \dots, r\}$. We have $\tau_j(Q(x)) > 0$, hence $\tau_j(x^2)\tau_j(D_0) < 0$. Thus τ_1 ramifies in K and $\tau_2, \dots, \tau_r$ are split. Moreover, $q_1(K^\times)$ fixes x_1 by definition of K. ■

Let us focus on the general case when V has dimension n. Fix $t \in F$ satisfying for all $j \in \{1, \dots, r\}$, $\tau_j(t) > 0$. $G(\widehat{\mathbf{Q}})$ acts on $\Omega_t = \{x \in V(F) \mid \quad Q(x) = t\}$ by conjugation.

Let φ be a Schwartz function on $V(\widehat{F})$. Assume $\Omega_t \neq \varnothing$ and fix $x \in \Omega_t$. Denote by $Z(y, \varphi; H)$ the sum

$$Z(t, \varphi; H) = \sum_{g \in G_x(\widehat{\mathbf{Q}}) \backslash G(\widehat{\mathbf{Q}}) / H\widehat{F}^\times} \varphi(g^{-1} \cdot x) Z(x, g; H).$$

Proposition 4.5 showed that for $n = 1$,

$$[Z(x, g; H)] = 0 \in H_{r-1}\big(\mathbf{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{C}\big).$$

A natural invariant to consider is the refined class

$$\{Z(t, \varphi; H)\} = \\ \omega \mapsto J_b^\beta \in \frac{(\mathrm{Harm}^r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}))^*}{\mathrm{Im}(H_r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{Z}) \to \mathrm{Harm}^r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}))^*)},$$

where $\mathrm{Harm}^r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}))$ is the set of harmonic differential forms on $\mathrm{Sh}_H(G/Z, X)(\mathbf{C})$.

In order to adapt the work of Yuan, Zhang, and Zhang, we need the following conjecture.

Conjecture 6.6 *In the situation of the above example* $(V, Q) = (B^{\mathrm{Tr}=0}, D_0 \cdot \mathrm{nr})$, *the sum*

$$\sum_{\substack{t \in \mathcal{O}_F \\ t \gg 0}} \{Z(t, \varphi; H)\} q^t$$

is a Hilbert modular form of weight 3/2.

In [31], the authors work by induction. To apply their method we would need to prove that the refined classes $\{Z(t, \varphi; H)\}$ are compatible with the tower of varieties attached to quadratic spaces $V_x \hookrightarrow V$ of signature $(n, 2) \times (n+1, 1)^{r-1} \times (n+2, 0)^{d-r}$ (in which case a generalization of [17] should imply that

$$\sum_{\substack{t \in \mathcal{O}_F \\ t \gg 0}} [Z(t, \varphi; H)] q^t$$

is a Hilbert modular form of weight $\frac{n}{2} + 1$ with coefficients in

$$H^{r+1}(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{C})).$$

6.3 A Gross–Kohnen–Zagier-type Conjecture

The Bruhat–Tits tree In this section we recall some basic facts about the Bruhat–Tits tree (see [4, 29]).

Let v be a finite place of F. The vertices of the Bruhat–Tits tree of $\mathrm{PGL}_2(F_v)$ are the maximal orders of $\mathrm{M}_2(F_v)$. Such maximal orders are endomorphism rings of lattices in F_v^2 ([29], lemme 2.1). There is an oriented edge between two vertices $\mathcal{O}_1$ and $\mathcal{O}_2$ if and only if there exist L_1, L_2 lattices in F_v^2 such that $\mathcal{O}_i = \mathrm{End}(L_i)$, $L_2 \subset L_1$ and $L_1/L_2 \simeq \mathcal{O}_{F_v}/\varpi_v\mathcal{O}_{F_v}$. The intersection of the source and the target of paths of length n correspond to level v^n Eichler orders.

Fix some quadratic extension K/F. This data allows us to organize the Bruhat–Tits tree. Let $\Psi\colon K_v \hookrightarrow \mathrm{M}_2(F_v)$ be a F_v-embedding of K_v. Let $\mathrm{M}_0(N)$ be the set of matrices in $\mathrm{M}_2(F_v)$ which are upper triangular modulo N. If

$$\Psi(\mathcal{O}_{K_v}) = \Psi(K_v) \cap \mathrm{M}_0(N),$$

we say that Ψ has level N. We can organize the vertices of the tree in "levels", by privileging a direction. Each level corresponds to a level of embedding relative to $\mathcal{O}_{K_v}$ *i.e.,* to orders that are in the same orbit under $K_v^\times$. The maximal orders in $\mathrm{PGL}_2(F_v)$ that are maximally embedded are on the bottom of the tree.

Figures 2, 3, and 4 illustrate the dependence on the ramification type of v in K.

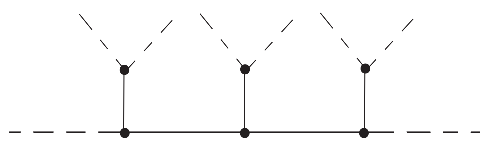

Figure 2: Bruhat-Tits tree of $\mathrm{PGL}_2(F_v)$ when v is split.

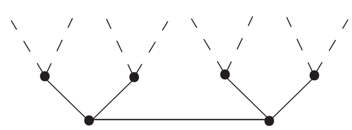

Figure 3: Bruhat-Tits tree $\mathrm{PGL}_2(F_v)$ when v is ramified.

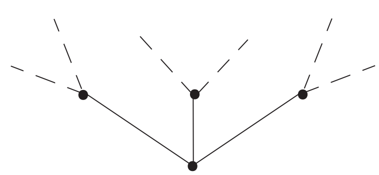

Figure 4: Bruhat-Tits tree of $\mathrm{PGL}_2(F_v)$ when v is inert.

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Darmon's Points, Kudla's Program, and a Gross–Kohnen–Zagier-type Theorem

Recall that $H = (R \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})^{\times}$, where R is an Eichler order of B of level N_+ and that $K = F + Fx$ satisfies the following Heegner hypothesis.

Hypothesis 6.7 *Each prime $\mathfrak{p} \mid N_+$ splits in K, and each prime $\mathfrak{p} \mid N_-$ is inert in K.*

The group G_x is isomorphic to $K^{\times}$, and $Z(x, 1; H)$ is the image of $K^{\times} \backslash D_x \times \widehat{K}^{\times}/H$ in $\mathrm{Sh}_H(G, X)(\mathbf{C})$. Note that

$$Z(x, 1; H) = \mathscr{T}_1^1 + t_1(\mathscr{T}_1^1),$$

where $\mathscr{T}_1^1 = [\cup_{u \in \pi_0(T(\mathbf{R}))} q(u) \cdot \mathscr{T}^{\circ}, 1]_{H\widehat{F}^{\times}}$.

Let $\varphi = \mathbf{1}_{\widehat{R}^{\mathrm{Tr}=0}}$. We are able to prove an analogue of [16, Proposition A.I.1] when $N = 1$, $B = \mathrm{M}_2(F)$, $R = \mathrm{M}_2(\mathcal{O}_F)$, $t = Q(x) = D_0 \mathrm{nr}(x) \in F$ and $K = F + Fx$ is such that $K \cap R = \mathcal{O}_K$ and $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F x$. Set $c_1(\mathscr{T}_1^1) = \{[t_1(x), b]_{H\widehat{F}^{\times}},\ b \in \widehat{B}^{\times}\}$.

Proposition 6.8 *If $N = 1$, $r = d$, $B = \mathrm{M}_2(F)$, $H = \widehat{R}^{\times}$ with $R = \mathrm{M}_2(\mathcal{O}_F)$ and if $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F x$, then $Z(t, \varphi; H)$ is equal to*

$$Z(x, 1; H) = \mathscr{T}_1^1 + c_1(\mathscr{T}_1^1) = \mathscr{T}_1^1 - \varepsilon \mathscr{T}_1^1.$$

Remark 6.9 Under the strong hypotheses above, $\varepsilon = (-1)^d$ and the cycle obtained is zero when d is even.

Proof By definition

$$Z(t, \varphi; H) = \sum_{g \in \widehat{K}^{\times} \backslash \widehat{B}^{\times}/\widehat{R}^{\times}} \mathbf{1}_{\widehat{R}^{\mathrm{Tr}=0}}(g^{-1} \cdot x) Z(x, g; H).$$

We have to determine $g \in \widehat{K}^{\times} \backslash \widehat{B}^{\times}/\widehat{R}^{\times}$ satisfying $g^{-1}xg \in \widehat{R}^{\mathrm{Tr}=0}$, *i.e.*, $x \in g\widehat{R}^{\mathrm{Tr}=0}g^{-1}$. As $F^{\times} \subset K^{\times}$,

$$\widehat{K}^{\times} \backslash \widehat{B}^{\times}/\widehat{F}^{\times}\widehat{R}^{\times} = \prod_v{}' K_v^{\times} \backslash B_v^{\times}/R_v^{\times} = \prod_v{}' K_v^{\times} \backslash B_v^{\times}/F_v^{\times} R_v^{\times}.$$

This allows us to work locally with $K_v^{\times} \backslash B_v^{\times}/F_v^{\times} R_v^{\times}$, which is identified to the $K_v^{\times}$-orbits of maximal orders of $\mathrm{PGL}_2(F_v)$. This gives the condition $x_v \in g_v R_v g_v^{-1}$.

First let us consider those $g_v \in B_v^{\times}/R_v^{\times} F_v^{\times}$ satisfying $x_v \in g_v R_v g_v^{-1}$. The ring $g_v R_v g_v^{-1}$ is a maximal order containing x_v. Using the fact that $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F x$, we have

$$x_v \in g_v R_v g_v^{-1} \Longleftrightarrow g_v R_v g_v^{-1} \cap K_v = \mathcal{O}_{K_v}.$$

Hence the maximal order $g_v R_v g_v^{-1}$ is maximally embedded in K_v. It is identified to a vertex at the lowest level of the Bruhat–Tits tree. As each vertex at the same level is in the same $K_v^{\times}$-orbit, we have for all v,

$$g_v = 1 \in K_v^{\times} \backslash B_v^{\times}/F_v^{\times} R_v^{\times}.$$

Thus $Z(t, \varphi; H) = Z(x, 1; H)$, and, as D_{x_1} is a set of two points, $Z(x, 1; H)$ is identified with $\mathscr{T}_1^1 + c_1(\mathscr{T}_1^1) = \mathscr{T}_1^1 - \varepsilon \mathscr{T}_1^1$, thanks to Proposition 6.1. ∎

We now consider the case when $N = N_+N_- \neq 1$ is prime to $d_{K/F}$. The following proposition is true even if $B \neq \mathrm{M}_2(F)$, but we still assume that R is an Eichler order of level N_+ and $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F x$.

Proposition 6.10 *Let N be the conductor of E. If N is prime to $d_{K/F}$, then*

$$Z(t,\varphi;H) = \prod_{v|N}(1 + \mathrm{inv}_v(B)\varepsilon_v)Z(x,1;H).$$

Proof The proof is analogous to the proof of Proposition 6.8. Let us first compute the number of terms in $Z(t,\varphi;H)$. We need to determine for each v the number of $K_v^\times$-orbits of oriented paths of length $\mathrm{ord}_v(N_+)$ in the Bruhat–Tits tree; this is equal to the number of g_v such that $x_v \in g_v R_v g_v^{-1}$.

- If $v \nmid N$, then the same argument as in Proposition 6.8 shows that there is only one orbit.
- If $v \mid N_-$, B_v is ramified and v is inert in K. Hence $K_v^\times \backslash B_v^\times / R_v^\times F_v^\times = \{1, \pi_v\}$ where $\pi_v \in B_v^\times$ is an element whose reduced norm has order 1 at v; π_v corresponds to the Atkin–Lehner involution.
- If $v \mid N_+$, v splits in K. Denote by v^δ the level of the order R_v. Each Eichler order of level v^δ is the intersection of the origin and the target of an oriented path of length δ. By hypothesis those orders are maximally embedded in K_v, and the path corresponding to $g_v R_v g_v^{-1}$ is contained in the lowest level of the tree. As $K_v^\times$ acts by translations on this level, there are exactly two $K_v^\times$-orbits corresponding to g_v depending on the orientation. We have g_v^+ and g_v^- that are exchanged by the Atkin–Lehner involution corresponding to $\left(\begin{smallmatrix} 0 & \varpi_v \\ 1 & 0 \end{smallmatrix}\right)$.

Let n be the number of prime ideals in the decomposition of N. The sum $Z(t,\varphi;H)$ has 2^n factors. Let W be the sets of these factors. By definition, $Z(x,g;H) = [\cdot g]Z(x,1;H)$. Using Proposition 6.1 we obtain

$$Z(t,\varphi;H) = \sum_{g\in W}[\cdot g]Z(x,1;H) = \prod_{v|N}\big(1 + \mathrm{inv}_v(B)\varepsilon_v\big)Z(x,1;H). \qquad \blacksquare$$

Let us conclude this paper with another conjecture. Assume that $E(F)$ has rank 1. Denote by P_0 some generator of $E(F)$ modulo torsion. For each $t \in \mathcal{O}_F$ totally positive such that (t) is square free and prime to $d_{K/F}$, denote by $K[t]$ the quadratic extension

$$K[t] = F(\sqrt{-D_0 t}),$$

which satisfies the hypothesis used to build Darmon's points. Let $P_{t,1}^1$ be Darmon's point obtained for $K[t]$, $b = 1$, and $\beta = 1$, and set $P_t = \mathrm{Tr}_{K[t]_1^+/F} P_{t,1}$. Assuming Conjectures 5.1 and 5.12, the point P_t lies in $E(F)$, and there exists an integer $[P_t] \in \mathbf{Z}$ such that $P_t = [P_t]P_0$ modulo torsion.

Proposition 6.10 together with Conjecture 6.6 suggest the following (as in [9, Conjecture 5.3]).

Conjecture 6.11 *There exists some Hilbert modular form g of level 3/2 such that the $[P_t]$s are proportional to some Fourier coefficients of g.*

Remark 6.12 Using the analogy with the Gross–Kohnen–Zagier theorem, the integers $[P_t]$ should be (proportional to) square roots of $L(E_{-D_0 t}, 1)$, where $E_{-D_0 t}$ is the twist of E by $-D_0 t$.

Let us end this paper with two open questions.

Question 6.13 Does Bruinier's generalization of Borcherds products [3] give anything interesting in this situation ?

It is natural to expect that results of Cornut and Vatsal [5,6] also hold for Darmon's points.

Question 6.14 Would it be possible to deduce such a result from suitable equidistribution properties for the real tori $\mathscr{T}_b^\circ$?

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