RATIONAL SURFACES WITH EXCEPTIONAL UNODES

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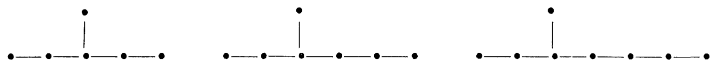
To Professor H. S. M. Coxeter on his sixtieth birthday

1. Introduction. Many years ago, I defined **(8)** three types of exceptional unode on an algebraic surface, which I called U^*_8, U^*_9, U^*_{10}, corresponding, on a non-singular model of the surface, to sets of six, seven, and eight rational curves, each of grade -2, with the intersection patterns represented by the Coxeter–Dynkin graphs now usually known as E_6, E_7, E_8:



E_6 E_7 E_8

where each dot represents a curve, and linked dots intersecting curves. In each case we shall denote the curves in the horizontal sequence by $s_1, s_2, \ldots$ from left to right, and the extra curve meeting s_3 by s^*. U^*_8 has been known since Cayley **(6)** to be one of the possible singularities of a cubic surface; and Herszberg **(10)** and Kirby **(12)** showed in different ways that the equations of surfaces with these three types of singularity were locally of the form

$$\text{(I)}\ z^2 = x^4 + y^3; \qquad \text{(II)}\ z^2 = y(x^3 + y^2); \qquad \text{(III)}\ z^2 = x^5 + y^3$$

respectively. As these singularities have been in the news again in the last few years in the work of Hirzebruch **(11)**, von Randow **(14)**, Brieskorn **(4)**, and Artin **(1)**, the last of whom rediscovered them independently, it is perhaps not out of place to point out that Herszberg's canonical forms are all rational surfaces, and to study their plane mappings, and the close relation of (I) to Cayley's cubic, and of (II) and (III) to each other.

2. Consecutive base points. In a plane π, on which all our mappings of rational surfaces are to be made, we consider a sequence of points

$$P_1, P_2, \ldots, P_n,$$

consecutive on an inflected branch. On a surface Π on which all these points are dilated, let s_i be the irreducible image of the neighbourhood of P_i, and s^* that of the inflexional tangent t. Then $s_1, \ldots, s_{n-1}$ are all of grade -2, and s_i, s_{i+1} intersect ($i = 1, \ldots, n-1$); and as t passes through P_1, P_2, P_3, s^* also has grade -2, and meets s_3. We thus have the intersection pattern E_n; for $n > 8$, of course, the intersection matrix is not negative definite, and the

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set of curves in question (without the exceptional curve s_n) cannot be contracted to a point; the cases of interest are $n = 6, 7, 8$. In these cases, the system of all curves in π, of order $3m$, with m-tuple base points at $P_1, \ldots, P_n$, has no free intersection with the neighbourhoods of $P_1, \ldots, P_{n-1}$, nor with t, so that the curves $s_1, \ldots, s_{n-1}, s^*$ are all fundamental to this system. Denoting by p_i the total neighbourhood of P_i, and by $|l|$ the linear system of all lines in π, we have

$$s_i = p_i - p_{i+1} \quad (i = 1, \ldots, n-1), \qquad s_n = p_n, \qquad s^* = l - p_1 - p_2 - p_3.$$

The least linear combination p of $s_1, \ldots, s_{n-1}, s^*$ that has non-positive intersection with each of them, and which consequently separates out with unit postulation from any system to which they are all fundamental, is in the three cases

$$(1) \quad \begin{cases} n = 6: & p = s_1 + 2s_2 + 3s_3 + 2s_4 + s_5 + 2s^* \\ & = 2l - p_1 - p_2 - p_3 - p_4 - p_5 - p_6, \\ n = 7: & p = 2s_1 + 3s_2 + 4s_3 + 3s_4 + 2s_5 + s_6 + 2s^* \\ & = 2l - p_2 - p_3 - p_4 - p_5 - p_6 - p_7, \\ n = 8: & p = 2s_1 + 4s_2 + 6s_3 + 5s_4 + 4s_5 + 3s_6 + 2s_7 + 3s^* \\ & = 3l - p_1 - p_2 - p_3 - p_4 - p_5 - p_6 - p_7 - 2p_8. \end{cases}$$

If F is the projective model of the system $|f| = (3m; m, \ldots, m)$, where the first integer in the parentheses is the order of the system and the others are its multiplicities at $P_1, \ldots, P_n$, the projection of F from its unode U^* is accordingly the projective model of $|f - p|$:

$$(2) \quad \begin{cases} n = 6: & (3m - 2; m - 1, \ldots, m - 1), \\ n = 7: & (3m - 2; m, m - 1, \ldots, m - 1), \\ n = 8: & (3m - 3; m - 1, \ldots, m - 1, m - 2). \end{cases}$$

$|f - p|$ has intersection unity with just one of the curves $s_1, \ldots, s_{n-1}, s^*$, namely s^* for $n = 6$, s_1 for $n = 7$, and s_7 for $n = 8$. It is clear in fact that $|f - p|$, as given by (2), has one variable intersection with t for $n = 6$, with the neighbourhood of P_1 for $n = 7$, and with that of P_7 for $n = 8$. This curve has in each case coefficient 2 in p, and represents the line on the projected surface which, counted twice, is the neighbourhood of the unode. The remaining curves are fundamental to $|f - p|$, and represent respectively a binode B_6 in the neighbourhood of U^*_8, an ordinary unode U_8 in that of U^*_9, and an exceptional unode U^*_9 in that of U^*_{10}.

For general positions of the base points, for $n = 6, 7, 8$ there would be on Π respectively 27, 56, 240 exceptional curves, which appear on F as rational curves of order m (for $n = 6$, $m = 1$ the 27 lines on the cubic surface; for $n = 7$, $m = 1$ the lines on the Geiser double plane which coincide by pairs in the bitangents of the branch curve; and for $n = 8$, $m = 2$ the conics on the Bertini double cone which coincide by pairs in the tritangent planes of the branch curve). In the present case, however, with $P_1, \ldots, P_n$ consecutive on an inflected branch, there is only one irreducible exceptional curve, s_n; the rest all reduce to linear combinations of this with $s_1, \ldots, s_{n-1}, s^*$; and on F there is only one rational curve of order m; this contains the unode, and combines with partial neighbourhoods of the unode to give formal equivalents of the remaining exceptional curves.

We shall take a coordinate system $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ in π, with P_1 at $(1, 0, 0)$ and t as the line $\boldsymbol{z} = 0$. Any cubic with an inflexion at P_1 and t as tangent there can be taken, for a suitable choice of the lines $\boldsymbol{x} = 0, \boldsymbol{y} = 0$, to be

$$f = \boldsymbol{x}^2\boldsymbol{z} - \boldsymbol{y}^3 + k\boldsymbol{y}\boldsymbol{z}^2 + l\boldsymbol{z}^3 = 0;$$

and the cubics $f + \boldsymbol{z}^2(a\boldsymbol{x} + b\boldsymbol{y} + c\boldsymbol{z}) = 0$ are those that meet $f = 0$ in at least six coincident points at P_1 (seven if $a = 0$ and all nine if $a = b = 0$). Taking $P_1, \ldots, P_8$ to be consecutive along $\boldsymbol{x}^2\boldsymbol{z} = \boldsymbol{y}^3$, we accordingly define the web $|C|$: $(3; 1, 1, 1, 1, 1, 1)$, the net $|G|$: $(3; 1, 1, 1, 1, 1, 1, 1)$, and the pencil $|B|$: $(3; 1, 1, 1, 1, 1, 1, 1, 1)$, with the bases

$$|C|: \quad \boldsymbol{x}^2\boldsymbol{z} - \boldsymbol{y}^3, \boldsymbol{z}^3, \boldsymbol{y}\boldsymbol{z}^2, \boldsymbol{x}\boldsymbol{z}^2,$$

$$|G|: \quad \boldsymbol{x}^2\boldsymbol{z} - \boldsymbol{y}^3, \boldsymbol{z}^3, \boldsymbol{y}\boldsymbol{z}^2,$$

$$|B|: \quad \boldsymbol{x}^2\boldsymbol{z} - \boldsymbol{y}^3, \boldsymbol{z}^3;$$

and a second net

$$|G'|: \quad \boldsymbol{x}^2\boldsymbol{z} - \boldsymbol{y}^3, \ \boldsymbol{y}\boldsymbol{z}^2, \ \boldsymbol{x}\boldsymbol{z}^2,$$

which has simple base points at $P_1, \ldots, P_6$ and a seventh at P': $(0, 0, 1)$. $|B|$ of course has the ninth associated base point P_9 consecutive to $P_1, \ldots, P_8$; but we regard this as unassigned, so that $|2B|$, $|3B|$ are the systems of sextics and nonics with eight, not nine, consecutive double and triple base points. $|C|$, $|G|$, $|G'|$ are the most general systems with the base points as specified, all such figures being projectively equivalent; but $|B|$ is not the most general pencil with all its base points consecutive on an inflected branch, being special in that all its curves are equi-anharmonic, except of course $\boldsymbol{x}^2\boldsymbol{z} = \boldsymbol{y}^3$ and $\boldsymbol{z}^3 = 0$ themselves.

Given any symbol such as $|A|$ for a linear system, we denote by $F(A)$ the surface which is the projective model of that system. Thus $F(C)$ is a cubic surface, $F(G)$, $F(G')$ are Geiser double planes, and $F(2B)$ is a Bertini double cone.

3. The Geiser and Bertini figures. It is convenient to recall briefly the relations between the surfaces $F(G)$, $F(2G)$, $F(2B)$, $F(3B)$ in the general case, i.e. when the base points are general in position.* In the first place, let $f = 0$, $g = 0$ be any two curves of the pencil $|B|$: $(3; 1, 1, 1, 1, 1, 1, 1, 1)$, and $h = 0$ any curve of the net $|G|$: $(3; 1, 1, 1, 1, 1, 1, 1)$ that is not in $|B|$. Then (f, g, h)

*For the classical theory of the Geiser and Bertini involutions see Geiser **(9)**, Bertini **(3)**, and Baker **(2)**; for the tacnodal quartic, the Castelnuovo surface, and the double plane, projection of the Bertini double cone, see Nöther **(13)**, Castelnuovo **(5)**, and Conforto **(7)**.

are a base for $|G|$, and can be taken as homogeneous coordinates in the plane γ that carries the Geiser double plane $F(G)$. Any equation of degree n in (f, g, h) is the equation of an n-ic curve in γ, and also (on expressing f, g, h in terms of x, y, z) of the curve in π, whose image is this curve in γ, doubled and branching at its intersections with the branch curve of $F(G)$. A linear system of polynomials in (f, g, h) thus defines a linear system of curves in π, and also one in γ, the projective model of the former being that of the latter doubled, and branching along the image of the branch curve of $F(G)$. This branch curve b_G is the image of the jacobian

$$j_G = \frac{\partial(f, g, h)}{\partial(x, y, z)} = 0$$

in π; and as b_G is a quartic in γ, there is an identity

$$j_G^2 = \phi_4(f, g, h), \tag{3}$$

where ϕ_4 is a quartic polynomial. Thus $j_G = 0$ is a curve of $|2G|$.

The octavic surface $F(2G)$ of Castelnuovo in S_6 thus has the parametrization

$$X_0 : X_1 : \ldots : X_6 = j_G : f^2 : fg : fh : g^2 : gh : h^2, \tag{4}$$

and its equations are those that express that the matrix

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ X_2 & X_4 & X_5 \\ X_3 & X_5 & X_6 \end{bmatrix}$$

is of rank 1, namely

$$\begin{aligned} &X_1X_4 - X_2^2 = 0, \quad && X_1X_5 - X_2X_3 = 0, \quad && X_1X_6 - X_3^2 = 0, \\ &X_2X_5 - X_3X_4 = 0, && X_2X_6 - X_3X_5 = 0, && X_4X_6 - X_5^2 = 0, \end{aligned} \tag{5}$$

together with

$$\Psi = X_0^2 - \psi_2(X_1, \ldots, X_6) = 0, \tag{6}$$

expressing the identity (3), where

$$\psi_2(f^2, fg, fh, g^2, gh, h^2) = \phi_4(f, g, h);$$

the quadratic form ψ_2 is of course only determined modulo the left-hand members of (5). The equations (5) are those of a Veronese surface V^4 in the S_5 $X_0 = 0$, projective model of the conics in γ; in S_6 they are the equations of the cone Γ_3^4 projecting V^4 from K_0 (where K_i is the point at which all the coordinates vanish except X_i). $F(2G)$ is the section of Γ_3^4 by the quadric $\Psi = 0$; it projects from K_0 into V^4 doubled and branching along the image of b_G, which is its section by the quadric $\psi_2(X_1, \ldots, X_6) = 0$; the double V^4 has the parametrization (4) with j_G, X_0 omitted.

The system $|G|$ appears on $F(2G)$ as a net of elliptic quartics; for instance, the image of $h = 0$ is in the S_3 $K_0K_1K_2K_4$: $X_3 = X_5 = X_6 = 0$, which cuts Γ_3^4 in the quadric cone $X_1X_4 = X_2^2$, and $F(2G)$ in the section of this cone by $\psi = 0$. The projection of $F(2G)$ from a plane α in this S_3 is a quartic surface with a tacnode; the curve $h = 0$ is contracted into the tacnode (whose first neighbourhood is a double line with four pinch points, image of $h = 0$) and the four points in which α meets the quartic curve $h = 0$ are dilated into lines, which form the section of the projected surface by the tangent plane at the tacnode. From its tacnode this surface projects into the double plane $F(G)$. In particular, if α is the plane $K_1 K_2 K_4$: $X_0 = X_3 = X_5 = X_6 = 0$, the projected surface has the parametrization

$$x\colon y\colon z\colon t = X_3\colon X_5\colon X_0\colon X_6 = fh\colon gh\colon j_G\colon h^2, \tag{7}$$

and the equation, expressing the identity (3),

$$z^2t^2 = \phi_4(x, y, t). \tag{8}$$

There are in general on $F(2G)$ 28 curves of $|G|$ that break up into two conics, images of two lines on $F(G)$ that coincide in a bitangent of b_G. If c, c' are the conics corresponding to the neighbourhood of P_7, and the unique curve of $|G|$ that has a double point at P_7, the projection of $F(2G)$ from the plane of c' is the cubic surface $F(C)$, where $|C| = (3; 1, 1, 1, 1, 1, 1, 0)$. The conic c is contracted in the projection into a simple point of $F(C)$, image of P_7, and c' becomes the section of $F(C)$ by the tangent plane at this point. From the image of P_7, $F(C)$ projects into $F(G)$.

On the other hand, from the tangent plane at the image of P_8, $F(2G)$ projects into the Bertini double cone $F(2B)$. Each of the elliptic quartics on $F(2G)$ that pass through the image of P_8, and hence also through its companion point in the Geiser involution, the image of the ninth associated point P_9, projects into a double line with four branch points, one of which is at the projection of P_9. These are the generators of the double cone $F(2B)$; it has an isolated branch point at the vertex, image of P_9, and a branch curve meeting each generator in three points and not passing through the vertex, i.e. a cubic section of the cone. If $k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = 0$ is any curve of $|2B|$ not passing through P_9 (i.e. not breaking up into two curves of $|B|$), $F(2B)$ can be parametrized as

$$x_0\colon x_1\colon x_2\colon x_3 = k\colon f^2\colon fg\colon g^2, \tag{9}$$

so that the equation of the cone that carries it is $x_1 x_3 = x_2^2$. The branch curve b_B is the image of a curve of united points of the Bertini involution in π with which $|2B|$ is compounded; this curve in π is one of the system $|3B|$, and its equation is $j_B = 0$, where j_B is the greatest common divisor of the four jacobians

$$\frac{\partial(f^2, fg, g^2)}{\partial(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})}, \quad \frac{\partial(k, fg, g^2)}{\partial(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})}, \quad \frac{\partial(k, f^2, g^2)}{\partial(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})}, \quad \frac{\partial(k, f^2, fg)}{\partial(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})}.$$

$F(3B)$ can be parametrized in the form

$$Y_0\colon Y_1\colon \ldots\colon Y_6 = j_B\colon fk\colon gk\colon f^3\colon f^2g\colon fg^2\colon g^3, \tag{10}$$

and among its equations are those that express that the matrix

$$\begin{bmatrix} Y_1 & Y_3 & Y_4 & Y_5 \\ Y_2 & Y_4 & Y_5 & Y_6 \end{bmatrix}$$

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is of rank 1, namely

$$(11)\quad \begin{array}{lll} Y_1 Y_4 - Y_2 Y_3 = 0, & Y_1 Y_5 - Y_2 Y_4 = 0, & Y_1 Y_6 - Y_2 Y_5 = 0, \\ Y_3 Y_5 - Y_4^2 = 0, & Y_3 Y_6 - Y_4 Y_5 = 0, & Y_4 Y_6 - Y_5^2 = 0. \end{array}$$

In the S_5 $Y_0 = 0$, (11) are the equations of a rational ruled quartic surface R^4 with directrix line $L_1 L_2$ (where L_i is the point at which all the coordinates except Y_i vanish); in S_6 they are those of the cone Λ_3^4 projecting R^4 from L_0. $F(3B)$ lies on Λ_3^4 and projects from the vertex L_0 into R^4 doubled; this double R^4 is parametrized by (10), with the omission of j_B and Y_0; and as this linear system is compounded with the Bertini involution, the double R^4 is also a model of this involution; R^4 is the projective model of the system of all rational cubics on the Bertini double cone, and the total branch curve of the double R^4 consists of the directrix line $L_1 L_2$, image of the vertex of the cone, together with a curve, image of b_B on $F(2B)$ and of $j_B = 0$ in π, which meets each generator three times and does not meet the directrix line $L_1 L_2$, and is thus the residual section of R^4 by a cubic primal through any three generators. As this curve is the projection of the prime section $Y_0 = 0$ of $F(3B)$, $F(3B)$ is the residual section of Λ_3^4 by a cubic primal through any three generating planes. The pencil $|B|$ appears on $F(3B)$ as a pencil of plane cubics in the generating planes of Λ_3^4, each with an inflexion at the vertex L_0 and inflexional tangent in the directrix plane $L_0 L_1 L_2$; L_0 is a simple point of the surface, whose tangent plane $L_0 L_1 L_2$ has three-point contact, i.e. every prime through the plane cuts the surface in a curve with a triple point at L_0 (breaking up in fact into three curves of $|B|$), and every curve on the surface passing simply through L_0 either is inflected there or has $L_0 L_1 L_2$ as osculating plane.

An important special case arises when P_8 is chosen on the jacobian curve $j_G = 0$ of $|G|$. In this case P_9 is consecutive to P_8, the curves of $|B|$ all touch each other here, except that one of them, which we take to be $g = 0$, has a double point at P_8 and does not pass through P_9. This means that in γ, $g = 0$ touches b_G at $f = g = 0$. Thus $gh = 0$ is a curve of $|2B|$, not passing through P_9, and can be taken as our curve $k = 0$; gh, f^2, fg, g^2 are a base for $|2B|$, which is thus compounded with the Geiser involution, i.e. the Geiser and Bertini involutions are the same in this case. The curve $g = 0$ is fundamental to $|2B|$, and $j_B = gj_G$; in the parametrizations

$$(12)\qquad x_0 : x_1 : x_2 : x_3 = gh : f^2 : fg : g^2,$$

$$(13)\qquad Y_0 : Y_1 : \ldots : Y_6 = gj_G : fgh : g^2h : f^3 : f^2g : fg^2 : g^3$$

of $F(2B)$, $F(3B)$, taking (f, g, h) merely as coordinates in γ, we see that the Bertini cone is the projective model of the conics in γ, with a simple base point at $f = g = 0$ and another consecutive to it on $g = 0$, and R^4 is that of the cubics in γ, with a double base point at $f = g = 0$ and a simple one consecutive to it on $g = 0$. $F(G)$ is the projection of $F(2B)$ from $(0, 1, 0, 0)$, represented by the exceptional curve $g = 0$; this is a double point of b_B, which projects into b_G.

4. The cubic surface and the tacnodal quartic surface (I). For the treatment of these it is convenient (for the present section only) to modify the coordinate system $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in π, so that the rational cubic $\mathbf{x}^2\mathbf{z} = \mathbf{y}^3$ becomes $2\mathbf{x}^2\mathbf{z} = \mathbf{y}^3$. We accordingly parametrize the cubic surface $F(C)$ in the form

$$\text{(14)} \qquad \mathrm{x}:\mathrm{y}:\mathrm{z}:\mathrm{t} = \mathbf{x}\mathbf{z}^2 : \mathbf{y}\mathbf{z}^2 : \mathbf{z}^3 : \mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z},$$

when its equation is easily seen to be

$$\text{(15)} \qquad \mathrm{z}^2\mathrm{t} + 2\mathrm{x}^2\mathrm{z} - \mathrm{y}^3 = 0.$$

This has a unode at (0, 0, 0, 1) with tangent cone $\mathrm{z}^2 = 0$. Since

$$\mathrm{x}:\mathrm{y}:\mathrm{z} = \mathbf{x}:\mathbf{y}:\mathbf{z},$$

the mapping on π is the obvious one by projection from the unode. As (15) can be written

$$(\mathrm{zt} + \mathrm{x}^2)^2 = \mathrm{x}^4 + \mathrm{y}^3\mathrm{t},$$

$F(C)$ projects from (0, 0, 1, 0), the image of P': (0, 0, 1) in π, into the Geiser double plane $F(G')$, with branch curve $\mathrm{x}^4 + \mathrm{y}^3\mathrm{t} = 0$; this is a rational quartic, with a triple point at (0, 0, 1), at which is one branch of order 3 with tangent $\mathrm{y} = 0$, and only simple points consecutive to it; and the curve has four-point contact with $\mathrm{t} = 0$ at (1, 0, 0).

$$j_{G'} = \frac{1}{6}\frac{\partial(\mathbf{x}\mathbf{z}^2, \mathbf{y}\mathbf{z}^2, \mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z})}{\partial(\mathbf{x}, \mathbf{y}, \mathbf{z})} = \mathbf{z}^3(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3);$$

the image of the point $(1, \mu, \mu^3)$ of $\mathbf{x}^2\mathbf{z} = \mathbf{y}^3$ is the point $(\mu^3, \mu^4, -1)$ of

$$\mathrm{x}^4 + \mathrm{y}^3\mathrm{t} = 0;$$

the factor $\mathbf{z}^3$ in $j_{G'}$ corresponds only to partial neighbourhoods of the point (0, 0, 0, 1) on $F(C)$, and also on $F(G')$.

The surface (I) in the homogeneous form

$$\text{(16)} \qquad z^2t^2 = x^4 + x^3t$$

is a quartic with a tacnode at (0, 0, 1, 0) (with tangent cone $t^2 = 0$), from which it projects into the same double plane $F(G')$. It is equivalent to $F(C)$ under the birational transformation

$$x:y:z:t = \mathrm{xt}:\mathrm{yt}:\mathrm{zt} + \mathrm{x}^2 : \mathrm{t}^2,$$

$$\mathrm{x}:\mathrm{y}:\mathrm{z}:\mathrm{t} = xt:yt:zt - x^2 : t^2,$$

which is regular at (0, 0, 0, 1) and does not affect the nature of the singularity there. Its parametrization is accordingly

$$x:y:z:t = \mathbf{x}\mathbf{z}^2(\mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z}) : \mathbf{y}\mathbf{z}^2(\mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z}) : \mathbf{z}^3(\mathbf{y}^3 - \mathbf{x}^2\mathbf{z}) : (\mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z})^2.$$

The relation between the two surfaces is best understood by noting that both are projections of the same Castelnuovo surface $F(2G')$ from different planes,

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both skew to the tangent S_3 of $\Gamma_3{}^4$ at the unode of $F(2G')$, so that in both cases the projection maps $\Gamma_3{}^4$ onto S_3 regularly at this point. As the parametrization of $F(G')$ is

$$t: x: y = \mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z}: \mathbf{x}\mathbf{z}^2: \mathbf{y}\mathbf{z}^2 = f: g: h \quad \text{say,}$$

that of $F(2G')$ is

$$X_0: X_1: \ldots : X_6 = \mathbf{z}^3(\mathbf{y}^3 - \mathbf{x}^2\mathbf{z}): (\mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z})^2: \mathbf{x}\mathbf{z}^2(\mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z}):$$
$$\mathbf{y}\mathbf{z}^2(\mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z}): \mathbf{x}^2\mathbf{z}^4: \mathbf{x}\mathbf{y}\mathbf{z}^4: \mathbf{y}^2\mathbf{z}^4,$$

and it is the section of $\Gamma_3{}^4$, with equations (5), by the quadric

$$X_0{}^2 = X_3X_6 + X_4{}^2 \tag{17}$$

expressing the identity $j_{G'}{}^2 = fh^3 + g^4$. As

$$x: y: z: t = X_2: X_3: X_0: X_1,$$

the surface (I) is the projection of $F(2G')$ from the plane

$$X_0 = X_1 = X_2 = X_3 = 0;$$

and as

$$\mathrm{x}: \mathrm{y}: \mathrm{z}: \mathrm{t} = X_2: X_3: X_0 - X_4: X_1,$$

$F(C)$ is its projection from the plane $X_0 - X_4 = X_1 = X_2 = X_3 = 0$. The S_3 $X_1 = X_2 = X_3 = 0$ meets $\Gamma_3{}^4$ in the quadric cone $X_4X_6 = X_5{}^2$, and meets the quadric (17) in the pair of planes $X_0 = \pm X_4$; hence it meets $F(2G')$ in two conics c, c', one in each of these planes, which touch each other and the plane $X_0 = 0$ at K_4. Thus $F(C)$ is the projection of $F(2G')$ from the plane of the conic c', and the conic c is contracted in the projection to the point $(0, 0, 1, 0)$ on $F(C)$, the image of P' in π. (I), on the other hand, is the projection of $F(2G')$ from a plane which meets it in four points at K_4, consecutive along the section of the surface by $X_0 = 0$, which is the image of $j_{G'} = 0$. The linear system of sextics

$$\lambda\mathbf{x}\mathbf{z}^2(\mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z}) + \mu\mathbf{y}\mathbf{z}^2(\mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z}) + \nu\mathbf{z}^3(\mathbf{y}^3 - \mathbf{x}^2\mathbf{z}) + \rho(\mathbf{y}^3 - 2\mathbf{x}^2\mathbf{z})^2 = 0$$

of which the surface (I) is the projective model has accordingly six double base points $P_1, \ldots, P_6$, consecutive along the inflected branch of $\mathbf{y}^3 = 2\mathbf{x}^2\mathbf{z}$ at $(1, 0, 0)$, and one double and four simple base points P', $Q_1, \ldots, Q_4$, consecutive along the cuspidal branch of $\mathbf{y}^3 = \mathbf{x}^2\mathbf{z}$ at $(0, 0, 1)$. The curve $\mathbf{y}^3 = 2\mathbf{x}^2\mathbf{z}$ is fundamental to this system, and represents the tacnode of (I), whose first neighbourhood consists of two coincident lines, images of this curve and of the neighbourhood of P' in π, and of the conics c', c on $F(2G')$.

5. The tacnodal quartic surface (II). We now return to the coordinate system of §2 in π, and define

$$f = \mathbf{x}^2\mathbf{z} - \mathbf{y}^3, \qquad g = \mathbf{z}^3, \qquad h = \mathbf{y}\mathbf{z}^2 \tag{18}$$

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as a base for $|G|$. The jacobian of these, omitting a numerical factor, is $j = \mathbf{xz}^5$, so that

$$(19) \qquad j^2 = fg^3 + gh^3.$$

The branch curve of the Geiser double plane $F(G)$ thus consists of the rational cubic b: $fg^2 + h^3 = 0$, which has a cusp at $g = h = 0$ with tangent $g = 0$, and an inflexion at $f = h = 0$, with tangent $f = 0$; together with the line $g = 0$, which is the unique line on $F(G)$, image of s_7. The involution I with which $|G|$ is compounded is that whose pairs are interchanged by the harmonic homology $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leftrightarrow (-\mathbf{x}, \mathbf{y}, \mathbf{z})$, and its loci of united points are $\mathbf{x} = 0$ and the neighbourhood of P_1: $(1, 0, 0)$. The former corresponds to the curve b, the image of $(0, 1, \mu)$ in π being $(-1, \mu^3, \mu^2)$ in γ; the latter, on Π, appears as a linear combination of $s_1, \ldots, s_7, s^*$, and on any surface, such as $F(G)$, on which all these are contracted except s_7, it appears as the image of s_7, in this case the line $g = 0$.

The corresponding Castelnuovo surface $F(2G)$ has accordingly the parametrization

$$(20) \quad X_0 : X_1 : \ldots : X_6 = \mathbf{xz}^5 : (\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)^2 : \mathbf{z}^3(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3) : \mathbf{yz}^2(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3) : \mathbf{z}^6 : \mathbf{yz}^5 : \mathbf{y}^2\mathbf{z}^4$$

by substitution from (19) in (4). Its equations are these of $\Gamma_3{}^4$ in the form (5), together with that of a quadric primal Ψ:

$$X_0^2 = X_2 X_4 + X_5 X_6$$

expressing the identity (19). The unode U^*_9 is at K_1, since any linear combination of the expressions on the right of (20), except $(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)^2$, is divisible by $\mathbf{z}^2$, and the remaining factor, a linear combination of $\mathbf{xz}^3$, $\mathbf{x}^2\mathbf{z}^2 - \mathbf{y}^3\mathbf{z}$, $\mathbf{x}^2\mathbf{yz} - \mathbf{y}^4$, $\mathbf{z}^4$, $\mathbf{yz}^3$, $\mathbf{y}^2\mathbf{z}^2$, represents a general curve of the system (4; 2, 1, 1, 1, 1, 1, 1), as required by (2). The tangent S_3 to $\Gamma_3{}^4$ at K_1 is spanned by the generator $K_1 K_0$ and the tangents $K_1 K_2$, $K_1 K_3$ to V^4, i.e. it is the S_3 $X_4 = X_5 = X_6 = 0$. The plane $K_4 K_5 K_6$: $X_0 = X_1 = X_2 = X_3 = 0$ is skew to this S_3, and from it $\Gamma_3{}^4$ projects into S_3 regularly at K_1, so that the unode U^*_9 at this point is not affected. The projected surface has the parametrization

$$\begin{aligned} x : y : z : t &= X_3 : X_2 : X_0 : X_1 = fh : fg : j : f^2 \\ &= \mathbf{yz}^2(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3) : \mathbf{z}^3(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3) : \mathbf{xz}^5 : (\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)^2 \end{aligned}$$

and the equation

$$z^2t^2 = x^3y + y^3t$$

expressing the identity (19). This, however, is the homogeneous form of the equation (II). Thus the surface (II) is the projection of $F(2G)$ from the plane $K_4 K_5 K_6$.

The S_3 $K_0 K_4 K_5 K_6$: $X_1 = X_2 = X_3 = 0$, cuts $F(2G)$ in the quartic curve of intersection of the quadric cones $X_5{}^2 = X_4 X_6$ (the section of $\Gamma_3{}^4$) and $X_0{}^2 = X_5 X_6$ (the section of Ψ). This is the image of the double line $f = 0$ on

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$F(G)$, and is a rational quartic with a cusp at K_4; putting $\mathbf{x}: \mathbf{y}: \mathbf{z} = \mu^3: \mu^2: 1$ in (20), we get

$$X_0: X_1: \ldots : X_6 = \mu^3: 0: 0: 0: 1: \mu^2: \mu^4.$$

The plane $X_0 = 0$ in this S_3 meets the curve in three coincident points at K_4 and one at K_6. It thus meets $F(2G)$ in three points at K_4 and one at K_6. The section of $F(2G)$ by $X_0 = 0$ is of course the image of the branch curve of $F(G)$, and consists of a conic c, image of the line $g = 0$ on $F(G)$ and of s_7 on Π, given by

$$X_0 = X_2 = X_4 = X_5 = 0, \qquad X_1: X_3: X_6 = f^2: fh: h^2,$$

and the sextic curve, image of b on $F(G)$ and of $x = 0$ in π,

$$X_0 = 0, \qquad X_1: \ldots : X_6 = 1: \tau^3: \tau^2: \tau^6: \tau^5: \tau^4 \quad (\tau = \mathbf{z}/\mathbf{y}),$$

with a cusp at K_1. K_4 is on the latter curve, and is the image of P' in π; K_6 is on the former, and is the image of P_8. The linear system of sextics

$$\lambda \mathbf{y}\mathbf{z}^2(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3) + \mu \mathbf{z}^3(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3) + \nu \mathbf{x}\mathbf{z}^5 + \rho(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)^2 = 0$$

whose projective model is the surface (II) has thus seven double base points and one simple one $P_1, \ldots , P_8$, consecutive on the inflected branch of $\mathbf{y}^3 = \mathbf{x}^2\mathbf{z}$ at (1, 0, 0), and three simple base points P', Q_1, R, consecutive on $\mathbf{x} = 0$ at (0, 0, 1).

6. The quintic surface (III). It is clear that the surfaces (II), (III) are very closely related, as we should expect from the fact that U^*_{10} has a U^*_9 in its first neighbourhood. On the one had, applying the ordinary dilating substitution of (x, xy, xz) for (x, y, z) in affine coordinates to $z^2 = x^5 + y^3$, we obtain (on removing the factor x^2 for the neighbourhood in S_3 of the double point) $z^2 = x^3 + xy^3$, which is (II), merely with x, y interchanged. On the other hand, the surface $F(3B) = F(9; 3, 3, 3, 3, 3, 3, 3, 3)$, on projection from its unode U^*_{10}, gives, as we have seen in §2, the surface $F(6; 2, 2, 2, 2, 2, 2, 2, 1)$, which is also the projection of $F(2G) = F(6; 2, 2, 2, 2, 2, 2, 2)$ from K_6, the image of P_8. This surface has on it two lines l, l', the images of s_7, s_8; in the projection from $F(2G)$, l arises from the unique conic c on $F(2G)$, and l' from the neighbourhood of K_6; in the projection from $F(3B)$, l arises from the neighbourhood of the unode, and l' from the unique exceptional curve on $F(3B)$, which is, as we shall see, a plane cubic with a cusp at the unode. The surface lies on a cone projecting a ruled cubic from a point, which is equally the projection of $\Gamma_3{}^4$ and $\Lambda_3{}^4$, the plane that arises from the neighbourhood of the projecting point being the directrix plane and a generating plane in the two cases; and the surface $F(6; 2, 2, 2, 2, 2, 2, 2, 1)$ is the residual section of this cone by a cubic primal through two generating planes.

As P_8 is in the neighbourhood of P_7, which as we have seen represents a constituent of the branch curve of $F(G)$, we have the special case envisaged at the end of §3. It is easily verified that $|2B|$ has the base gh, f^2, fg, g^2, where f, g, h are still given by (18); and the jacobians by threes of

$$gh = \mathbf{y}\mathbf{z}^5, \qquad f^2 = (\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)^2, \qquad fg = \mathbf{z}^3(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3), \qquad g^2 = \mathbf{z}^6$$

with respect to $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ have the greatest common divisor $gj = \mathbf{x}\mathbf{z}^8$. The parametrization (12) of the Bertini double cone $F(2B)$ is thus

$$(21) \qquad x_0\colon x_1\colon x_2\colon x_3 = \mathbf{y}\mathbf{z}^5\colon (\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)^2\colon \mathbf{z}^3(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)\colon \mathbf{z}^6.$$

The branch curve is obtained by putting $\mathbf{x} = 0$ in this, and is the rational sextic curve

$$x_0\colon x_1\colon x_2\colon x_3 = \tau^5\colon 1\colon -\tau^3\colon \tau^6 \qquad (\tau = \mathbf{z}/\mathbf{y}),$$

the section of the Bertini cone $x_2^2 = x_1 x_3$ by the cubic cone

$$x_0^3 + x_2 x_3^2 = 0.$$

It has a triple point at (0, 1, 0, 0), with a double point consecutive to it along $x_0 = x_3 = 0$. The double cone projects from (0, 0, 0, 1), the point $\tau = \infty$ of the branch curve and image of P' in π, into the double plane

$$(22) \qquad x_0\colon x_1\colon x_2 = gh\colon f^2\colon fg$$

and the branch curve into the quintic

$$x_0\colon x_1\colon x_2 = \tau^5\colon 1\colon -\tau^3,$$

whose equation is

$$x_2^5 + x_0^3 x_1^2 = 0,$$

still with a triple point at (0, 1, 0) and a double point consecutive to it along $x_0 = 0$. This plane quintic has also a cusp of the second kind at (1, 0, 0), with two double and one simple point consecutive on $x_1 = 0$. Also, as the point (0, 0, 0, 1) from which we project is on the branch curve of the double cone, the line $x_1 = 0$ into which it is dilated by the projection is a constituent of the branch curve of the double plane; the total branch curve of the latter is thus

$$(23) \qquad x_1(x_2^5 + x_0^3 x_1^2) = 0,$$

a sextic with two consecutive triple points at (1, 0, 0), as we expect for the plane projection of any Bertini double cone.

The parametrization (13) of $F(3B)$ becomes

$$(24) \quad Y_0\colon Y_1\colon \ldots\colon Y_6 = \mathbf{x}\mathbf{z}^8\colon \mathbf{y}\mathbf{z}^5(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)\colon \mathbf{y}\mathbf{z}^8\colon (\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)^3\colon \mathbf{z}^3(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)^2\colon \mathbf{z}^6(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)\colon \mathbf{z}^9,$$

and the projection of $F(3B)$ from L_3 has the parametrization

$$Y_0\colon Y_1\colon Y_2\colon Y_4\colon Y_5\colon Y_6 = j\colon fh\colon gh\colon f^2\colon fg\colon g^2 = X_0\colon X_3\colon X_5\colon X_1\colon X_2\colon X_4$$

by (4), and is thus the same surface as the projection of $F(2G)$ from K_6. L_3 is accordingly the unode U^*_{10} on $F(3B)$.

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If Λ_3^4 is parametrized as

$$Y_0 : Y_1 : \ldots : Y_6 = w : uv : v : u^3 : u^2 : u : 1, \tag{25}$$

the generating planes are u = constant, $v = \infty$ is the directrix plane, and (for general u, v) $w = \infty$ at the vertex L_0. Comparing this with (13), we have

$$u = f/g, \qquad v = h/g, \qquad w = j/g^2,$$

so that the equation of $F(3B)$, expressing the identity (19) in terms of these parameters, is

$$w^2 = u + v^3. \tag{26}$$

Defining now the four cubic forms

$$\Phi_0 = Y_0^2 Y_3 - Y_4 Y_5^2 - Y_1^3, \qquad \Phi_1 = Y_0^2 Y_4 - Y_3 Y_6^2 - Y_1^2 Y_2,$$

$$\Phi_2 = Y_0^2 Y_5 - Y_4 Y_6^2 - Y_1 Y_2^2, \qquad \Phi_3 = Y_0^2 Y_6 - Y_5 Y_6^2 - Y_2^3,$$

we see that the general linear combination

$$\lambda\Phi_0 + \mu\Phi_1 + \nu\Phi_2 + \rho\Phi_3 = 0$$

of these is the equation of a primal cutting Λ_3^4 in the surface

$$(w^2 - u - v^3)(\lambda u^3 + \mu u^2 + \nu u + \rho) = 0,$$

consisting of $F(3B)$ together with a general set of three generating planes; and the equations of $F(3B)$ can be taken to be (11), together with $\Phi_0 = \Phi_3 = 0$.

The generating planes of Λ_3^4 trace on $F(3B)$ a pencil of plane cubics, with a simple base point at L_0, which is an inflexion on each of them; these are the images of the pencil $|B|$, and they are all equi-anharmonic, being given by (26) for each constant value of u, except those in the planes $u = \infty$: $L_0 L_1 L_3$, and $u = 0$: $L_0 L_2 L_6$. These are both cuspidal, as putting $Y_2 = Y_4 = Y_5 = Y_6 = 0$ in $\Phi_0 = 0$ we have $Y_0^2 Y_3 = Y_1^3$, and putting $Y_1 = Y_3 = Y_4 = Y_5 = 0$ in $\Phi_3 = 0$ we have $Y_0^2 Y_6 = Y_2^3$. The former is the unique exceptional curve on $F(3B)$, image of s_8, and its cusp is at the unode L_3; the latter is the image of $f = 0$, and is not exceptional, but has virtual genus 1 on the surface; its cusp is at L_6, the image of P', and corresponds to the cusp of $f = 0$ in π, the mapping of $F(3B)$ on π being regular here.

The tangent S_3 to Λ_3^4 at the unode L_3 is spanned by the generating plane $L_0 L_1 L_3$ and the tangent $L_3 L_4$ to the curve $\mu = \nu = 0$. It is thus the S_3

$$Y_2 = Y_5 = Y_6 = 0.$$

The plane $L_2 L_5 L_6$: $Y_0 = Y_1 = Y_3 = Y_4 = 0$ is skew to this S_3, so that the projection of Λ_3^4 onto S_3 from this plane is regular at L_3. The projected surface can be parametrized as

$$\begin{aligned} x : y : z : t &= Y_4 : Y_1 : Y_0 : Y_3 = f^2 g : fgh : gj : f^3 \\ &= \mathbf{z}^3(\mathbf{x}^3\mathbf{z} - \mathbf{y}^3)^2 : \mathbf{y}\mathbf{z}^5(\mathbf{x}^2\mathbf{z} - \mathbf{y}^3) : \mathbf{x}\mathbf{z}^8 : (\mathbf{x}^2\mathbf{z} - \mathbf{y}^3)^3 \end{aligned} \tag{27}$$

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and its equation, expressing the identity (19), is

$$z^2t^3 = x^5 + y^3t^2, \tag{28}$$

the homogeneous form of (III). Thus the surface (III) is the projection of $F(3B)$ from the plane $L_2 L_5 L_6$.

The section of $F(3B)$ by the prime $Y_0 = 0$ is the image of the branch curve of $F(2B)$, of the partial branch curve b of $F(G)$, and of the line $\boldsymbol{x} = 0$ in π. Putting $\boldsymbol{x} = 0$ in (24) we find the parametrization of this curve:

$$Y_0\colon Y_1\colon \ldots\colon Y_6 = 0\colon \tau^5\colon \tau^8\colon 1\colon \tau^3\colon \tau^6\colon \tau^9 \qquad (\tau = \boldsymbol{z}/\boldsymbol{y}), \tag{29}$$

so that it has a triple point at L_3 with a double point consecutive to it on $L_3 L_4$. The plane $L_2 L_5 L_6$ meets this curve in four consecutive points at $\tau = \infty$, the point L_6, image of P', since any linear equation in (Y_0, Y_1, Y_3, Y_4) is by (29) at most quintic in τ. Thus the linear system of nonic curves

$$\lambda\boldsymbol{z}^3(\boldsymbol{x}^2\boldsymbol{z} - \boldsymbol{y}^3)^2 + \mu\boldsymbol{y}\boldsymbol{z}^5(\boldsymbol{x}^2\boldsymbol{z} - \boldsymbol{y}^3) + \nu\boldsymbol{x}\boldsymbol{z}^8 + \rho(\boldsymbol{x}^2\boldsymbol{z} - \boldsymbol{y}^3)^3 = 0 \tag{30}$$

of which (III) is the projective model, as well as its eight double base points $P_1, \ldots, P_8$ consecutive on the inflected branch of $\boldsymbol{x}^2\boldsymbol{z} - \boldsymbol{y}^3 = 0$ at $(1, 0, 0)$, has four simple base points P', Q_1, R, R' consecutive on $x = 0$ at P': $(0, 0, 1)$.

The surface (III), being a quintic surface with plane sections of genus 4, must have a double curve of total order 2; and as the surface is rational, it must have singularities which debar it from having an effective adjoint plane. In fact, it is evident from (28) that the line $x = t = 0$ is a cuspidal edge of the second kind, the plane $t = 0$ cutting the surface in two double lines and a simple line, all consecutive. This is the image of the neighbourhood of the final simple base point R' of the mapping system (30), and is the only line on the surface. $f = 0$ is fundamental to the system (30), and with the neighbourhoods of P', Q_1, R represents a triple point of the surface at $(0, 0, 1, 0)$, from which it projects into the double plane with branch curve $x^5t + y^3t^3 = 0$, which is the same as the projection of $F(2B)$ from the image of P', as comparing (21) with (27),

$$x\colon y\colon t = fg\colon gh\colon f^2 = x_2\colon x_0\colon x_1.$$

The tangent cone at this triple point is $t^3 = 0$. The dilating transformation

$$x\colon y\colon z\colon t = x'y'\colon y'z'\colon z'^2\colon y't',$$

$$x'\colon y'\colon z'\colon t' = xz\colon y^2\colon yz\colon zt$$

maps the neighbourhood of $(0, 0, 1, 0)$ on the plane $y' = 0$ (with singularity of the mapping on the line $y' = z' = 0$) and in particular the point consecutive on $x = t = 0$ on the point $(0, 0, 1, 0)$, and the line consecutive in the plane $t = 0$ on the line $y' = t' = 0$. It transforms the surface (III) into the surface F':

$$z'^4t'^3 = y'^2(x'^5 + z'^3t'^2)$$

on which the line $y' = t' = 0$ is a cuspidal edge with the tangent plane $t' = 0$ at all its points. Thus (III) has not only the two consecutive double lines in the plane $x = 0$, and the triple point $(0, 0, 1, 0)$, but a double line in the neighbourhood of $(0, 0, 1, 0)$ in the tangent plane $t = 0$; and these are sufficient to reduce the genus of the surface to 0. It may be noted that it has a further triple point consecutive to $(0, 0, 1, 0)$ on $x = t = 0$, and further singularities in the neighbourhood of this; but the detailed analysis of these hardly seems worth while.

References

1. M. Artin, *Some numerical criteria for contractability of curves on an algebraic surface*, Amer. J. Math., *84* (1962), 485–496.
2. H. F. Baker, *Principles of geometry* (Cambridge, 1933), pp. 121–131.
3. E. Bertini, *Ricerche sulle trasformazioni univoche involutorie nel piano*, Ann. di Mat., *8* (1877), 244–286.
4. E. Brieskorn, *Uber die Auflösung gervisser Singularitäten von holomorphen Abbildungen*, Math. Ann., *166* (1966), 76–102.
5. G. Castelnuovo, *Sulle superficie algebriche le cui sezioni sono curve di genere 3*, Atti R. Ist. Sci. Torino, *25* (1880), 695–715.
6. A. Cayley, *A memoir on cubic surfaces*, Philos. Trans. Roy. Soc., *159* (1869), 231–236; Coll. Papers, *6* (1893), 359–455.
7. F. Conforto, *Le superficie razionali* (Bologna, 1939), pp. 187–193.
8. P. Du Val, *On isolated singularities of surfaces which do not affect the conditions of adjunction*, Proc. Cambridge Philos. Soc., *30* (1934), 453–459.
9. A. Geiser, *Ueber zwei geometrische Probleme*, J. für Math., *67* (1876), 78–83.
10. J. Herszberg, *Algebraic characterisation of types of unodes of surfaces in S_3*, J. London Math. Soc., *32* (1957), 187–198.
11. F. Hirzebruch, *The topology of normal singularities of an algebraic surface*, Sém. Bourbaki, 15e année, No. 250 (1962–63).
12. D. Kirby, *The structure of isolated multiple points of a surface, I, II, III*, Proc. London Math. Soc. (3), *6* (1956), 597–609 and *7* (1957), 1–28.
13. M. Nöther, *Uber eine Classe von auf die einfache Ebene abbildbaren Doppelebenen*, Math. Ann., *33* (1889), 525–544; *Ueber die rationalen Flächen vierter Ordnung*, Math. Ann., *33* (1889), 545–571.
14. R. von Randow, *Zur Topologie der Baummannigfaltigkeiten*, Bonn. Math. Schr., *14* (Bonn, 1962).

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