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Bounded-to-1 factors of an aperiodic shift of finite type are 1-to-1 almost everywhere factors also

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Abstract We show that if π $\Sigma_G \to \Sigma_H$ is a bounded-to-1 factor map from an irreducible shift of finite type Σ_G with period p_G to a shift of finite type Σ_H with period p_H, then there is a factor map $\hat{\pi}$ $\Sigma_G \to \Sigma_H$ that is (p_G/p_H)-to-1 almost everywhere Moreover, if π is right closing, then $\hat{\pi}$ may be taken to be right closing also

1 *Introduction*

We prove the following result

THEOREM 1 1 *If* π $\Sigma_G \to \Sigma_H$ *is a bounded-to-*1 *factor map from an irreducible shift of finite type* Σ_G *with period* p_G *to a shift of finite type* Σ_H *with period* p_H, *then there is a factor map* $\hat{\pi}$ $\Sigma_G \to \Sigma_H$ *that is* (p_G/p_H)*-to-*1 *almost everywhere Moreover, if* π *is right closing, then* $\hat{\pi}$ *may be taken to be right closing also*

In particular, if Σ_G is aperiodic, then $\hat{\pi}$ $\Sigma_G \to \Sigma_H$ is 1-to-1 almost everywhere It is easy to show that p_G/p_H is the smallest possible degree of a factor map from a shift of period p_G to a shift of period p_H

This result generalizes a result in [**AGW**] where the range shift is the full n-shift

As was pointed out to me by Bruce Kitchens and Brian Marcus, this result simplifies the proof of the main theorem in [**AM**] that topological entropy and period are a complete set of invariants for almost topological conjugacy

2 *Background*

We assume some familiarity with shifts of finite type § 3 of [**AM**] and § 2 of [**BMT**] are good introductions We make some definitions here in order to establish notation

Given a strongly connected directed graph G with a finite set of states $\mathscr{S}$ and at most one edge from any state to any other, we define the shift of finite type Σ_G by

$$\Sigma_G = \{s \in \mathscr{S}^{\mathbb{Z}} \quad s_i s_{i+1} \text{ is an edge in } G \text{ for } i \in \mathbb{Z}\}$$

This definition follows [**AM**] rather than [**BMT**] In [**BMT**] the defining graph G may have many parallel edges from one state to another, and the symbols in the shift Σ_G are the *edges* of G, not the *states* of G The definitions are equivalent up to conjugacy

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The set Σ_G is topologized by the product of the discrete topologies on its coordinate spaces

The shift map σ $\Sigma_G \to \Sigma_G$ defined by

$$(\sigma x)_i = x_{i+1}$$

is a homeomorphism

The *period* of Σ_G is the greatest common divisor of all cycle lengths in the graph G

Given a finite path of states $s_1 s_2 \quad s_k$ in the graph G, we denote

$$_n[s_1 s_2 \quad s_k]_{n+k-1} = \{x \in \Sigma_G \quad x_{n+i-1} = s_i, 1 \le i \le k\}$$

This set called a *k-block* of Σ_G

Given $y \in \Sigma_G$, we denote the finite path $y_i y_{i+1} \quad y_j$ in G by $_i(y)_j$

A *k-block* map π $\Sigma_G \to \Sigma_H$ is a shift-commuting map such that there is some l for which

$$(\pi y)_0 = (\pi y')_0 \quad \text{if } {}_{l-k+1}(y)_l = {}_{l-k+1}(y')_l$$

In the 1-block case we require merely for notational convenience that $l = 0$ In the 1-block case we have

$$(\pi y)_0 = (\pi y')_0 \quad \text{if } y_0 = y'_0$$

Thus π is defined by a map from single states of G to single states of H that we again call π In this case we say that a path of states $s_1 s_2 \quad s_k$ in G is *π-labelled* by $\pi(s_1)\pi(s_2) \quad \pi(s_k) = \pi(s_1 s_2 \quad s_k)$

A *bounded-to-*1 factor map π $\Sigma_G \to \Sigma_H$ is a k-block map such that the set of positive integers $\{\#\pi^{-1}(y) \quad y \in \Sigma_H\}$ is bounded from above

A 1-block map π $\Sigma_G \to \Sigma_H$ is *right-closing* if it never identifies two distinct left asymptotic points if s, $s' \in \Sigma_G$ have an $l_0 \in \mathbb{Z}$ such that $s_l = s'_l$ for all $l \le l_0$ and $\pi(s) = \pi(s')$ then $s = s'$

A 1-block map π $\Sigma_G \to \Sigma_H$ is *right-resolving* if for every path $t_1 t_2$ of length 2 in H, and for every state s_1 of G with $\pi(s_1) = t_1$, there is a unique state s_2 such that $s_1 s_2$ is an edge of G and $\pi(s_2) = t_2$

3 *Resolving blocks*

If π $\Sigma_G \to \Sigma_H$ is bounded-to-1, then the minimum d of $\{\#\pi^{-1}(y) \quad y \in H\}$ is the generic degree of π except for a set of measure zero in Σ_G (with respect to the measure of maximal entropy) π is a d-to-1 map **[KMT]** We call d the *degree* of π after **[B]**

The degree of a 1-block factor map π $\Sigma_G \to \Sigma_H$ is the smallest integer d such that there is a path $m_1 m_2 \quad m_k$ in the graph H, an integer l, $1 \le l \le k$, and a set $\{r^1, r^2, \quad, r^d\}$ of d states in the graph G such that every path $s_1 s_2 \quad s_k$ in G with $\pi(s_1 s_2 \quad \cdot s_k) = m_1 m_2 \cdot \cdot \quad m_k$ has $s_l \in \{r^1, r^2, \quad, r^d\}$ **[KMT]** The path $m_1 m_2 \quad m_k$ is a *resolving block* for the map π We use the following construction from **[KMT]** to reduce to a convenient special case of Theorem 1 1

Given a shift of finite type Σ_H define the *k-block presentation* of Σ_H to be the shift of finite type $\Sigma_H^{[k]}$ whose symbols are the paths of length k in H, with a transition from symbol $s_1 s_2 \quad s_k$ to symbol $t_1 t_2 \quad t_k$ iff $s_2 s_3 \quad s_k = t_1 t_2 \quad t_{k-1}$ The k-block

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map ψ_k $\Sigma_H \to \Sigma_H^{[k]}$ defined by mapping the path $s_1 s_2 \quad s_k$ in H to the symbol $s_1 s_2 \quad s_k$ in $\Sigma_H^{[k]}$ is a conjugacy

Given a 1-block map π $\Sigma_G \to \Sigma_H$ and integers k and l with $1 \le l \le k$, define the shift of finite type $\Sigma_G^{k,l}$ as follows The symbols to $\Sigma_G^{k,l}$ are the equivalence classes of paths of length k in G where path $s_1 s_2 \quad s_k$ is equivalent to $s'_1 s'_2 \quad s'_k$ iff

(i) $$\pi(s_1 s_2 \quad s_k) = \pi(s'_1 s'_2 \quad s'_k)$$

and

(ii) $$s_l = s'_l$$

There is a transition in $\Sigma_G^{k,l}$ from equivalence class s to equivalence class t iff there is a path $s_1 s_2 \quad s_k s_{k+1}$ in G such that $s_1 s_2 \quad s_k \in s$ and $s_2 s_3 \quad s_{k+1} \in t$ The k-block map $\varphi_{k,l}$ $\Sigma_G \to \Sigma_G^{k,l}$ taking a path of length k in G to the equivalence class containing it is a conjugacy

Define the 1-block map $\pi_{k,l}$ $\Sigma_G^{k,l} \to \Sigma_H^{[k]}$ by taking a symbol of $\Sigma_G^{k,l}$ (which is an equivalence class of paths of length k in G) to the common π-label of its elements

THEOREM 3 1 ([**KMT**]) *The diagram*

$$\begin{array}{ccc} \Sigma_G & \xrightarrow{\varphi_{k,l}} & \Sigma_G^{k,l} \\ \pi \downarrow & & \downarrow \pi_{k,l} \\ \Sigma_H & \xrightarrow{\psi_k} & \Sigma_H^{[k]} \end{array}$$

commutes Moreover, if $m_1 m_2 \quad m_k$ is a path in H that is a resolving block for π, and l, $1 \le l \le k$, is as in the definition of a resolving block, then $m_1 m_2 \quad m_k$ is a resolving symbol for $\pi_{k,l}$

We also use the following lemma essentially contained in [**KMT**] regarding bounded-to-1 factor maps

PERMUTATION LEMMA 3 2 *Let π $\Sigma_G \to \Sigma_H$ be a degree d 1-block map with resolving symbol m Let $m^1, m^2, \quad, m^d$ be the states in G with $\pi(m^i) = m$, $1 \le i \le d$ For each path of the form mum in H, there are paths $u^1, u^2, \quad, u^d$ in G and a permutation τ_u of $\{1, 2, \quad, d\}$ such that the paths of G π-labelled by mum are exactly $m^i u^i m^{\tau_u(i)}$, $1 \le i \le d$*

4 *Proof of the main theorem*

In case the entropy of Σ_G is zero, Σ_G and Σ_H each consist of a single finite orbit and Theorem 1 1 holds trivially The rest of this section treats the positive entropy case

We first reduce to a special case Suppose the given map π $\Sigma_G \to \Sigma_H$ is a k-block map

As composition (on either side) with conjugacies preserves both degree and the property of being right-closing, we can reduce to the case where π $\Sigma_G \to \Sigma_H$ is a

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1-block map by replacing π with $\varphi\circ\pi$ where

$$\varphi\quad \Sigma_G^{[k]}\to\Sigma_G$$

is the 1-block conjugacy mapping the word $s_1s_2\quad s_k$ in Σ_G to the symbol s_k

Using Theorem 3 1, we further reduce to the case that $\pi\quad \Sigma_G\to\Sigma_H$ is a 1-block map with a resolving symbol m We may assume, by increasing k in Theorem 3 1 if necessary, that the resolving symbol m in H has at least two incoming edges and at least two outgoing edges in H The motive here will not become apparent until later

Assuming $\pi\quad \Sigma_G\to\Sigma_H$ has degree exceeding p_G/p_H, we will construct a bounded-to-1 factor map $\hat{\pi}\quad \Sigma_G\to\Sigma_H$ that has lower degree than π Moreover $\hat{\pi}$ will be right-closing if π is Since any factor map from Σ_G to Σ_H has degree at least p_G/p_H, this will prove that there is a factor map from Σ_G to Σ_H with degree exactly p_G/p_H

First we construct $\hat{\pi}$ and then show that it has the desired properties

If the graph G has period p_G, then the states of G are partitioned into p_G equivalence classes $\mathscr{C}_0, \mathscr{C}_1, \quad, \mathscr{C}_{p_G-1}$, where a state s is equivalent to a state t iff there is a path sut in G with $|ut|$ a multiple of p_G

Let $m^1, m^2, \quad, m^d$ be the symbols in G with $\pi(m^i)=m$, $1\le i\le d$ Since Σ_H has period p_H, any cycle based at m has length a multiple of p_H So we may assume that all the symbols $m^1, m^2, \quad, m^d$ occur in the equivalence classes

$$\mathscr{C}_0, \mathscr{C}_{p_H}, \mathscr{C}_{2p_H}, \quad, \mathscr{C}_{(p_G/p_H-1)p_H}$$

Thus d objects are placed in p_G/p_H pigeon holes If we assume $d>p_G/p_H$, then two of $m^1, m^2, \quad, m^d$ lie in the same equivalence class We may assume these two are m^1 and m^2 and that $m^1, m^2\in\mathscr{C}_0$

Fix $N_0>0$ such that for any $0\le i, j<p_G$, any state s in $\mathscr{C}_i$ and any state t in $\mathscr{C}_j$, there is a path of length $(j-i)+N_0p_G$ from state s to state t

Let $e_2e_3\quad e_L$ be a (possible empty) path in H such that $me_2e_3\quad e_L$ is a simple cycle in H Denote $m=e_1$ Recall that m has at least two incoming and at least two outgoing edges Choose states f and h in H so that mf and hm are edges of H not occurring on the cycle $e_1e_2\quad e_L$

Choose an integer p such that $pL+1\ge p_G+N_0p_G+1$

By the Permutation Lemma 3 2 there is a path $c_1c_2\quad c_{pL+1}s_0=cs_0$ in G with $c_1=m^1$ and with π-label $(e_1e_2\quad e_L)^pe_1f$ Again by the permutation Lemma 3 2, there is a state s_N of G such that s_Nm^1 is an edge of G and $\pi(s_N)=h$ Similarly, there is a state $\bar{s}_N$ of G such that $\bar{s}_Nm^2$ is an edge of G and $\pi(\bar{s}_N)=h$

Now $s_0\in\mathscr{C}_{pL+1}$ and $s_N, \bar{s}_N\in\mathscr{C}_{-1}$ (indices are $\bmod p_G$) Fix I_0 with $I_0\equiv -1-(pL+1)\bmod p_G$ and $0\le I_0<p_G$ Set $N=I_0+N_0p_G$ We may choose a path $s_0s_1\cdot\ s_{N-1}s_N$ from state s_0 to state s_N and a path $s_0\bar{s}_1\quad \bar{s}_{N-1}\bar{s}_N$ from state s_0 to state $\bar{s}_N$ Denote $s_0=\bar{s}_0$ By the choice of p, we have

$$pL+1\ge p_G+N_0p_G+1\ge I_0+N_0p_G+2=N+2,$$

an inequality we will use in the proof of Lemma 4 1 below

Denote $M=pL+1+N+2$ and

$$t=t_1t_2\quad t_M=(c_1c_2\quad c_{pL+1})(s_0s_1\quad s_N)m^1$$

and

$$\bar{t} = \bar{t}_1\bar{t}_2 \quad \bar{t}_M = (c_1c_2 \quad c_{pL+1})(\bar{s}_0\bar{s}_1 \quad \bar{s}_N)m^2$$

Denote $\pi(s_1s_2 \quad s_{N-1}) = g$, $\pi(\bar{s}_1\bar{s}_2 \quad \bar{s}_{N-1}) = \bar{g}$, and $(e_1e_2 \quad e_L)^pe_1 = e$ Note that $\pi(t) = efghm$ and $\pi(\bar{t}) = ef\bar{g}hm$ Thus $g \neq \bar{g}$ by Lemma 3 2

The paths t and $\bar{t}$ in G were chosen in part to make the following lemma true

LEMMA 4 1 *The two paths $\pi(t) = mvm = efghm$ and $\pi(\bar{t}) = m\bar{v}m = ef\bar{g}hm$ in H non-trivially overlap each other or themselves only at their end symbols, m*

Proof Say path u *encroaches upon* path w by n if $u = u's$, $w = sw'$ and $|s| = n$ Since the edge hm does not occur in the path $e = (e_1e_2 \quad e_L)^pm$, neither $efghm$ nor $ef\bar{g}hm$ can encroach upon itself or the other by any n with $2 \leq n \leq pL+1$ If $pL+2 \leq n \leq M-1$, and one of $efghm$ or $ef\bar{g}hm$ encroached upon the other by n, then the edge mf would occur in the path e for the following reason Since $|ghm| = |\bar{g}hm| = N+1$, the edge mf would occur ending at position $n-(N+1)$ in the encroached-upon path But

$$2 \leq (pL+2)-(N+1) \leq n-(N+1) \leq (M-1)-(N+1) = pL+1,$$

which puts the edge mf in the path e Thus neither $efghm$ nor $ef\bar{g}hm$ can encroach upon itself or the other by any n with $2 \leq n \leq M-1$ Now $efghm \neq ef\bar{g}hm$, so neither can encroach upon the other by M Since $|efghm| = M$, this proves the lemma □

We define $\hat{\pi}\ \Sigma_G \to \Sigma_H$ as follows Let $x \in \Sigma_G$

If the block t occurs in x, say ${}_{i-M+1}(x)_i = t$, then

$${}_{i-M+1}(\hat{\pi}(x))_i = \pi(\bar{t}) = m\bar{v}m,$$

if the block $\bar{t}$ occurs in x, say ${}_{i-M+1}(x)_i = \bar{t}$, then

$${}_{i-M+1}(\hat{\pi}(x))_i = \pi(t) = mvm,$$

and for any coordinate x_i of x not occurring in a block t or $\bar{t}$, set

$$(\hat{\pi}(x))_i = \pi(x_i)$$

By Lemma 4 1, the strings mvm and $m\bar{v}m$ in H nontrivially overlap themselves or each other only at their end symbols Thus the strings t and $\bar{t}$ can overlap each other or themselves in at most that many ways (in fact fewer ways), so $\hat{\pi}$ is well-defined as a $(2M-1)$-block map $\hat{\pi}\ \Sigma_G \to \Sigma_H$

Define functions f and $\bar{f}$ with domain and range $\{1, 2, \quad , d\}$ by

$$f(i) = \begin{cases} \tau_v(i) & \text{if } i \neq 1, \\ 2 & \text{if } i = 1, \end{cases}$$

and

$$\bar{f}(i) = \begin{cases} \tau_{\bar{v}}(i) & \text{if } i \neq 1, \\ 1 & \text{if } i = 1 \end{cases}$$

Note that $\tau_v(1) = 1$, so 1 is not in the range of f Similarly, $\tau_{\bar{v}}(1) = 2$, so 2 is not in the range of $\bar{f}$

Let $m'v'm^{\tau_v(i)}$, $1 \le i \le d$, be the paths in G given by Lemma 3 2 that are π-labelled by mvm Similarly, let $m'\bar{v}'m^{\tau_{\bar{v}}(i)}$ be the paths in G that are π-labelled by $m\bar{v}m$

We may denote

$$m'\hat{v}'m^{f(i)} = \begin{cases} mv'm^{\tau_v(i)} & \text{if } i \neq 1, \\ \bar{t} = m^1\bar{v}^1m^2 & \text{if } i = 1, \end{cases}$$

and

$$m'\hat{\bar{v}}'m^{\bar{f}(i)} = \begin{cases} m'\bar{v}'m^{\tau_{\bar{v}}(i)} & \text{if } i \neq 1, \\ t = m^1v^1m^1 & \text{if } i = 1 \end{cases}$$

For a string w, denote ${}_0[w] = {}_0[w]_{|w|-1}$

LEMMA 4 2

$$\hat{\pi}^{-1}({}_0[mvm]) = \bigcup_{i=1}^{d} {}_0[m'\hat{v}'m^{f(i)}]$$

and

$$\hat{\pi}^{-1}({}_0[m\bar{v}m]) = \bigcup_{i=1}^{d} {}_0[m'\hat{\bar{v}}'m^{\bar{f}(i)}]$$

Proof The $2d$ paths $m'v'm^{\tau_v(i)}$ and $m'\bar{v}'m^{\tau_{\bar{v}}(i)}$, $1 \le i \le d$ in the graph G are π-labelled by mvm or $m\bar{v}m$, so by Lemma 4 1 each of these paths non-trivially overlaps another or itself at most by one symbol (some m') In particular, each non-trivially overlaps t and $\bar{t}$ by at most one symbol Thus, for $2 \le i \le d$,

$$\hat{\pi}({}_0[m'v'm^{\tau_v(i)}]) \subseteq \pi({}_0[m'v'm^{\tau_v(i)}]) = {}_0[mvm],$$

and

$$\hat{\pi}({}_0[m^1\bar{v}^1m^2]) \subseteq {}_0[mvm]$$

so

$$\bigcup_{i=1}^{d} {}_0[m'\hat{v}'m^{f(i)}] \subseteq \hat{\pi}^{-1}({}_0[mvm])$$

On the other hand, if ${}_0(\hat{\pi}(x))_{|mvm|-1} = mvm$ then either ${}_0(x)_{|mvm|-1} = \bar{t} = m^1\bar{v}^1m^2$ or ${}_0(x)_{|mvm|-1}$ overlaps t and $\bar{t}$ by at most one symbol, in which case $\hat{\pi}$ agrees with π on ${}_0(x)_{|mvm|-1}$, giving that ${}_0(x)_{|mvm|-1} = m'v'm^{\tau_i(i)}$, where $2 \le i \le d$ This shows

$$\hat{\pi}^{-1}({}_0[mvm]) \subseteq \bigcup_{i=1}^{d} {}_0[m'\hat{v}'m^{f(i)}]$$

The barred version is proved similarly □

Lemma 4 2 is the base case for an induction used to prove Lemma 4 3 below

Let w be any path in the graph H beginning and ending with the strings mvm or $m\bar{v}m$ We can express

$$w = m(w_1m)(w_2m) \quad (w_km)\tilde{v}m,$$

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where
(1) $\tilde{v} = v$ or $\tilde{v} = \bar{v}$,
(2) each $w_j m$ begins with vm or $\bar{v}m$,
(3) the strings mvm and $m\bar{v}m$ do not occur in any $w_j m$, $1 \le j \le k$.
Note that $k = 0$ if $w = mvm$ or $m\bar{v}m$ There is a unique decomposition satisfying (1), (2), and (3) because mvm and $m\bar{v}m$ non-trivially overlap each other and themselves only in a single symbol (m)

LEMMA 4 3 *Let w be any path in H beginning and ending with mvm or $m\bar{v}m$ Let*

$$w = m(w_1 m)(w_2 m) \cdot \quad (w_k m)\tilde{v}m$$

be the decomposition defined above Then for $1 \le i \le d$,

$$\begin{aligned} &{}_0[m^i] \cap \hat{\pi}^{-1}{}_0[w] \\ &\quad = {}_0[m^i(w_1^i m^{f_1(i)})(w_2^i m^{f_2 \circ f_1(i)}) \cdot \quad (w_k^i m^{f_k \circ \quad \circ f_1(i)})\tilde{v}^i m^{h \circ f_k \circ \quad \circ f_1(i)}], \end{aligned}$$

where w_j^i, $1 \le j \le k$, and $\tilde{v}^i$ are paths in G and

$$f_j = \begin{cases} f & \text{if } w_j m = vm \\ \bar{f} & \text{if } w_j m = \bar{v}m \\ \tau_u \circ f & \text{if } w_j m = vmum \\ \tau_u \circ \bar{f} & \text{if } w_j m = \bar{v}mum \end{cases}$$

and

$$h = \begin{cases} f & \text{if } \tilde{v} = v, \\ \bar{f} & \text{if } \tilde{v} = \bar{v} \end{cases}$$

Proof The proof is by induction on k. If $k = 0$, then $w = mvm$ or $w = m\bar{v}m$ and this case follows from the equality

$$\hat{\pi}^{-1}({}_0[mvm]) = \bigcup_{i=1}^{d} {}_0[m^i \hat{v}^i m^{f(i)}]$$

or

$$\hat{\pi}^{-1}({}_0[m\bar{v}m]) = \bigcup_{i=1}^{d} {}_0[m^i \hat{\bar{v}}^i m^{\bar{f}(i)}]$$

given by Lemma 4 2 Now suppose the lemma is true for all $0 \le k < l$ and that

$$w = m(w_1 m)(w_2 m) \quad (w_l m)\tilde{v}m$$

Suppose that $w_l m$ begins with vm (The argument for $\bar{v}m$ is similar) Set

$$u = m(w_1 m)(w_2 m) \quad (w_{l-1} m)vm$$

Then u satisfies the inductive hypothesis, so

$${}_0[m^i] \cap \hat{\pi}^{-1}{}_0[u] = {}_0[m^i \quad m^{g(i)} v^i m^{f \circ g(i)}],$$

where $g = f_{l-1} \circ \quad \circ f_1$ There are two cases to consider
(1) $w_l m = vm$,
(2) $w_l m = vmum$, where neither mvm nor $m\bar{v}m$ occurs in mum
In case (1),

$$\begin{aligned} {}_0[m^i] \cap \hat{\pi}^{-1}{}_0[w] &= {}_0[m^i] \cap \hat{\pi}^{-1}{}_0[u] \cap \sigma^{-|u|+1} \hat{\pi}^{-1}{}_0[m\tilde{v}m] \\ &= {}_0[m^i \quad m^{f \circ g(i)}]_{|u|-1} \cap {}_{|u|-1}[m^{f \circ g(i)} \tilde{v}^i m^{h \circ f \circ g(i)}], \end{aligned}$$

so $f_i = f$ in this case In case (2),

$$
\begin{aligned}
&{}_0[m' \cdots] \cap \hat{\pi}^{-1}{}_0[w] \\
&= {}_0[m' \quad m^{g(i)}v'm^{f\circ g(i)}]_{|u|-1} \cap \sigma^{-|u|+|mvm|}\hat{\pi}^{-1}{}_0[mvmum\tilde{v}m] \\
&= {}_0[m' \quad m^{g(i)}v'm^{f\circ g(i)}]_{|u|-1} \cap {}_{|u|-|mvm|}[m^{g(i)}v'm^{f\circ g(i)}u'm^{\tau_u\circ f\circ g(i)}\tilde{v}'m^{h\circ\tau_u\circ f\circ g(i)}],
\end{aligned}
$$

so $f_i = \tau_u \circ f$ in this case □

COROLLARY 4 4 *The map* $\hat{\pi}$ $\Sigma_G \to \Sigma_H$ *is onto*

Proof By Lemma 4 3, each path of the form $(mvm)u(mvm)$ in H is the image by $\hat{\pi}$ of a path $(m'v'm^{f(i)})u'(m^{g(i)}v''m^{f\circ g(i)})$ in G Thus, by the irreducibility of H, any finite path in H is the image by $\hat{\pi}$ of some path in G It follows that the image of $\hat{\pi}$ in Σ_H is dense, and by the compactness of Σ_G, that the image of $\hat{\pi}$ is all of Σ_H □

COROLLARY 4 5 *The map* $\hat{\pi}$ $\Sigma_G \to \Sigma_H$ *is bounded-to-*1

Proof As π $\Sigma_G \to \Sigma_H$ is bounded-to-1, Σ_G and Σ_H have the same entropy [**CP**] It follows from this, Corollary 4 4, and [**CP**] that $\hat{\pi}$ is bounded-to-1 □

COROLLARY 4 6 *If* π *is right-closing, then so is* $\hat{\pi}$

Proof Let x, $x' \in \Sigma_G$ be left asymptotic points with $\hat{\pi}(x) = \hat{\pi}(x')$ We must show $x = x'$ We may assume $x_i = x_i'$ for $i \le 0$ We may also assume (by replacing ${}_{-\infty}(x)_0$ by some other past and shifting if necessary) that ${}_{-|mvm|+1}(\hat{\pi}(x))_0 = mvm$ If words from $\{mvm, m\bar{v}m\}$ occur infinitely often in ${}_0(\hat{\pi}(x))_\infty$ then $x = x'$ b an induction and Lemma 4 3 If words from $\{mvm, m\bar{v}m\}$ occur a finite number of times in ${}_0(\hat{\pi}(x))_\infty$, let ${}_{k-|mvm|+1}(\pi(x))_k$ be the final occurrence Then $x_i = x_i'$ for $i \le k$ by Lemma 4 3 Now ${}_k(\hat{\pi}(x))_\infty = {}_k(\pi(x))_\infty$ by the definition of $\hat{\pi}$ off the blocks t and $\bar{t}$ Similarly, ${}_k(\hat{\pi}(x'))_\infty = {}_k(\pi(x'))_\infty$ So ${}_k(\pi(x'))_\infty = {}_k(\pi(x))_\infty$, so $x = x'$ because π is right-closing □

COROLLARY 4 7 *The map* $\hat{\pi}$ $\Sigma_G \to \Sigma_H$ *has lower degree than* π *has*

Proof Because the map $\hat{\pi}$ is not a 1-block map, we cannot apply verbatim the characterization of degree we gave in terms of the pre-image of a resolving block in H However, we may choose an integer q so that $|(mv)^q m| > 2M$ and observe that

$$
\begin{aligned}
\hat{\pi}^{-1}{}_0[mv(mv)^q m] &= \bigcup_{i=1}^{d} {}_0[m'v'm^{f(i)} \quad m^{f^{q+1}(i)}] \\
&\subseteq \bigcup_{i=1}^{d} \sigma^{-|mv|}{}_0[m^{f(i)} \quad m^{f^{q+1}(i)}],
\end{aligned}
$$

the last set being a disjoint union of at most $d-1$ $|(mv)^q m|$-blocks Thus we may apply the criterion directly to the 1-block map

$$\hat{\pi} \circ \psi_{2M}^{-1} \quad \Sigma_G^{[2M]} \to \Sigma_H,$$

to conclude that the degree of $\hat{\pi}$ is at most $d-1$ □

From the assumption that the bounded-to-1 factor map π $\Sigma_G \to \Sigma_H$ has degree exceeding p_G/p_H, we have constructed a bounded-to-1 factor map $\hat{\pi}$ $\Sigma_G \to \Sigma_H$ with degree less than the degree of π Since the smallest possible degree of a factor map $\pi' = \Sigma_G \to \Sigma_H$ is p_G/p_H, this shows that one could iterate the construction to get a factor map degree exactly p_G/p_H, proving Theorem 1 1

5 *The sofic case*

A sofic system is a symbolic system that is a factor of a shift of finite type In fact any sofic system is a factor by a 1-to-1 almost everywhere map of a shift of finite type [**F**]

Theorem 1 1 can be generalized to the case of sofic domain and range

THEOREM 5 1 *If π $S \to T$ is a bounded-to-1 factor map from an irreducible sofic system S with period p_S to an irreducible sofic system T with period p_T, then there is a factor map $\hat{\pi}$ $S \to T$ that is (p_S/p_T)-to-1 almost everywhere Moreover, if π is right closing, then $\hat{\pi}$ may be taken to be right closing also*

Here, the period of a sofic system is the period of any 1-to-1 almost everywhere finite type extension

The proof of Theorem 5 1 is largely the same as the proof of Theorem 1 1 The only real change is that we replace resolving blocks by their appropriate generalization in the sofic setting *markov magic words* [**B**]

We follow [**B**] in the following two definitions

Given a sofic system T, a *markov word* for T is an allowable word w such that if uw and wv are allowable words in T, then so is uwv

Given a bounded-to-1 1-block factor map π $S \to T$ from an irreducible sofic system S to an irreducible sofic system T, define $\mathscr{W}$ to be the set of allowable words w in T for which

(i) w is a markov word for T,

(ii) $\pi^{-1}{}_0[w] \subseteq \bigcup_{i=1}^{d} {}_j[w^i]_k$, where $k \geq j$ and $w^1, w^2, \quad , w^d$ are markov words for S

In [**B**] it is shown that $\mathscr{W}$ is non-empty Any $w \in \mathscr{W}$ for which d in (ii) is minimal is called a *markov magic word* for π $S \to T$ The minimal d is the degree of the factor map π $S \to T$ [**B**]

We may use [**B**, Proposition 1 4] and a construction similar to that of § 3 above (from [**KMT**]) to reduce to the case where π $S \to T$ has a markov magic *symbol* m Then [**B**, Proposition 1 4] gives the following generalization of the permutation Lemma 2 2

LEMMA 5 2 *Let π $S \to T$ be a degree d 1-block map with markov magic symbol m Let $m^1, m^2, \quad , m^d$ be the symbols in S with $\pi^{-1}{}_0[m] = \bigcup_{i=1}^{d} {}_0[m^i]$ For each allowable word of the form mum in T, there are d words $u^1, u^2, \quad , u^d$ in S and a permutation τ_u of $\{1, 2, \quad , d\}$ such that*

$$\pi^{-1}{}_0[mum] = \bigcup_{i=1}^{d} {}_0[m^i u^i m^{\tau_u(i)}]$$

As in the shift of finite type case, we may assume that the symbol m in T has at least two predecessors and two successors

The period of T is

$$gcd\{|mum|-1 \quad mum \text{ is a word in } T\}$$

and the period of S is

$$gcd\{|m^1um^1|-1 \quad m^1um^1 \text{ is a word in } S\}$$

The construction of $\hat{\pi}$ $S \to T$ using Lemma 5 2 follows much the same lines as the shift of finite type case

6 *The Markov chain case*

If the irreducible shift of finite type (Σ_G, σ) is given a Markov measure μ_G defined by a stochastic matrix $P \geq 0$ via

$$\mu_G({}_0[st]) = P_{st}\mu_G({}_0[s]),$$

then $(\Sigma_G, \sigma, \mu_G)$ is called a *Markov chain*

Following [**PS**], define the *weight* of a cycle $s_0s_1 \quad s_{p-1}$ in the graph G as

$$w_G(s_0s_1 \quad s_{p-1}) = P_{s_0s_1}P_{s_1s_2} \quad P_{s_{p-1}s_0},$$

and the multiplicative subgroup Δ_G of $\mathbb{R}^+$ by

$$\Delta_G = \left\{\frac{W_G(s)}{W_G(s')} \quad s, s' \text{ are cycles in } G \text{ with } |s| = |s'|\right\}$$

In [**PS**] it is shown that if

$$\pi \quad (\Sigma_G, \sigma, \mu_G) \to (\Sigma_H, \sigma, \mu_H)$$

is measure-preserving, then $\Delta_G \subseteq \Delta_H$, moreover, if π is 1-to-1 almost everywhere, then $\Delta_G = \Delta_H$

As was pointed out to me by Brian Marcus, the construction of $\hat{\pi}$ used in the proof of Theorem 1 1 can be adapted to work in the category of Markov measure-preserving block maps to give a partial converse to the [**PS**] result

THEOREM 6 1 *If π $(\Sigma_G, \sigma, \mu_G) \to (\Sigma_H, \sigma, \mu_H)$ is a measure-preserving factor map from the Markov chain Σ_G with period p_G to a Markov chain Σ_H with equal period $p_H = p_G$, and if $\Delta_G = \Delta_H$, then there is a measure-preserving factor map $\hat{\pi}$ $(\Sigma_G, \sigma, \mu_G) \to (\Sigma_H, \sigma, \mu_H)$ that is 1-to-1 almost everywhere*

Sketch of proof In the proof of Theorem 1 1, we construct paths t and $\bar{t}$ in the graph G such that their images $\pi(t) = mvm$ and $\pi(\bar{t}) = m\bar{v}m$ in the graph H overlap by at most one symbol The map $\hat{\pi}$ $\Sigma_G \to \Sigma_H$ is defined by "switching the images" of t and $\bar{t}$

Now vm and $\bar{v}m$ are both cycles in the graph H If $w_H(vm) = w_H(\bar{v}m)$ then $\hat{\pi}$, like π, will be measure-preserving Otherwise the ratio

$$\frac{w_H(vm)}{w_H(\bar{v}m)} = \rho \in \Delta_H = \Delta_G$$

is equal to a ratio

$$\frac{w_G(\bar{r})}{w_G(r)} = \rho \in \Delta_G,$$

where $r = s_0 r_1 r_2 \cdots r_k$ and $\bar{r} = s_0 \bar{r}_1 \bar{r}_2 \cdots \bar{r}_k$ are cycles in the graph G based at the state s_0 of G defined in the proof of Theorem 1.1.

Now interpolate the cycle $\bar{r}$ into the path $\bar{t}$ at state s_0, and interpolate the cycle r into the path t at state s_0, and extend the common prefix $c_1 c_2 \cdots c_{pL+1}$ of t and $\bar{t}$ (by choosing a larger L if necessary) to ensure that the two modified paths t' and $\bar{t}'$, like t and $\bar{t}$, non-trivially overlap themselves or each other only by one symbol. Denote $\pi(t') = mv'm$ and $\pi(\bar{t}') = m\bar{v}'m$.

Now

$$w_H(\pi(r)) = w_G(r)$$

and

$$w_H(\pi(\bar{r})) = w_G(\bar{r}),$$

so

$$\frac{w_H(v'm)}{w_H(\bar{v}'m)} = \frac{w_H(vm) w_G(r)}{w_H(\bar{v}m) w_G(\bar{r})} = 1,$$

by the choice of the cycles r and $\bar{r}$. Hence if we define $\hat{\pi}: \Sigma_G \to \Sigma_H$ by 'switching the images' of t' and $\bar{t}'$ (which we can do since t' and $\bar{t}'$ non-trivially overlap each other or themselves by at most one symbol), then $\hat{\pi}$, like π, will be measure-preserving. As in the proof of Theorem 1.1, $\hat{\pi}$ will have lower degree than π. □

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REFERENCES

[AGW] R. Adler, L. W. Goodwyn & B. Weiss. Equivalence of topological Markov Shifts. *Israel J. Math.* **27** (1977) 49–63.

[AM] R. Adler & B. Marcus. Topological entropy and equivalence of dynamical systems. *Memoirs Amer. Math. Soc.* **219** (1979).

[B] M. Boyle. Constraints on the degree of a sofic homomorphism and the induced multiplication of meaures on unstable sets. *Israel J. of Math.* **53** (1986) 52–68.

[BKM] M. Boyle, B. Kitchens & B. Marcus. A note on minimal covers for sofic systems. *Proc. Amer. Math. Soc.* **95** (1985) 403–411.

[BMT] M. Boyle, B. Marcus & P. Trow. Resolving maps and the dimension group for shifts of finite type. *Memoirs AMS* **377** (1987).

[CP] E. Coven & M. Paul. Endomorphisms of irreducible shifts of finite type. *Math. Systems Theory* **8** (1974) 167–175.

[F] R. Fischer. Sofic systems and graphs. *Monatch. fur Math.* **80** (1975) 179–186.

[KMT] B. Kitchens, B. Marcus & P. Trow. Eventual factor maps and compositions of closing maps, in preparation.

[PS] W. Parry & K. Schmidt. Natural coefficients and invariants for Markov shifts. *Invent. Math.* **76** (1984) 15–32.

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