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WEIGHTED FOURIER TRANSFORM INEQUALITIES VIA MIXED NORM HAUSDORFF-YOUNG INEQUALITIES

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ABSTRACT. Wiener-Lorentz amalgam spaces are introduced and some of their interpolation theoretic properties are discussed. We prove Hausdorff-Young theorems for these spaces unifying and extending Hunt's Hausdorff-Young theorem for Lorentz spaces and Holland's theorem for amalgam spaces. As consequences we prove weighted norm inequalities for the Fourier transform and show how these inequalities fit into a natural class of weighted Fourier transform estimates.

1. **Introduction.** The Hausdorff-Young-Titchmarsh theorem and weighted extensions due to Hardy-Littlewood-Paley and to Pitt were among the first applications of Riesz-Thorin and Marcinkiewicz interpolation theorems. Stein [St] (following Hirschman [Hi]) noted that stronger interpolation techniques lead to strengthened "rearrangement invariant" versions of Pitt's theorems. This observation in turn led to generalizations due to P. G. Rooney [R] and T. Flett [Fl], again using interpolation. Thus power weights were replaced by more general positive weight functions satisfying some weak-type integrability conditions. Finally, in the early 1980's there was a strong push to generalize the weight classes in extensions of the Hausdorff-Young theorem, *e.g.*, [AH; H; S] culminating in several independent solutions of the following problem of characterizing the pairs of weights having the property that the Fourier transform is a "rearrangement invariant" continuous map from one weighted Lebesgue space to another:

REARRANGEMENT FOURIER TRANSFORM PROBLEM. Characterize those pairs (u, v) of locally integrable weights for which u, $1/v$ are radially decreasing, and exponents $1 \le p \le q < \infty$ such that for some fixed constant C,

$$\text{(RFT)} \qquad \forall f \in L^1 \cap L^p_v(\mathbb{R}^d), \quad \Big(\int_{\mathbb{R}^d} |\hat{f}^{\circledast}(\xi)|^q u(\xi)\, d\xi\Big)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}^d} |f^{\circledast}(x)|^p v(x)\, dx\Big)^{\frac{1}{p}}.$$

The symbol $\circledast$ appearing in the inequality (RFT) denotes the "symmetrically decreasing rearrangement," $f^{\circledast}(x) = f^*(\Omega_d|x|^d)$, and f^* is the ordinary equimeasurable decreasing rearrangement of f, see *e.g.*, [SW]. The norm on L^p_v is the expression on the right hand side of (RFT) (without the $\circledast$). Characterizations of the pairs of weights for which

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(RFT) holds were obtained independently by Benedetto and Heinig [H; BH], Muckenhoupt [M1; M2], and Jurkat and Sampson [JS1; JS2]. The main tool in [H] is essentially a weighted extension of the Marcinkiewicz interpolation theorem. Direct use of interpolation is avoided in [BH], but the main tools used are Hardy inequalities and rearrangement methods that also underlie interpolation techniques. The Jurkat-Sampson approach also avoids interpolation per se, but again is heavily based on Hardy type estimates and convexity methods. The approach in Muckenhoupt involves a clever reduction to the Hausdorff-Young inequality. In summary, it is fair to say that solutions of (RFT) are based on the same machinery that drives interpolation theory. Our statement of the characterization of (RFT) most closely resembles the Jurkat-Sampson statement.

THEOREM 1.1 (RFT). *Let $1 \le p \le q < \infty$ and let w, v be nonnegative, radial weight functions. Set $W(\Omega_d|x|^d) = w(x)$ and similarly define V in terms of v. A necessary condition for the inequality (without rearrangements),*

$$\text{(FT)} \qquad \forall f \in L^1 \cap L^p_v(\mathbb{R}^d), \quad \Big(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^q u(\xi)\, d\xi\Big)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, dx\Big)^{\frac{1}{p}},$$

is that

$$\text{(UP}^*\text{)} \qquad \sup_{s>0}\Big(\int_0^{\frac{1}{s}} W(t)\, dt\Big)^{\frac{1}{q}} \Big(\int_0^{s} V^{1-p'}(t)\, dt\Big)^{\frac{1}{p'}}.$$

Conversely, if w is decreasing and v is increasing then (UP*) is sufficient for (RFT).

The condition (UP*) signifies the (rearrangement) uncertainty principle (see [B] for an explanation of this terminology). As pointed out in [M2, Theorem 7], there are examples of weighted Fourier transform inequalities for weights whose decreasing rearrangements do not exist (that is, whose distribution functions are never finite). Moreover, restriction theorems are examples of weighted Fourier transform inequalities with singular measure weights and there is no sensible way of defining an equimeasurable rearrangement of such a weight. Nonetheless, weighted Fourier transform estimates in a rearrangement dependent context have been shown to have important applications to the study of pseudodifferential operators [J], spherical summation methods [CD], and unique continuation properties [K], among other things.

To establish versions of (FT) which are not rearrangement invariant one would like to keep as much machinery available as possible from the rearrangement invariant case. One possible approach, then, is to look for inequalities which are invariant under the action of some restricted class of measure preserving transformations. This approach has already been used in work of Bloom, Jurkat, and Sampson [BJS], where weighted Fourier transform inequalities in $\mathbb{R}^d$ are proved for "sectionally decreasing weights" and are thus invariant under "sectional-type" rearrangements. Their results reduce to Theorem 1.1 in the one-dimensional case. To extend the one-dimensional result, one looks for a class of measure preserving transformations that are well-adapted to the geometry of the Fourier transform. The condition (UP*) reflects the fact that the Fourier transform exchanges local and global behavior. Therefore it is natural to look for Fourier transform inequalities invariant under rearrangements that are the composition of local and global transformations.

LOCAL/GLOBAL REARRANGEMENT FOURIER TRANSFORM PROBLEM. Characterize those pairs of weights u, v for which one has a weighted norm inequality

$$\forall f \in L^1 \cap L^p_V(\mathbb{R}^d), \quad \|\hat{f}\|_{L^q_U} \leq C_{u,v}\|f\|_{L^p_V}.$$

Here U is any "local-global rearrangement" of u in the sense that $U(x) = u\big(\tau \circ \sigma(x)\big)$ where σ is a permutation of the cubes Q_n and τ is a measure preserving transformation that maps each cube Q_n to itself. We similarly define V in terms of v. Here $Q_n = [0,1)^d + n$ where $n \in \mathbb{Z}^d$.

In this paper we prove partial results for this problem, which may be viewed as local/global Pitt-type inequalities. A local/global version of the Hausdorff-Young theorem, which has been known for quite some time, is given in terms of the Wiener amalgam spaces, defined as follows.

DEFINITION 1.2. Given $1 \leq p, q < \infty$, the *Wiener amalgam space* $W(L^p, l^q)(\mathbb{R}^d)$ is the Banach space of functions f for which

$$\|f\|_{W(L^p,l^q)} = \left(\sum_{n\in\mathbb{Z}^d}\left(\int_{Q_n}|f(x)|^p\,dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty.$$

Again, Q_n denotes the translate by $n \in \mathbb{Z}^d$ of the unit cube $[0,1)^d$. The natural adjustments are made to define $W(L^p, l^q)(\mathbb{R}^d)$ in case p or q is infinite. Notice that the scale of Wiener amalgam spaces is decreasing as p increases and increasing as q increases. The space $W(L^\infty, l^1)(\mathbb{R})$ was introduced by Wiener in [W, p. 21]. For a detailed discussion of these spaces we refer to [FS]. One has

THEOREM 1.3 (HAUSDORFF-YOUNG). *Given $1 \leq p, q \leq 2$. The Fourier transform satisfies*

$$\forall f \in W(L^p, l^q)(\mathbb{R}^d), \quad \|\hat{f}\|_{W(L^{q'}, l^{p'})} \leq C_{p,q}\|f\|_{W(L^p,l^q)}.$$

Theorem 1.3 was first proved for $\mathbb{R}$ by Holland [Ho]. Later generalizations include those of Bertrandias and Dupuis [BD], Fournier [Fo], Feichtinger [F1; F2], and others. Versions of the Wiener amalgam spaces were used inherently in work of Aguilera and Harboure involved in finding necessary conditions for weighted Fourier transform estimates. *These spaces are not themselves rearrangement invariant, but rather are amalgams of rearrangement invariant spaces.* The Hausdorff-Young theorem for Wiener amalgam spaces furnishes a nice illustration in terms of norm inequalities of the exchange between local and global behavior of a function and its Fourier transform.

Our paper is outlined as follows. In Section 2 we present a necessary uncertainty principle condition, (UP), for weighted Fourier transform inequalities. This sharpens (UP*) in the sense that it takes into account the translation structure of the Fourier transform as well as the dilation structure. We give an example to show that this new and improved weight condition, however, still fails to characterize those pairs of weights for which local/global rearrangement invariant Fourier transform inequalities hold. Nevertheless, this counterexample indicates the sort of pathologies that make the sufficiency of (UP) for (FT) fail and allows us to refine our interpretation of the meaning of (UP*) in the rearrangement invariant context. In Section 3 we show how weighted Fourier transform estimates may be easily obtained from various versions of the Hausdorff-Young theorem by means of Hölder's inequality. We indicate the role of Wiener amalgam spaces in proving weighted norm inequalities that are not rearrangement invariant. In Section 4 we introduce Wiener-Lorentz amalgam spaces as real interpolation spaces between Wiener amalgam spaces. Such interpolation is possible because the Wiener amalgam spaces are isomorphic to certain mixed norm Lebesgue spaces. Hausdorff-Young theorems for these spaces then arise as applications of vector-valued versions of the Marcinkiewicz interpolation theorem. We thereby generalize the Hausdorff-Young theorem for Wiener amalgam spaces as well as the version for Lorentz spaces. These results are applied to prove weighted Fourier transform estimates of "Pitt-type" when weights satisfy certain local/global weak-type integrability conditions.

We should point out that Feichtinger [F1; F2] has found different generalizations of the Hausdorff-Young theorem, also by means of Wiener amalgam spaces, but by making use of the complex method of interpolation. Although the motivation for our work is quite different from Feichtinger's, the idea of using Wiener amalgam spaces and interpolation to sharpen the Hausdorff-Young theorem is inspired by Feichtinger's work.

Finally, we compare our sufficient conditions with the necessary condition (UP) and outline strategies for closing the gap between necessary and sufficient conditions in the local/global rearrangement invariant setting.

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2. **Necessary conditions.** As pointed out, (UP*) is always necessary for the weighted norm inequality (FT) but does not take into account the full Euclidean structure of the Fourier transform. To introduce a strengthened uncertainty principle condition we need some extra terminology. ${\mathbb{R}^d}_+$ denotes the (strictly) positive cone of $\mathbb{R}^d$ consisting of those elements $t = (t_1, \ldots, t_d)$ for which $t_j > 0, j = 1, \ldots, d$. For $t \in {\mathbb{R}^d}_+$ we denote by $1/t$ the vector $(1/t_1, \ldots, 1/t_d)$. Given $x_0 \in \mathbb{R}^d$ and $t \in {\mathbb{R}^d}_+$, $R(x_0; t)$ denotes the rectangle $|x_j - x_{0j}| < t_j$. The inequality (FT) still makes sense if we take a measure weight on the transform side. We denote the class of such weights by $M_+(\mathbb{R}^d)$.

THEOREM 2.1. *Given weights $\mu \in M_+(\mathbb{R}^d)$ and $v \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and exponents $1 \leq p, q < \infty$. Suppose that $L^p_v(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$ and one has the weighted Fourier transform norm inequality,*

$$\forall f \in L^1 \cap L^p_v(\mathbb{R}^d), \quad \Big(\int |\hat{f}(\xi)|^q \, d\mu(\xi)\Big)^{\frac{1}{q}} \leq C\Big(\int |f(x)|^p v(x)\, dx\Big)^{\frac{1}{p}}.$$

Then

$$\text{(UP)} \qquad \sup_{(x_0,\xi_0;t)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d{}_+}\Big(\int_{R(\xi_0;\frac{1}{t})} d\mu(\xi)\Big)^{\frac{1}{q}}\Big(\int_{R(x_0;t)} v^{1-p'}(x)\,dx\Big)^{\frac{1}{p}} < \infty.$$

PROOF. The fact that $v^{1-p'} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ follows from the inequality dual to (FT) which states that (*cf.*, [L1])

$$\forall g \in L^q_\mu(\mathbb{R}^d), \quad \Big(\int |\widehat{(g\mu)}(x)|^{p'} v^{1-p'}(x)\,dx\Big)^{\frac{1}{p'}} \le C\Big(\int |g(\xi)|^{q'}\,d\mu(\xi)\Big)^{\frac{1}{q'}}.$$

Now fix $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and consider the function

$$f(x) = e^{2\pi i x\cdot\xi_0} v^{1-p'}(x)\chi_{R(x_0;t)}(x).$$

Then $f \in L^1 \cap L^p_v(\mathbb{R}^d)$, and

$$\|f\|_{L^p_v} = \Big(\int_{R(x_0;t)} v^{1-p'}(x)\,dx\Big)^{\frac{1}{p}}.$$

On the other hand,

$$\begin{aligned}
|\hat f(\xi)| &= \Big|\int_{R(x_0;t)} v^{1-p'}(x)e^{-2\pi i(\xi-\xi_0)\cdot x}\,dx\Big| \\
&= \Big|e^{-2\pi i(\xi-\xi_0)\cdot x_0}\int_{R(x_0;t)} v^{1-p'}(x)e^{-2\pi i(\xi-\xi_0)\cdot(x-x_0)}\,dx\Big| \\
&= \Big|\int_{R(x_0;t)} v^{1-p'}(x)e^{-2\pi i(\xi-\xi_0)\cdot(x-x_0)}\,dx\Big| \\
&\ge \Big|\int_{R(x_0;t)} v^{1-p'}(x)\cos 2\pi(\xi-\xi_0)\cdot(x-x_0)\,dx\Big|.
\end{aligned}$$

This shows that

$$\begin{aligned}
\|\hat f\|^q_{L^q_\mu} &\ge \int\Big|\int_{R(x_0;t)} v^{1-p'}(x)\cos 2\pi(\xi-\xi_0)\cdot(x-x_0)\,dx\Big|^q d\mu(\xi) \\
&\ge \int_{R(\xi_0;\frac{1}{8dt})}\Big|\int_{R(x_0;t)} v^{1-p'}(x)\cos 2\pi(\xi-\xi_0)\cdot(x-x_0)\,dx\Big|^q d\mu(\xi) \\
&\ge \cos\frac{\pi}{4}\Big(\int_{R(\xi_0;\frac{1}{8dt})} d\mu(\xi)\Big)\Big(\int_{R(x_0;t)} v^{1-p'}(x)\,dx\Big)^q.
\end{aligned}$$

The inequality follows by reducing the domain of integration and from the fact that $\cos 2\pi(\xi-\xi_0)\cdot(x-x_0) \ge \cos\frac{\pi}{4}$ if $(x,\xi) \in R(x_0;t) \times R(\xi_0;\frac{1}{8dt})$. Taking q-th roots and using the Fourier transform inequality, we conclude that

$$\Big(\int_{R(\xi_0;\frac{1}{8dt})} d\mu(\xi)\Big)^{\frac{1}{q}}\Big(\int_{R(x_0;t)} v^{1-p'}(x)\,dx\Big)^{\frac{1}{p}} \le \frac{C}{\cos\frac{\pi}{4}}.$$

Since the inequality does not depend on (x_0, ξ_0) we may use Minkowski's inequality to replace $R(\xi_0;\frac{1}{8dt})$ by $R(\xi_0;\frac{1}{t})$. This proves the theorem. ∎

One might conjecture that condition (UP) is sufficient for local/global rearrangement invariant Fourier transform inequalities—at least in certain situations. For example, on $\mathbb{R}$, suppose $u(x)$ is symmetric, that for each $n = 0, 1, \ldots, u$ is decreasing from n to $n+1$, and that $u([n, n+1])$ is decreasing in n. Suppose that v^{-1} behaves similarly. This is seemingly analogous to the weight conditions in Theorem 1.1, except that now one has separated the local and global behaviors of the weights. We give a counterexample to show that the uncertainty principle inequality (UP) is not always sufficient for the inequality (FT)—even under such relatively nice circumstances.

EXAMPLE 2.2. Take $p = q = 2$ and $d = 1$. We exhibit a pair (u, v) of weights for which (UP) holds with $p = q = 2$ but the corresponding weighted Fourier transform inequality fails.

Let $v(x) = |\sin x|^{1-\alpha}$ where $0 < \alpha < 1$ is fixed, and let

$$u(\xi) = \begin{cases} 1, & \xi \in \left[n - \frac{1}{|n|}, n + \frac{1}{|n|}\right], n \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly one has

$$\sup_{s>0} \Big(\int_{|\xi_0-\xi|<\frac{1}{s}} u\Big)\Big(\int_{|x_0-x|<s} v^{-1}\Big) \leq C$$

where C is independent of x_0 and ξ_0. This follows since when s is large, the integral involving v^{-1} is bounded by a constant times s, whereas the integral involving u is bounded by a constant times $1/s$. On the other hand, when s is small the integral involving v^{-1} is bounded by a constant times s^{α}, whereas the integral involving u is bounded by a constant times $|\log s|$.

Now define $f_0(x) = x^{-\beta}\chi_{[0,1]}(x)$. Then $f_0 \in L^2_v(\mathbb{R})$ provided $\beta < (2-\alpha)/2$. By [T, Theorem 126] it follows that $\hat{f}_0(\xi) \approx c|\xi|^{\beta-1}$ for some constant c whenever ξ is large. Next set $f_m(x) = \sum_{k=0}^{m} f_0(x-k)$. Thus $\|f_m\|^2_{L^2_v} \sim m+1$. On the other hand,

$$\hat{f}_m(\xi) = \Big(\sum_0^m e^{-2\pi i k\xi}\Big)\hat{f}_0(\xi).$$

But

$$\Big|\sum_0^m e^{-2\pi i k\xi}\Big| \geq \sum_0^m \cos 2\pi k\xi \geq \sum_0^m \cos\frac{\pi k}{4m} \geq (m+1)\cos\frac{\pi}{4}$$

whenever $|\xi| < \frac{1}{8m}$. By periodicity it follows that

$$\Big|\sum_0^m e^{-2\pi i k\xi}\Big| \geq (m+1)\cos\frac{\pi}{4} \text{ whenever } |\xi - n| < \frac{1}{8m}, \quad n \in \mathbb{Z}.$$

Thus,

$$\begin{aligned}
\|\hat{f}_m\|^2_{L^2_u} &= \int_{\mathbb{R}} |\sum_0^m e^{-2\pi i k\xi}|^2 |\hat{f}_0(\xi)|^2 u(\xi)\, d\xi \\
&\geq Cm^2 \sum_{n\geq m} \int_{|\xi-n|<\frac{1}{8m}} |\hat{f}_0(\xi)|^2 u(\xi)\, d\xi \\
&\geq C'm^2 \sum_{n\geq m} \int_{|\xi-n|<\frac{1}{8n}} |\xi|^{2(\beta-1)}\, d\xi \\
&\geq C''m^2 \sum_{n\geq m} n^{2\beta-3} \sim m^{2\beta}.
\end{aligned}$$

That is for large m we have $\|\hat{f}_m\|^2_{L^2_u} \geq Cm^{2\beta}$. Hence

$$\frac{\|\hat{f}_m\|_{L^2_u}}{\|f_m\|_{L^2_v}} \geq Cm^{\beta-\frac{1}{2}}.$$

Therefore, if $1/2 < \beta < (2-\alpha)/2$ the weighted Fourier transform norm inequality fails.

The Fourier transform inequality fails because f_m is concentrated where v is small, whereas $\hat{f}_m$ is concentrated where u is large. One might guess that the failure of (FT) is related to the fact that the symmetrically decreasing rearrangement function of u is a constant function. This is not really the issue, since we could have replaced u by the weight that is one when $|\xi - n| < 1/n^{1+\epsilon}$, and shown the inequality still fails if ϵ is small enough—depending on β. The symmetrically decreasing rearrangement of this new weight is essentially the characteristic function of a disc, and (FT) holds if we replace u by $u^{\circledast}$ in this case. The real problem here is that we cannot define the decreasing rearrangement of $|\sin x|^{\alpha-1}$ since the distribution function is always infinite. We therefore need some condition on u, for example a pointwise decay condition that takes into account the fact that the singularities of v^{-1} add up in a nontrivial way. For example, it is known that if we take $v(x) = |\sin x|^{1-\alpha}$ and $u(\xi) = 1/(1+|\xi|)^{\alpha}$, then the weighted Fourier transform estimate (FT) holds when $p = q = 2$, *cf.*, [L1; KS].

3. **Weighted norm inequalities via Hausdorff-Young inequalities.** The first theorem we present is really a corollary of Theorem 1.1. We present the argument in order to exhibit the technique we will use later to get inequalities in the rearrangement dependent case.

THEOREM 3.1. *Given $1 \leq p \leq 2$ and weights u, v for which u and $\frac{1}{v}$ both belong to* wk-$L^p(\mathbb{R}^d)$. *Then*

$$\forall f \in L^2_v(\mathbb{R}^d), \quad \|\hat{f}\|_{L^2_u} \leq C\|f\|_{L^2_v}.$$

PROOF. We have

$$\begin{aligned}
\|\hat{f}\|^2_{L^2_u} &\leq C_p\|u\|_{\text{wk-}L^p}\|\hat{f}\|^2_{L^{2p',2}} \\
&\leq C_u C'_p\|f\|^2_{L^{\frac{2p}{p+1}},2} \\
&= C'\|fv^{\frac{1}{2}}v^{-\frac{1}{2}}\|^2_{L^{\frac{2p}{p+1}},2} \\
&\leq C'\|v^{-1}\|_{\text{wk-}L^p}\|f\|^2_{L^2_v}.
\end{aligned}$$

Here we have made use of Hölder's inequality for Lorentz spaces, *cf.*, [Hu] along with the Hausdorff-Young theorem for Lorentz spaces. ∎

Notice that the conditions on u and $\frac{1}{v}$ imply that their symmetrically decreasing rearrangements are bounded by constants times $|x|^{\frac{1}{p}}$ so that the condition (UP*) is easy to check in this case. More refined local/global versions of the Hausdorff-Young theorem, together with corresponding Hölder inequalities, will yield weighted Fourier transform estimates with corresponding local/global conditions on the weights. We illustrate our point of view with the following theorem.

THEOREM 3.2. *Given exponents $1 \leq p, q \leq 2$ and weights u, v such that $u \in W(L^{\frac{q}{2-q}}, l^{\frac{p}{2-p}})(\mathbb{R}^d)$ and $v^{-1} \in W(L^{\frac{p}{2-p}}, l^{\frac{q}{2-q}})(\mathbb{R}^d)$. Then*

$$\forall f \in L^2_v(\mathbb{R}^d), \quad \|\hat{f}\|_{L^2_u} \leq C\|f\|_{L^2_v}.$$

PROOF. We prove the result in the case where p, $q < 2$, but the natural adjustments can be made when $p = 2$ or $q = 2$

$$\begin{aligned}
\|\hat{f}\|^2_{L^2_u} &\leq \sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n}|\hat{f}|^{q'}\Big)^{\frac{2}{q'}}\Big(\int_{Q_n}u^{\frac{q}{2-q}}\Big)^{\frac{2-q}{q}} \\
&\leq \bigg(\sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n}|\hat{f}|^{q'}\Big)^{\frac{p'}{q'}}\bigg)^{\frac{2}{p'}}\bigg(\sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n}u^{\frac{q}{2-q}}\Big)^{\frac{2-q}{q}\frac{p}{2-p}}\bigg)^{\frac{2-p}{p}} \\
&= \|u\|_{W(L^{\frac{q}{2-q}}, l^{\frac{p}{2-p}})}\|\hat{f}\|^2_{W(L^{q'}, l^{p'})} \\
&\leq C_u C_{p,q}\|f\|^2_{W(L^p, l^q)} \\
&\leq C\bigg(\sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n}|f|^2 v\Big)^{\frac{q}{2}}\Big(\int_{Q_n}v^{-\frac{p}{2-p}}\Big)^{\frac{q}{2}\frac{2-p}{p}}\bigg)^{\frac{2}{q}} \\
&\leq C\Bigg(\bigg(\sum_{n\in\mathbb{Z}^d}\int_{Q_n}|f|^2 v\bigg)^{\frac{q}{2}}\bigg(\sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n}v^{-\frac{p}{2-p}}\Big)^{\frac{q}{2-q}\frac{2-p}{p}}\bigg)^{\frac{2-q}{2}}\Bigg)^{\frac{2}{q}} \\
&= C\|v^{-1}\|_{W(L^{\frac{p}{2-p}}, l^{\frac{q}{2-q}})}\|f\|^2_{L^2_v} \\
&= C'\|f\|^2_{L^2_v}
\end{aligned}$$

■

This technique can easily be extended to obtain weighted Fourier transform estimates with other exponents as long as simple compatibility conditions (which allow application of Hölder's inequality) are met. For example,

THEOREM 3.3. *Given $1 \leq p \leq 2$, $1 \leq q \leq p'$, and $u \in W(L^\infty, l^1)(\mathbb{R}^d)$. One has*

$$\forall f \in L^p_v(\mathbb{R}^d), \quad \|\hat{f}\|_{L^q_u} \leq C\|f\|_{L^p_v}$$

if and only if $v^{1-p'} \in W(L^1, l^\infty)(\mathbb{R}^d)$.

The necessary condition in this result follows from Theorem 2.1, and the sufficiency follows from mimicking the proof of the previous theorem. In fact, one may prove a converse of this particular result which states that if the constant in the inequality above depends only on $\|v^{-1}\|_{W(l^1, l^\infty)}$, then $u \in W(L^\infty, l^1)(\mathbb{R}^d)$, *cf.*, [L2]. One also sees that

[M2, Theorem 7] is an immediate consequence of Theorem 3.3. The proof in [M2] relies on the theory of fractional integrals. We may interpret the local/global weight conditions in the theorem above as saying that the local condition on $v^{1-p'}$ should coincide with the global condition on u and that the global condition on $v^{1-p'}$ should be the same as the local condition on u.

4. **The Hausdorff-Young theorem for Wiener-Lorentz amalgam spaces.** Our goal in this section is to prove weighted norm inequalities for the Fourier transform where the weights satisfy mixed norm (or quasinorm) weak-type integrability conditions. We do this by first establishing mixed norm versions of the Hausdorff-Young theorem, then applying the same simple methods used in the proofs of Theorems 3.1 and 3.2. All that is required is the simple observation that we may apply real interpolation methods between Wiener amalgam spaces. All of the interpolation theoretic background may be found in [Tr] or [BB].

Our first observation is that the Wiener amalgam space $W(L^p, l^q)(\mathbb{R}^d)$ may be identified with the mixed norm space $l^q(L^p)(Q\times\mathbb{Z}^d)$ consisting of functions $g(x,n)$ defined on $Q\times\mathbb{Z}^d$ (where Q denotes the unit cube $[0,1)^d$) having finite norm

$$\|g(x,n)\|_{l^q(L^p)} = \Big(\sum_{n\in\mathbb{Z}^d}\Big(\int_Q |g(x,n)|^p\,dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}.$$

The isomorphism with $W(L^p, l^q)(\mathbb{R}^d)$ is simply given by the mapping $g(x,n) = f(x+n)$, *cf.*, [FS]. Again the usual adjustments are made when p and or q are infinite.

Next we observe that the mixed norm spaces are special instances of the vector valued sequence spaces $l^q(A)$ of functions $g(\cdot,n)$ having values in a Banach space A with norm $\big\|\{\|g(\cdot,n)\|_A\}\big\|_{l^q}$. For such spaces there are interpolation theorems where one interpolates between compatible pairs of underlying Banach spaces, A_1, A_2, as well as theorems where one fixes the space A and interpolates between the l^q components. We shall need both types of results. In the first result the underlying Banach spaces will be Lorentz spaces $L^{p,r}(Q)$, $1\le p, r\le\infty$, with quasinorm

$$\|g(\cdot,n)\|_{L^{p,q}} = \Big[\int_0^\infty \big(t^{\frac{1}{p}} g(\cdot,n)^*(t)\big)^q \frac{dt}{t}\Big]^{\frac{1}{q}}$$

when $0<p, q<\infty$, and

$$\|g(\cdot,n)\|_{\text{wk-}L^p} = \sup_{t>0} t\{\lambda_{g(\cdot,n)}(t)\}^{\frac{1}{p}} = \sup_{t>0} t^{\frac{1}{p}} g(\cdot,n)^*(t)$$

when $q=\infty$. Here $\lambda_f(t) = |\{x : |f(x)| > t\}|$ is the distribution function of f. The rearrangement is taken with n fixed so that $g(\cdot,n)$ is considered as a function defined on Q. On the other hand, one may define a Lorentz space norm in the $\mathbb{Z}^d$ component by $l^{q,s}(A)$ with quasinorm

$$\|g\|_{l^{q,s}(A)} = \Big(\sum_{j=1}^{\infty}\big(\|\{g\}\|_A^*(j)\big)^s j^{\frac{s}{q}-1}\Big)^{\frac{1}{s}}$$

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when $0 < p,\, q < \infty$. Here the rearrangement $\|\{g\}\|_A^*(j)$ denotes the reordering of the lattice norms $\|g(\cdot,n)\|_A$ in decreasing order of magnitude. When $q = \infty$ we take the wk l^p quasinorm to be

$$\sup_{s>0} s[\#\{n : \|g(\cdot,n)\|_A > s\}]^{\frac{1}{p}}.$$

The first interpolation theorem is a special case of [Tr, p.128].

THEOREM 4.1. *Fix exponents* $1 \leq p_1 < p_2 \leq \infty$ *and* $1 \leq q_1,\, q_2 < \infty$. *For* $0 < \theta < 1$ *write*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \textit{ and } \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then the real interpolation method yields

$$\big(l^{q_1}(L^{p_1})(Q\times\mathbb{Z}^d), l^{q_2}(L^{p_2})(Q\times\mathbb{Z}^d)\big)_{\theta,q} = l^q(L^{p,q})(Q\times\mathbb{Z}^d). \tag{4.1}$$

The next result tells us what happens when we fix the underlying Banach space and interpolate between the l^q components. The result is a special case of [Tr, pp. 125–127].

THEOREM 4.2. *Fix* $1 \leq p \leq \infty$, $1 \leq q_1 < q_2 \leq \infty$, *and* $1 \leq s \leq \infty$. *For* $0 < \theta < 1$ *write*

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then

$$\big(l^{q_1}(L^{p})(Q\times\mathbb{Z}^d), l^{q_2}(L^{p})(Q\times\mathbb{Z}^d)\big)_{\theta,s} = l^{q,s}(L^{p})(Q\times\mathbb{Z}^d). \tag{4.2}$$

The isomorphism between the Wiener amalgam spaces and the mixed norm spaces now allows us to define Wiener-Lorentz amalgam spaces.

DEFINITION 4.3. Given $0 < p,q,r,s \leq \infty$. One says that f belongs to the *Wiener-Lorentz amalgam space* $W(L^{p,r}, l^{q,s})(\mathbb{R}^d)$ if the function $g(x,n) = f(x+n)$ defined for $(x,n) \in Q\times\mathbb{Z}^d$ belongs to the mixed norm space $l^{q,s}(L^{p,r})(Q\times\mathbb{Z}^d)$. The amalgam space quasinorms are those inherited from the corresponding mixed quasinorm spaces.

The obvious adjustments are made in the case where any of the exponents are infinite—provided the corresponding Lorentz spaces are defined. Strictly speaking, Theorem 4.1 only gives us "diagonal spaces" $W(L^{p,r}, l^r)(\mathbb{R}^d)$. Off-diagonal versions can be obtained by substituting $L^{p,r}$ for L^p in Theorem 4.2. The Wiener-Lorentz spaces are normable whenever both components are normable.

REMARKS 4.4. The containment properties of the spaces $W(L^{p,r}, l^{q,s})(\mathbb{R}^d)$ are like those for Lorentz spaces in that they are increasing in the indices r, s, decreasing in p, and increasing in q. In the case of Wiener amalgam spaces one has $W(L^p, l^p) \equiv L^p$ so it is interesting to compare the global Lorentz space $L^{p,q}$ with the spaces $W(L^{p,q}, l^{p,q})$,

$W(L^{p,q}, l^p)$, and $W(L^p, l^{p,q})$. To simplify the illustration consider, in one dimension, the case where $q = \infty$.

(i) In this case wk-L^p is a proper subset of $W(\text{wk-}L^p, \text{wk}\, l^p)$. For example, set $f_k = x^{-\frac{1}{p}} k^{-\frac{1}{p}} \chi_{[0,1]}$, and $f = \sum_{k>0} f_k(x-k)$. Then $f \in W(\text{wk-}L^p, \text{wk}\, l^p)$ but $f \notin \text{wk-}L^p$.

(ii) On the other hand, one has $W(\text{wk-}L^p, l^p) \subseteq \text{wk-}L^p$. This is just a statement of the fact that

$$\|f\|_{\text{wk-}L^p}{}^p = \sup_{s>0} s^p \lambda_f(s) \leq \sum_k \sup_{t>0} t^p \lambda_{f_k}(t) = \|f\|^p_{W(\text{wk-}L^p, l^p)}.$$

(iii) There are no containment relations between wk-L^p and $W(L^p, \text{wk}\, l^p)$. Clearly, wk-L^p is not locally contained in L^p. On the other hand, set $f_k = \chi_{[k,k+\alpha_k]}$ such that $\sum_k \alpha_k = 1$. Then for $f = \sum_k f_k$ the decreasing rearrangement of f is simply $\chi_{[0,1]}$, independent of the choice of $\{\alpha_k\}$, whereas this sequence can be chosen to give f arbitrarily small norm in $W(L^p, \text{wk}\, l^p)$.

All of these properties persist when we replace $q = \infty$ by any $q > p$. We leave the details to the reader.

Of the possible versions of the Marcinkiewicz interpolation theorem that hold for these spaces, we single out two cases which can be applied to the Fourier transform, see *e.g.*, [BB, p. 180].

THEOREM 4.5. *Given exponents* $1 \leq p_1, p_2, p \leq \infty$, $1 \leq q_1, q_2, q \leq \infty$, $1 \leq r_1, r_2, r \leq \infty$, *and* $1 \leq s_1, s_2, s < \infty$. *For* $0 < \theta < 1$ *write*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}, \quad \frac{1}{s} = \frac{1-\theta}{s_1} + \frac{\theta}{s_2}.$$

a) If $q_1 \neq q_2$, $r_1 \neq r_2$, *and* T *is a linear operator such that*

$$T\colon W(L^p, l^{q_1}) \longrightarrow W(L^{r_1}, l^{s_1}), \quad \textit{with norm } M_1,$$
$$T\colon W(L^p, l^{q_2}) \longrightarrow W(L^{r_2}, l^{s_2}), \quad \textit{with norm } M_2,$$

then

$$T\colon W(L^p, l^{q,s}) \longrightarrow W(L^{r,s}, l^s), \quad \textit{with norm } CM_1^{1-\theta} M_2^{\theta}. \tag{4.3}$$

b) If $p_1 \neq p_2$, $q_1 \neq q_2$, *and* T *is a linear operator such that*

$$T\colon W(L^{p_1}, l^{s_1}) \longrightarrow W(L^r, l^{q_1}), \quad \textit{with norm } N_1,$$
$$T\colon W(L^{p_2}, l^{s_2}) \longrightarrow W(L^r, l^{q_2}), \quad \textit{with norm } N_2,$$

then

$$T\colon W(L^{p,s}, l^s) \longrightarrow W(L^r, l^{q,s}), \quad \textit{with norm } CN_1^{1-\theta} N_2^{\theta}. \tag{4.4}$$

Applying Theorem 4.5 to the Fourier transform we have the following extension of the Hausdorff-Young inequality.

COROLLARY 4.6 (HAUSDORFF-YOUNG). *Given $1 < p < 2$ and $1 \le q \le 2$, the Fourier transform $\mathcal{F}$ extends to a continuous map*

$$\mathcal{F}\colon W(L^{p,q}, l^q)(\mathbb{R}^d) \longrightarrow W(L^{q'}, l^{p',q}). \tag{4.5}$$

By duality one also has the continuity of

$$\mathcal{F}\colon W(L^{p}, l^{q,p'})(\mathbb{R}^d) \longrightarrow W(L^{q',p'}, l^{p'}). \tag{4.6}$$

The result follows by interpolating between one component in the Hausdorff-Young theorem for Wiener amalgam spaces when the other component is held fixed. Off-diagonal versions of this Hausdorff-Young theorem will be established below once we have desired endpoint versions.

Corollary 4.5 provides a version of the Hausdorff-Young theorem for a finer scale of spaces than previously considered. Furthermore, in view of Remarks 4.4 we note that in the case where $q = p$ this result neither implies nor is implied by Hunt's theorem for Lorentz spaces.

The following calculation shows how Hölder's inequality for the corresponding Wiener-Lorentz amalgam spaces can be applied to obtain weighted Fourier transform inequalities—when the weights lie in appropriate Wiener-Lorentz spaces and exponents are chosen appropriately.

Given $f \in L^2_v$ we have

$$\begin{aligned}
\|\hat{f}\|^2_{L^2_u} &= \sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n} |\hat{f}|^2 u\Big) \\
&\le \sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n} |\hat{f}|^{q'}\Big)^{\frac{2}{q'}}\Big(\int_{Q_n} u^{\frac{q}{2-q}}\Big)^{\frac{2-q}{q}} \\
&= \sum_{n\in\mathbb{Z}^d} \|\hat{f}\chi_{Q_n}\|^2_{L^{q'}} \|u\chi_{Q_n}\|_{L^{\frac{q}{2-q}}} \\
&\le C\Big\|\{\|\hat{f}\chi_{Q_n}\|^2_{L^{q'}}\}_n\Big\|_{l^{\frac{p'}{2},\frac{q'}{2}}} \Big\|\{\|u\chi_{Q_n}\|_{L^{\frac{q}{2-q}}}\}_n\Big\|_{l^{\frac{p}{2-p},\frac{q}{2-q}}} \\
&= \|\hat{f}\|^2_{W(L^{q'},l^{p',q'})} \|u\|_{W(L^{\frac{q}{2-q}},l^{\frac{p}{2-p},\frac{q}{2-q}})} \\
&\le C_u \|\hat{f}\|^2_{W(L^{q'},l^{p',q})} \\
&\le C_u \|f\|_{W(L^{p,q},l^q)} \\
&= C_u \|f v^{\frac{1}{2}} v^{-\frac{1}{2}}\|_{W(L^{p,q},l^q)} \\
&\le C_u C' \Big\|\{\|f v^{\frac{1}{2}}\chi_{Q_n}\|_{L^2} \|v^{-\frac{1}{2}}\chi_{Q_n}\|_{L^{\frac{2p}{2-p},\frac{2q}{2-q}}}\}_n\Big\|^2_{l^q} \\
&= C_u C' \Big\|\{\|f v^{\frac{1}{2}}\chi_{Q_n}\|_{L^2} \|v^{-1}\chi_{Q_n}\|^{\frac{1}{2}}_{L^{\frac{p}{2-p},\frac{q}{2-q}}}\}_n\Big\|^2_{l^q} \\
&= C_u C' \Big(\sum_{n\in\mathbb{Z}^d} (\|f v^{\frac{1}{2}}\chi_{Q_n}\|_{L^2})^q (\|v^{-1}\chi_{Q_n}\|_{L^{\frac{p}{2-p},\frac{q}{2-q}}})^{\frac{q}{2}}\Big)^{\frac{2}{q}}
\end{aligned}$$

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$$\leq C_u C' \Big(\sum_{n\in\mathbb{Z}^d} (\|fv^{\frac{1}{2}}\chi_{Q_n}\|_{L^2})^2 \Big) \Big(\sum_{n\in\mathbb{Z}^d} (\|v^{-1}\chi_{Q_n}\|_{L^{\frac{p}{2-p},\frac{q}{2-q}}})^{\frac{q}{2}\frac{2}{2-q}} \Big)^{\frac{2}{q}\frac{2-q}{2}}$$
$$= C_u C' \|f\|^2_{L^2_v} \|v^{-1}\|_{W(L^{\frac{p}{2-p},\frac{q}{2-q}}, l^{\frac{q}{2-q}})}$$
$$= C_u C_v \|f\|^2_{L^2_v}.$$

In this calculation we have used the Hölder's inequality for $L(p,q)$ spaces along with the fact that $(f^*)^\alpha = (f^\alpha)^*$ whenever f is nonnegative and $\alpha > 0$.

Thus we have unified the results in Section 3.

THEOREM 4.7. *Given exponents $1 < p < 2$ and $1 \leq q \leq 2$, and weights u, v such that $u \in W(L^{\frac{q}{2-q}}, l^{\frac{p}{2-p},\frac{q}{2-q}})$ and $v^{-1} \in W(L^{\frac{p}{2-p},\frac{q}{2-q}}, l^{\frac{q}{2-q}})$. Then*

$$\forall f \in L^2_v(\mathbb{R}^d), \quad \|\hat{f}\|_{L^2_u} \leq C_u C_v \|f\|_{L^2_v}. \tag{4.7}$$

A similar calculation, using instead the second inequality in Corollary 4.6 yields a corresponding weighted norm inequality in the case where we reverse the local and global weight conditions. Of course we could modify the argument slightly to prove weighted norm inequalities of the form

$$\forall f \in L^r_v(\mathbb{R}^d), \quad \|\hat{f}\|_{L^s_u} \leq C\|f\|_{L^r_v}$$

where $r, s \in [1,\infty)$. The result in Theorem 4.7 is sharpest when $q = 2$, since this is the only case where the estimate

$$\|\hat{f}\|_{W(L^{q'}, l^{p',q'})} \leq \|\hat{f}\|_{W(L^{q'}, l^{p',q})}$$

used in the calculation before Theorem 4.7 is sharp.

EXAMPLE 4.8. Take $v(x) = |\sin x|^{\frac{2-p}{p}}$ and set $u(\xi) = (1+|\xi|)^{\frac{p-2}{p}}$, where $1 < p < 2$, and take $q = 2$. Thus $v^{-1} \in W(L^{\frac{p}{2-p},\infty}, l^\infty)$ and $u \in W(L^\infty, l^{\frac{p}{2-p},\infty})$. Theorem 4.7 applies in this case to give us a weighted norm inequality. Notice that the Fourier domain weight of Example 2.2 fails to satisfy the weight condition in Theorem 4.7.

So far we have only established $W(L^{p,q}, l^q)$ estimates for the Fourier transform. To establish $W(L^{p,q}, l^r)$ estimates, where $q \neq r$, we need an extension of the "endpoint" result involving $W(L^p, l^1)$ to one involving $W(L^{p,q}, l^1)$. Then the desired estimates can be obtained by interpolating the global component with the local $L^{p,q}$ space held fixed. We begin with the following lemma, which is really [Ho, Lemma, p. 301] restated.

LEMMA 4.9. *Given $g \in W(L^\infty, l^1)(\mathbb{R}^d)$, and $1 \leq s < \infty$, one has*

$$\forall h \in L^s(\mathbb{R}^d), \quad \|h * g\|_{W(L^\infty, l^s)} \leq 2^{\frac{1}{s}} \|g\|_{W(L^\infty, l^1)} \|h\|_{L^s}.$$

By means of the Marcinkiewicz interpolation theorem (for vector-valued functions with values in L^∞), we have

COROLLARY 4.10. *Given* $1 \leq s < \infty$, $1 \leq t \leq \infty$,

$$\forall h \in L^{s,t}(\mathbb{R}^d), \quad \|h * g\|_{W(L^\infty, l^{s,t})} \leq C_{g,s,t}\|h\|_{L^{s,t}}.$$

We wish to apply this last corollary to get a new endpoint version of the Hausdorff-Young inequality. Begin with a function $f \in W(L^{p,q}, l^1)(\mathbb{R}^d)$, and write $f_n = f\chi_{Q_n}$. Let $k_n = \tau_n k$ where k is a smooth cutoff function which is one on Q_0 (so that $\hat{k} \in W(L^\infty, l^1)(\mathbb{R}^d)$). Thus $\hat{f}_n = \hat{f}_n * \hat{k}_n$. Hence,

$$\begin{aligned}
\|\hat{f}\|_{W(L^\infty, l^{p',q})} &= \|\sum_n \hat{f}_n\|_{W(L^\infty, l^{p',q})} \\
&= \|\sum_n \hat{f}_n * \hat{k}_n\|_{W(L^\infty, l^{p',q})} \\
&\leq C \sum_n \|\hat{f}_n * \hat{k}_n\|_{W(L^\infty, l^{p',q})} \\
&\leq C \sum_n \|\hat{f}_n\|_{L^{p',q}} \\
&\leq C' \sum_n \|f_n\|_{L^{p,q}} = C'\|f\|_{W(L^{p,q}, l^1)}.
\end{aligned}$$

Thus we have proven

COROLLARY 4.11. *Given* $1 < p \leq 2$ *and* $1 \leq q \leq \infty$, *one has*

$$\forall f \in W(L^{p,q}, l^1)(\mathbb{R}^d), \quad \|\hat{f}\|_{W(L^\infty, l^{p',q})} \leq C\|f\|_{W(L^{p,q}, l^1)}. \tag{4.8}$$

One can apply the Marcinkiewicz interpolation theorem now to get inequalities of the form

$$\|\hat{f}\|_{(W(L^\infty, l^{p',q}), W(L^{q'}, l^{p',q}))_{\theta,s}} \leq C\|f\|_{(W(L^{p,q}, l^1), W(L^{p,q}, l^q))_{\theta,s}}.$$

Unfortunately, there seems to be no simple identification of the intermediate spaces appearing in this inequality. Nevertheless, if we apply the Marcinkiewicz interpolation theorem between this inequality and (4.6), we get

$$\|\hat{f}\|_{W((L^\infty, l^{q'})_{\theta,p'}, l^{p'})} \leq C\|f\|_{W(L^p, (l^1, l^q)_{\theta,p'})}.$$

One can use Corollary 4.11 together with Corollary 4.6 to get a local improvement of the classical Hausdorff-Young theorem. That is, from Corollary 4.11, together with containment properties of Lorentz spaces we have (when $1 < p \leq q \leq 2$)

$$\forall f \in W(L^{p,q}, l^1)(\mathbb{R}^d), \quad \|\hat{f}\|_{W(L^\infty, l^{p'})} \leq \|\hat{f}\|_{W(L^\infty, l^{p',q})} \leq C\|f\|_{W(L^{p,q}, l^1)},$$

and

$$\|\hat{f}\|_{W(L^{q'}, l^{p'})} \leq \|\hat{f}\|_{W(L^{q'}, l^{p',q})} \leq C\|f\|_{W(L^{p,q}, l^q)}.$$

Interpolating between these two inequalities yields

COROLLARY 4.12. *Given* $1 < p \leq 2$ *and* $r \leq q \leq 2$.

$$\forall f \in W(L^{p,q}, l^1)(\mathbb{R}^d), \quad \|\hat{f}\|_{W(L^{r'}, l^{p'})} \leq C\|f\|_{W(L^{p,q}, l^r)}, \tag{4.9}$$

REMARKS 4.13. As mentioned, the weight conditions in Theorem 4.7 are not expected to be sharp except in the case $q = 2$. Nevertheless, the condition gives a rough explanation of why the necessary condition (UP) fails to be sufficient (*i.e.*, it does not require the weight u to satisfy the same weight condition as $v^{1-p'}$ where the local and global conditions are reversed). We may reinterpret the rearrangement condition (where the weights are decreasing) as saying that u and v^{-1} have the correct sort of local/global duality.

The results in this paper may be viewed as Hardy-Littlewood-Paley, or Pitt type theorems, in the sense that the weights considered must satisfy some weak-type integrability condition—either locally or globally. More refined interpolation techniques will likely provide improved versions of Corollary 4.6. For example, it is likely that versions of (4.5) and (4.6) hold with Lorentz space norms in both the local and global components simultaneously. However, to actually solve the local/global rearrangement Fourier transform problem we need to find the correct analogues of tools used in the solutions of the rearrangement Fourier transform problem. One might look for local/global analogues of Hardy inequalities used in [BH] and [JS2], or one might try to reduce the Fourier transform problem to a form of the Hausdorff-Young theorem as in [M2].

In addition to the particular amalgam spaces considered here, one can also consider Wiener-Lorentz amalgam spaces in the setting of other groups, or by considering other decompositions, such as the dyadic decomposition of $\mathbb{R}^d$. The latter is a natural setting for considering endpoint versions of Pitt-type inequalites as studied, for example, in [BL].

As pointed out in the introduction, a solution of the local/global rearrangement problem still would not be the end of the story, since the problem does not cover the case of weighted Fourier transform inequalities with measure weights (for example, restriction theorems). Nonetheless, such a solution would significantly refine our understanding of the uncertainty principle.

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