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WHEN ARE QUASI-INJECTIVES INJECTIVE?

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We call a ring R (associative and with identity) for which every quasi-injective right R-module is injective a *QII*-ring. Similarly R is called an *SSI*-ring when every semisimple right R-module is injective. Clearly every semisimple, artinian ring is a *QII*-ring and every *QII*-ring is an *SSI*-ring. One then asks whether these inclusions among classes of rings are proper. The purpose of this note is to point out an instance when *SSI* implies *QII*. It is then easy to see that an example of Cozzens shows that the class of *QII*-rings properly contains the class of semisimple, artinian rings.

The aim of this note is to prove the following:

PROPOSITION. *If R is an SSI-ring for which right ideals are principal and if R has the ascending chain condition on principal left ideals then R is a QII-ring.*

The proposition is easily proved by making use of the following results:

(A) A ring R is right noetherian if and only if countable direct sums of injective hulls of simple right R-modules are injective [5].

(B) If R is a prime ring with ascending chain condition on complement right ideals and annihilator right ideals and if R is a V-ring (i.e. simple, right R-modules are injective) then R is simple [2, p. 130].

(C) If R is a prime ring and every right ideal of R is principal then R is isomorphic to D_n, the ring of $n \times n$ matrices over some right Ore domain D [3].

Proof of Proposition. (A) implies that *SSI*-rings are the same as right noetherian V-rings. (B) implies that for any *SSI*-ring the prime ideals are the same as the maximal ideals. It is well known that any V-ring has zero Jacobson radical and hence is semiprime. Thus *SSI*-rings are right Goldie rings and there is a finite set $\{P_1, \ldots, P_n\}$ of prime ideals with $\cap_{k=1}^{n} P_k = 0$ [6, p. 112]. Since these are maximal ideals we may express R as a direct sum of the simple rings R/P_k as in [7, p. 59]. Thus we may assume R is a simple ring. From (C) we then have $R \simeq D_n$ where D is a right Ore domain with ascending chain condition on principal left ideals. Since the categories of D-modules and D_n-modules are equivalent D is an *SSI*-ring.

Matlis [8] has shown that the injective D-modules are direct sums of indecomposable injectives of the form $E(D/I)$, i.e. each isomorphic to the injective hull of a module D/I where I is a meet-irreducible right ideal of D. As in [4, p. 30] we

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know that since D has a.c.c. on principal left ideals then any module D/I where $I\neq 0$ is right artinian. Thus since D is a V-ring and a right Ore domain the indecomposable injectives are the simple D-modules together with $E(D)_D$, the injective hull of D_D. As is well known, $E(D)_D$ is the right classical quotient ring of D and is a division ring.

The quasi-injective D-modules are those modules M with the property that $\Lambda M\subseteq M$ where $\Lambda=\text{End}_D E(M)$, i.e. the D-modules that are invariant in their injective hull. Thus a quasiinjective D-module is a direct sum of invariant submodules of indecomposable injectives, i.e. direct sums of modules which are either simple or invariant submodules of $E(D)_D$. But invariant submodules of $E(D)_D$ are those submodules closed under left action by $\text{End}_D E(D)=\text{End}_{E(D)}E(D)\cong E(D)$, i.e. $E(D)-D$ submodules of ${}_{E(D)}E(D)_D$. Thus invariant submodules of $E(D)_D$ are left ideals of $E(D)$ hence either 0 or $E(D)$. Clearly then D is a QII-ring.

From the category equivalence it follows that $R\cong D_n$ is a QII-ring. This completes the proof of the proposition.

REMARK. With minor changes the same proof yields that any SSI-ring where left ideals are principal is a QII-ring.

EXAMPLE. In [1] Cozzens has exhibited a simple (noncommutative) domain D which is a V-ring having all its one-sided ideals principal which is not a division ring. From the proposition it is clear that Cozzens' ring is a QII-ring so QII-rings need not be (semisimple) artinian.

Obvious open questions are:

(a) What is the structure of a simple QII-ring?

(b) Is every SSI-ring also a QII-ring?

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