SUBGROUPS WITH CENTRE IN HNN GROUPS

A. KARRASS and D. SOLITAR*

Abstract

A subgroup with non-trivial centre in a one-relator group is shown to be a treed HNN group (graph product) with infinite cyclic vertices. Moreover, subgroups with non-trivial centre in HNN groups are also examined.

1. Introduction

Karrass, Pietrowski and Solitar (1974) have proved the following:

If G is a one-relator group and H is a finitely generated subgroup with non-trivial centre, then H is a treed HNN group with infinite cyclic vertices. (A treed HNN group is an HNN group whose base is a tree product and whose associated subgroups are contained in vertices of the tree product base).

Here we establish the result for H infinitely generated. More generally, *if G is a group built up from the infinite cyclic group by repeatedly forming free products and HNN extensions always using free amalgamated and associated subgroups, and H is a subgroup with nontrivial centre, then H is a treed HNN group with infinite cyclic vertices.*

As a corollary we obtain that H is an extension of a free group by a subgroup of the additive rationals; this was first proved by Cossey and Smythe (1975). As another corollary we have that if $H_1 < H$ and H_1 has trivial intersection with the centre of H, then H_1 is a free group; this result was first proved for H_1 finitely generated in Karrass, Pietrowski and Solitar (1974) and extended to infinitely generated H_1 in Bieri and independently in Bagherzadeh (1976).

Moreover, let G be a generalized free product

(1) $G = (A * B ; U)$

or an HNN group

(2) $G = \, < t, K ; \, tLt^{-1} = M >$

and let H be a subgroup of G with g a central element of H not contained in a conjugate of U or of L. If G is as in (1), *then H is in a conjugate of A or B, or H is an infinite cyclic extension of a subgroup of a conjugate of U. Similarly, if G is as in* (2), *then H is in a conjugate of K, or H is an infinite cyclic extension of a subgroup of a conjugate of L.*

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2. Parity and sign in HNN groups

Let G be as in (2) and let L_ε denote L when $\varepsilon = 1$ and M when $\varepsilon = -1$. A *reduced form* in G is a product

(3) $k_0 t^{\varepsilon_1} k_1 t^{\varepsilon_2} \cdots t^{\varepsilon_n} k_n$

where $\varepsilon_i = \pm 1$, $k_i \in K$, and if $\varepsilon_{i+1} = -\varepsilon_i$ then $k_i \notin L_{\varepsilon_i}$; n is called the *t-length* of (3). [Britton's lemma implies that a reduced form (3) defines an element $\neq 1$ if either $n > 0$ or $k_0 \neq 1$.] A *block B* is a product (3) where $n > 0$, $k_0 = k_n = 1$, $\varepsilon_i = \varepsilon$ for $i = 1, \cdots, n$; ε is the *sign* of B. Clearly, a reduced form (3) can be written in *block form*

$$k_0 B_1 k_{i_1} B_2 k_{i_2} \cdots B_r k_{i_r}$$

where the B_j alternate in sign and $k_{i_j} \neq 1$ for $j \neq r$; r is the *block-length* of (3).

Suppose right coset representative systems for K mod L and K mod M, respectively, are chosen. Then a reduced form (3) is called *normal* if k_i is a representative for K mod L_{ε_i}, $i > 0$; every element in G has a unique normal form (this follows easily from Britton's lemma).

LEMMA 1. *Let the reduced form*

(4) $p_0 t^{\delta_1} p_1 t^{\delta_2} \cdots t^{\delta_m} p_m$

define the same element as the reduced form (3). *Then* (3) *and* (4) *have the same t-length (i.e., $m = n$), and $\varepsilon_i = \delta_i$; moreover, k_i and p_i determine the same double coset of K mod $(L_{\varepsilon_i}, L_{-\varepsilon_{i+1}})$ where $L_{r_0} = L_{-\varepsilon_{n+1}} = 1$. In particular, any two reduced forms of an element of an HNN group have the same block length and corresponding blocks have the same t-length and sign.*

PROOF. It is easily shown by induction on the t-length of (3), that if (4) is the normal form for (3), then the conclusions of the lemma are satisfied.

Specifically, (3) is normalized by first normalizing $k_1 t^{\epsilon_2} k_2 \cdots t^{\epsilon_n} k_n$ to $p_1' t^{\epsilon_2} p_2 \cdots t^{\epsilon_n} p_n$ and then $k_0 t^{\epsilon_1} p_1' t^{\epsilon_2} p_2 \cdots t^{\epsilon_n} p_n$ is normalized to $p_0 t^{\epsilon_1} p_1 t^{\epsilon_2} p_2 \cdots t^{\epsilon_n} p_n$. □

If $g \in G$, then $\tau(g), \beta(g)$ denote the t-length and block-length of g. The *sign* of g is the sign of the first block of g; the *cosign* of g is the sign of the last block of g. An element of G not in K is *odd* or *even* according as it has odd or even block length; an element of G not in K is *positive* or *negative* according as its sign is positive or negative.

LEMMA 2. *Let G be as in* (2). *Then an odd element cannot commute with an even element, and a positive even element cannot commute with a negative even element.*

PROOF. It is easy to see that if v, w are in $G - K$, and cosign $v = $ sign w, then $\tau(vw) = \tau(v) + \tau(w)$, $\beta(vw) = \beta(v) + \beta(w) - 1$, sign $vw = $ sign v, and cosign $vw = $ cosign w. On the other hand, if cosign $v \neq $ sign w, then $\tau(vw) < \tau(v) + \tau(w)$ or $\beta(vw) = \beta(v) + \beta(w)$.

Suppose $vw = wv$. Then cosign $v = $ sign w iff cosign $w = $ sign v.

Hence, if v is odd, sign $v = $ cosign v and so sign $w = $ cosign w; hence w is odd. Also if v is positive even, then w is even; if w were negative, then sign $v = $ sign $vw = $ sign $wv = $ sign w, which implies w is positive. □

COROLLARY 1. *Let G be as in* (2) *with $K \neq L$. Then the centre of G is in L.*

PROOF. Since $K \neq L$, G contains an even element tkt^{-1}, $k \notin L$, as well as the odd element t. Therefore, the centre of G is in K and hence in $t^{-1}Kt$. Since $L = K \cap t^{-1}Kt$, the centre of G is in L. □

The following corollary, although not needed for the proofs of Theorems 1 or 2, is also a consequence of Lemma 2. It was obtained independently in Bagherzadeh (1976) and by L. Comerford (unpublished).

COROLLARY 2. *Let G be as in* (2) *with $L \neq K$ or $M \neq K$. If G is a non-trivial direct product $C \times D$, then C or D is in L.*

PROOF. Since $t = cd$ where $c \in C$ and $d \in D$, c and d commute with t; hence c or d is odd. To be specific suppose d is odd. Then by Lemma 2, C cannot contain even elements. We show, however, that C does contain even elements if $C \not\subseteq L$.

Clearly, $C \not\subseteq K$ since $K \cap t^{-1}Kt = L$. Assume $K \neq L_\epsilon$. If C contains an element $k_0 B_1$ of block length one and sign ϵ, then C contains $k_0 B_1 k B_1^{-1} k_0^{-1} k^{-1}$ where $k \notin L_\epsilon$; hence if $L \neq K (M \neq K)$, C would contain a positive even (negative even) element. If $g \in C$ with $g = k_0 B_1 \cdots B_n$, $n \geq 2$, n odd, sign $B_n = \epsilon$, then g is cyclically reduced of block length ≥ 2; but cyclic

permutations of a cyclically reduced element of block length ≥ 2, will yield positive and negative even elements.   □

COROLLARY 3. *If a one-relator group G is a proper direct product, then G is finite cyclic or free abelian of rank two.*

PROOF. Let $G = C \times D$, $C \neq 1 \neq D$, and suppose G is not cyclic. We may Nielsen transform the generators of G so that the corresponding relator is in pre-abelianized form (see p. 142 in Magnus, Karrass, and Solitar (1966)) and will have zero exponent sum on one of the generators, say t; moreover, since G cannot be a proper free product, when rewritten in terms of the conjugates of the other generators by t, this relator will involve distinct conjugates of at least one generator. Then by the standard method for writing a one-relator group as an HNN extension of another one-relator group (see Moldavanski (1967)), $G = \langle t, K ; tLt^{-1} = M \rangle$ where L is free, $K = gp(L, M)$, and K is a one-relator group on rank L + rank $G - 1$ generators (provided we include in K at least two consecutive conjugates of every generator $\neq t$ of G).

If $K \neq L$ or $K \neq M$, then by the preceding corollary, we may suppose $C < L$. Since L is free, $D \cap L = 1$ and so $C = L$. But $C \triangleleft G$, so that $M = L = K$.

Thus $L = K = M$ and hence rank $G \leq$ rank $K + 1 -$ rank $L + 1 = 2$. On the other hand, C and D must be free (Corollary 2 to Theorem 2 in Karrass, Pietrowski and Solitar (1974)), and homomorphic images of G; but if C or D had rank 2 than $G = C \times D$ would have rank ≥ 3. Consequently C and D must be cyclic and G free abelian of rank two.   □

LEMMA 3. *Let G be as in (2), $H < G$, and suppose H contains an odd element. Then H contains a positive even element unless H is conjugate (by an element of K) to an HNN group $\langle s, L' ; sL's^{-1} = L'' \rangle$ where $L'' < L' < L$; if $H - K$ contains only odd elements, then $L' = L''$.*

PROOF. By induction on $\tau(u)$, one can easily show that if u, v are in $G - K$ and $\tau(u) \geq \tau(v)$, then one of the following three possibilities holds: sign $uv =$ sign u and cosign $uv =$ cosign v; or $\tau(uv) < \tau(u)$ and sign $uv =$ sign u; or $uv \in K$.

Now let s be a positive odd element of shortest t-length in H, and suppose H contains no positive even elements. We show that $K \cap H = kL'k^{-1}$ where $L' < L$, $k \in K$, and that $H = gp(s, K \cap H)$. If s has reduced form as in (3), and $k' \in K \cap H$, then $sk's^{-1}$ will be positive even unless $k_n k' k_n^{-1} \in L$; hence $K \cap H = k_n^{-1} L' k_n$ where $L' < L$. Moreover, if u is any positive element in H, then u is odd and $\tau(u) \geq \tau(s)$. Since us^{-1} is not positive even, our first remark implies that $us^{-1} \in K$, or $\tau(us^{-1}) < \tau(u)$ and us^{-1} is positive. Hence by induction on the t-length of u, every positive element of H

has the form ps^n, $n > 0$, $p \in K \cap H$; moreover $\tau(ps^n) = n\tau(s)$. Therefore $s(K \cap H)$, which consists of positive elements having the same t-length as s, is contained in $(K \cap H)s$, so that $s(K \cap H)s^{-1} = K' < K \cap H$. Hence the HNN group $R = \langle s, K \cap H; s(K \cap H)s^{-1} = K' \rangle$ can be mapped homomorphically onto $H_1 = gp(s, K \cap H)$ in H; thus the normal subgroup generated by $K \cap H$ in H_1 is just the ascending union $s^{-m}(K \cap H)s^m$, $m \geqq 0$, and so has trivial intersection with $gp(s)$ in H_1 since $gp(s) \cap (K \cap H) = 1$; thus the homomorphism of R onto H_1 has kernel in the normal closure of $K \cap H$ in R. But this is mapped isomorphically onto the normal closure of $K \cap H$ in H_1; hence, R is mapped isomorphically onto H_1.

To show that $H_1 = H$, it remains to consider the negative elements of H. Clearly, any negative odd element of H is in H_1 since it has the form $s^{-n}p^{-1}$, $n > 0$, $p \in K \cap H$. Let q be a negative even element of smallest t-length in $H - H_1$. If $\tau(q) \leqq \tau(s)$, then by our initial remark, $sq \in H_1$ contrary to $q \notin H_1$; if $\tau(q) > \tau(s)$ then $qs^{-1} \in H_1$, or $\tau(qs^{-1}) < \tau(q)$ and qs^{-1} is negative even, contrary to the definition of q.

Finally, if H contains no negative even element, then $s^{-1}(K \cap H)s < K \cap H$, since $(s^{-1}k)s$ cannot be negative and of smaller t-length than s. $\quad\square$

An analogous result clearly holds for subgroups H of G which contain no negative even elements.

Using similar reasoning we obtain analogous results for an amalgamated product (1). Specifically, call the elements of $A - U$ positive and of $B - U$ negative; a non-trivial product $g_1 g_2 \cdots g_n$ with factors g_i alternating from $A - U$ and $B - U$ has the sign of g_1 and the parity of n.

LEMMA 4. Let G be as in (1). Then an even element cannot commute with an odd element, and a positive odd element cannot commute with a negative odd element.

LEMMA 5. Let G be as in (1), $H < G$, and suppose H contains an even element. Then H contains both a positive odd element and a negative odd element unless H is conjugate to an HNN group $\langle s, U'; sU's^{-1} = U'' \rangle$ where $U'' < U'$ and $U' = H \cap U$; if $H - U$ contains only even elements then $U' = U''$.

3. Subgroups with centre

THEOREM 1. (i) Suppose G is as in (1), $H < G$, g is in the centre of H, and g has no conjugate in U. Then H is conjugate to an HNN group $\langle s, U'; sU's^{-1} = U' \rangle$ where $U' < U$ (i.e., H is an infinite cyclic extension of a subgroup of a conjugate of U) unless H is in a conjugate of a factor.

(ii) Suppose G is as in (2), $H < G$, g is in the centre of H, and g has no

conjugate in L. *Then* H *is conjugate to an HNN group* $\langle s, L'; sL's^{-1} = L' \rangle$ *where* $L' < L$ *unless* H *is in a conjugate of* K.

PROOF. (i) If g is in a factor then $C(g)$, the centralizer of g, is also in a factor. If no conjugate of g is in a factor, then we may assume g is cyclically reduced and even. Hence by Lemma 4, H has no odd elements and so H has the asserted structure by Lemma 5.

(ii) If $g \in K$ then again an easy argument shows that $C(g) < K$. Suppose no conjugate of g is in K. If some conjugate H^x of H contains an odd element, then by Lemma 2, g^x is odd and so H has the asserted structure by Lemmas 2 and 3.

Suppose no conjugate of H contains an odd element. Now some cyclic rearrangement of a cyclically reduced element of G is odd unless all of its blocks have t-length 1. Hence an element of G having no odd conjugate must be in a conjugate of $Q = \langle K * tKt^{-1}; M = tLt^{-1} \rangle$. Thus $H < K^G$. We may assume $g \in Q$, and show $H < Q$. Now K^G is a tree product of three groups Q_{-1}, Q, Q_1, where Q_{-1} is generated by all $t^{-n}Kt^n$ for $n \geq 1$, Q_1 is generated by all t^mKt^{-m} for $m \geq 2$, with $t^{-1}Mt = L$ amalgamated between Q_{-1} and Q, and $tMt^{-1} = t^2Lt^{-2}$ amalgamated between Q and Q_1. Since g is not in a conjugate of L, the amalgamations are proper and so by the first remark in the proof of (i) above, $H < Q$. Since g is not in a conjugate of K, (i) applies again to show that H has the asserted form. $\square$

COROLLARY. *If* g *is an odd element in* (2), *then* $C(g)$, *the centralizer of* g, *is an infinite cyclic extension of a subgroup of a conjugate of* L.

4. One relator groups and generalizations

We next wish to describe the structure of subgroups with non-trivial centre of one-relator groups. As is well known (and follows from Lemma 4) the centre of an amalgamated product $G = (A * B; U)$ is in U if $A \neq U \neq B$. Similarly, the centre of a tree product is in each edge if the tree product is proper, i.e., if it has at least two vertices and for each edge U_{ij} joining A_i to A_j, $A_i \neq U_{ij} \neq A_j$. On the other hand, if the tree product is not proper, its centre need not be contained in any edge, e.g., consider a properly ascending union of groups. However, this is essentially the only exception.

LEMMA 6. *Let* G *be given as a tree product* P. *If* G *is not an ascending union of some of its vertices, then* G *can be expressed as a tree product* P'' *of a subset of the original vertex groups and edge groups of* P *in such a way that the centre of* G *is in each edge group; moreover, each of the vertices of* P *is contained in a vertex group of* P''.

PROOF. We wish to write P as a proper tree product. Let A_0 be a vertex

group of P and define the level $\lambda(A_i)$ of the vertex group A_i as the length of the shortest path in P from A_0 to A_i. Clearly, every subtree of P has a unique vertex of minimal level.

An edge of P joining A_i to A_j with $\lambda(A_i) < \lambda(A_j)$ is said to be *descending* (*strictly ascending*) if A_j is contained in A_i (A_j strictly contains A_i); a subgraph is *descending* (*strictly ascending*) if each edge of it is descending (strictly ascending). If we delete all edges of P but the descending ones (strictly ascending ones), then the resulting connected components are maximum descending (strictly ascending) subtrees. Clearly the tree product of every maximum descending subtree is just its vertex of minimal level.

We shall now eliminate the descending edges from P by contracting the maximum descending subtrees. By the generalized associative law for tree products (Theorem 1 of Karrass and Solitar (1970)), G is the tree product P' of each of the vertices of P of minimal level in some maximum descending subtree of P; moreover the edges of P' are the edges of P which join vertices from different maximum descending subtrees of P. Let λ' be the level function for the tree P' with $\lambda'(A_0) = 0$. Then P' has no descending edges. For, suppose A_i, A_j are joined by an edge in P' and $A_i < A_j$. Then in P there are neighboring vertices A'_i and A'_j such that there exists (in P) a descending path from A_i to A'_i and from A_j to A'_j. Now in P, the simple path joining A_i to A_j generates its tree product, and so A_i is contained in each of the vertices of this simple path. Thus $A'_i < A_i < A'_j$, and hence $\lambda(A'_i) < \lambda(A'_j)$ since the edge joining A'_i and A'_j cannot be descending. But since A'_j can be joined by an edge to a unique vertex of smaller level, the vertex A'_j must coincide with the vertex A_j. Therefore in P the simple path from A_0 to A_j consists of the simple path from A_0 to A_i, the descending path from A_i to A'_i, and the edge from A'_i to A_j; moreover, $A_i(A_j)$ is the only vertex on this simple path joining A_0 to $A_i(A_j)$, which lies in the maximum descending subtree containing $A_i(A_j)$. When this path is projected into P' and made simple, there results a simple path in P' joining A_0 to A_j and passing through A_i; thus $\lambda'(A_j) > \lambda'(A_i)$. Consequently, P' has no descending edges.

We next contract slightly the maximum strictly ascending subtrees. Clearly P' is the tree product of its maximum strictly ascending subtrees, with edges the proper edges of P'. Let M be a maximum strictly ascending subtree of P'. If M is a chain and some proper edge of P' has a vertex in M, let A_i be the vertex of M of minimum level on a proper edge; let M' result from M by contracting the vertices below A_i into A_i. If M is not a chain, then there exists a unique vertex A_i of minimum level which is joined in M to at least two vertices of higher level; let M' result from M by contracting the vertices below A_i into A_i. Let P'' result from P' by replacing each M by M'.

Suppose now that G is not an ascending union of vertices. Then Z, the centre of G, will be contained in each edge of P''. For, Z is contained in each proper edge of P'' and so is in each maximum strictly ascending subtree M' of P''. If M' is a chain then, by construction, there is a proper edge adjacent to the lowest vertex of M' and so Z is contained in each vertex and edge of M'. If M' is not a chain then $M' = (C * D; U)$ where U is the vertex of minimum level in M' and C, D each contain a vertex of M' joined to U. Since M' is strictly ascending, $C \neq U \neq D$; hence $Z < U$. But each vertex of M' contains U and so each edge of M' contains Z. $\square$

COROLLARY. *Let G be given as a tree product P. If G is not an ascending union of vertices of P, then G can be expressed as a proper tree product Q whose edge groups are edge groups from P and whose vertices are strictly ascending trees with vertex groups and edge groups from P.*

LEMMA 7. *Let G be a treed HNN group with tree product base B having vertices $\{A_i\}$ where each A_i is itself a treed HNN group such that each amalgamated and associated subgroup of G incident with A_i is conjugate in A_i to a subgroup of a vertex of A_i. Then G is a treed HNN group whose base C is a tree product with vertices conjugate in B to the vertices of the various A_i and whose amalgamated subgroups are conjugate in B to the original amalgamated subgroups of G and to the amalgamated subgroups of the various A_i; finally, the new associated subgroups are conjugate in B to the original associated subgroups of G and to the associated subgroups of the various A_i.*

PROOF. Let A_0 be a vertex of B and let λ be the level function for B with $\lambda(A_0) = 0$. For each vertex A_1 in B of level one, there exists an edge $U_{01} = U_{10}$ joining A_0 to A_1. By hypothesis $a_{01} U_{01} a_{01}^{-1}$ and $a_{10} U_{10} a_{10}^{-1}$ are in vertices of A_0 and A_1 respectively, where $a_{ij} \in A_i$. If we define $A_1' = a_{01} a_{10}^{-1} A_1 a_{10} a_{01}^{-1}$, then $gp(A_0, A_1) = gp(A_0, A_1') = (A_0 * A_1'; a_{01} U_{01} a_{01}^{-1} = a_{01} U_{10} a_{01}^{-1})$, and the amalgamated subgroups are in vertices of the treed HNN groups A_0 and A_1'. Similarly, each vertex A_2 of level two can be joined by an edge $U_{21} = U_{12}$ to a vertex A_1 of level one, where $a_{12} U_{12} a_{12}^{-1}$ and $a_{21} U_{21} a_{21}^{-1}$ are in vertices of A_1 and A_2, respectively. If we define $A_2' = a_{01} a_{10}^{-1} a_{12} a_{21}^{-1} A_2 a_{21} a_{12}^{-1} a_{10} a_{01}^{-1}$, then

$$gp(A_0, A_1, A_2) = gp(A_0, A_1', A_2') = \langle A_0 * A_1' * A_2'; a_{01} U_{01} a_{01}^{-1} =$$

$$= a_{01} U_{10} a_{01}^{-1}, a_{01} a_{10}^{-1} a_{12} U_{12} a_{12}^{-1} a_{10} a_{01}^{-1} = a_{01} a_{10}^{-1} a_{12} U_{21} a_{12}^{-1} a_{10} a_{01}^{-1} \rangle$$

where the amalgamated subgroups are in vertices of the treed HNN groups A_0, A_1', A_2'. Continuing in this way we obtain that B is a treed HNN group whose tree product base C has as vertices conjugates in B of the vertices of

the various A_i; moreover, the amalgamated subgroups of C are conjugate in B to the original amalgamated subgroups of B and the amalgamated subgroups of the various A_i. The associated subgroups of B are conjugate in B to the associated subgroups of the various A_i.

Finally, consider any pair L, M of associated subgroups of G and the corresponding free generator t of G. By hypothesis, L, M are conjugate in B to subgroups of vertices of C. If $b_1 L b_1^{-1}$ and $b_2 M b_2^{-1}$ are in vertices of C where b_1, $b_2 \in B$, replace t by $t_1 = b_2 t b_1^{-1}$; the pair of associated subgroups corresponding to t_1 are $b_1 L b_1^{-1}$ and $b_2 M b_2^{-1}$. In this way G can be expressed as a treed HNN group of the type asserted in Lemma 7. $\square$

Lemma 7 also appears in Cohen (1973).

LEMMA 8. *Suppose G/N is a treed HNN group with vertices A_i, amalgamated subgroups $U_{ij} = U_{ji}$, and associated subgroups L_k, M_k corresponding to the generator t_k. If φ is the natural homomorphism of G onto G/N, then G is a treed HNN group with vertices $\varphi^{-1}(A_i)$, amalgamated subgroups $\varphi^{-1}(U_{ij}) = \varphi^{-1}(U_{ji})$, and associated subgroups $\varphi^{-1}(L_k), \varphi^{-1}(M_k)$ corresponding to the generator t_k' where t_k' is an element of $\varphi^{-1}(t_k)$.*

PROOF. First we observe that the group $G = (A * B; U)$ can be characterized internally by the fact that $G = gp(A, B), A \cap B = U$, and $g_1 g_2 \cdots g_n \neq 1$ where the g_i are alternately from $A - U$ and $B - U$; the group $G = \langle t, K; t L_1 t^{-1} = L_{-1} \rangle$ can be characterized internally by the fact that $G = gp(t, K), t L_1 t^{-1} = L_{-1}$, and $k_0 t^{\varepsilon_1} k_1 t^{\varepsilon_2} k_2 \cdots t^{\varepsilon_n} k_n \neq 1$, where $k_i \in K, \varepsilon_i = \pm 1$, and $k_i \not\in L_\varepsilon$ if $\varepsilon_i = \varepsilon$ and $\varepsilon_{i+1} = -\varepsilon$.

If $G/N = (A * B; U = V)$ or $\langle t, K; t L t^{-1} = M \rangle$, then the result follows from the internal characterizations of such groups. If G/N is a treed HNN group with finitely many vertices and free part of finite rank, then the result follows by induction. Finally, one can characterize a general treed HNN group internally by specifying that each finite subtree of vertices and a finite number of free generators whose associated subgroups are incident with these vertices generate the natural treed HNN group. $\square$

THEOREM 2. *Let H be a subgroup of a torsion-free one relator group G where H has non-trivial centre Z. Then H is a treed HNN group with infinite cyclic vertices; moreover, if H is not locally cyclic and not free abelian of rank two, then Z is contained in each vertex and associated subgroup of this treed HNN group.*

PROOF. The proof is by induction on the length of the relator of G. If G is not infinite cyclic, then G is a subgroup of an HNN group $\langle t, K; t L t^{-1} = M \rangle$ where K is a group with a single defining relator of shorter length than that of

G, and L is free. Therefore H is a treed HNN group with vertices in conjugates of K and amalgamated and associated subgroups in conjugates of L (by Theorem 1 of Karrass, Pietrowski and Solitar (1974)).

Suppose first that H has non-trivial free part and Z is contained in each associated subgroup of H. Then Z must be infinite cyclic and contained in some vertex of the tree product base B of H. Hence by Lemma 6 we may assume that Z is contained in each vertex and amalgamated subgroup of B. By inductive hypothesis each vertex V of H is a treed HNN group with infinite cyclic vertices V_i each of which contains Z (this holds even if V is locally cyclic or free abelian of rank two). If E is any amalgamated or associated subgroup incident with V in the treed HNN representation of H, then E is infinite cyclic since it is free and has a non-trivial centre and E/Z is a finite subgroup of the treed HNN group V/Z, which has vertices V_i/Z. Hence a conjugate of E/Z in V/Z is in some vertex V_j/Z. Therefore some conjugate of E in V lies in some vertex V_i. It follows from Lemma 7 that H is a treed HNN group with infinite cyclic vertices such that Z is contained in each vertex and amalgamated and associated subgroup.

If the free part F of H has rank $\geqq 2$, or if F is infinite cyclic and the base of H is different from either associated subgroup, then by Corollary 1 to Lemma 2, the preceding argument applies.

Hence if $F \neq 1$ we may assume H is an infinite cyclic extension of a free group B; moreover, H/Z is a finite extension of the free group BZ/Z. Therefore by Scott (1974), H/Z is a treed HNN group with finite vertices, and so by Lemma 8, H is a treed HNN group with infinite cyclic vertices and Z is contained in each vertex and associated subgroup.

Finally, suppose $F = 1$. If $H = B$ is not a single vertex, then Lemma 6 shows that H is locally cyclic or that Z is contained in each vertex V of B, and so as in the first case H has the desired form. If $H = B$ is a single vertex, then the inductive hypothesis applies. $\square$

COROLLARY 1. *Let A, B, K be groups such that any subgroup H with non-trivial centre has the structure described in the preceding theorem. If $G = (A * B; U)$ or $G = \langle t, K; tLt^{-1} = M \rangle$ where U, L are free, then any subgroup of G with non-trivial centre has the structure described.* $\square$

Thus, for example, the centralizer of an element in $(A * B; U)$ where A, B are free, is a treed HNN group with infinite cyclic vertices.

COROLLARY 2. *Let G be as in the preceding corollary, and let C be the centralizer of an element $x \neq 1$ in G. If H is a subgroup of C such that $H \cap gp(x) = 1$, then H is a free group.*

PROOF. Same as for Corollary 2 of Theorem 2 in Karrass, Pietrowski and Solitar (1974).

COROLLARY 3. *Let G be as in the preceding corollary and let H = C × D < G, C ≠ 1 ≠ D. Then C, D are free groups.*

PROOF. If $1 \neq d \in D$, then C is in the centralizer of D, and $C \cap gp(d) = 1$, and so Corollary 2 applies. □

In particular, if G is a torsion-free one-relator group, and $C \times D < G$, $C \neq 1 \neq D$, then C, D are free.

To obtain the result of Cossey–Smythe referred to in the introduction we must establish that a treed HNN group with infinite cyclic vertices and non-trivial centre is an extension of a free group by a subgroup of the additive rationals. For this purpose choose some fixed generator x_0 of a vertex (referred to as level zero) and map this into the rational number 1. If x_{n+1} is a generator of a vertex A_{n+1} of level $n + 1$ and x_n is a generator of a vertex A_n of level n joined to a_{n+1} by an edge and if the amalgamated subgroups of this edge are $x_n^p = x_{n+1}^q$ and x_n has been mapped into r_n then map x_{n+1} into pr_n/q; map each free generator t into 1. If c is a non-trivial central element then $c = x_n^m = x_s^j$ and so if $tx_n^\alpha t^{-1} = x_s^\beta$ then $tx_n^{\alpha m} t^{-1} = x_s^{\beta m} = tc^\alpha t^{-1} = c^\alpha = x_s^{\alpha j}$ and therefore $\beta m = \alpha j$. But if x_i goes into the rational r_i then $mr_n = jr_s$. Hence, $tx_n^\alpha t^{-1}$ goes into αr_n while x_s^β goes into βr_s; but $\alpha/\beta = m/j = r_s/r_n$ so that $\alpha r_n = \beta r_s$ and the homomorphism is well-defined. Since the mapping is one-one on the vertices it follows that the kernel is a free group.

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Department of Mathematics,
York University,
Downsview, Ontario M3J 1PS,
Canada.