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Prehomogeneity on Quasi-Split Classical Groups and Poles of Intertwining Operators

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Abstract. Suppose that $P = MN$ is a maximal parabolic subgroup of a quasisplit, connected, reductive classical group G defined over a non-Archimedean field and A is the standard intertwining operator attached to a tempered representation of G induced from M. In this paper we determine all the cases in which $\mathrm{Lie}(N)$ is prehomogeneous under $\mathrm{Ad}(m)$ when N is non-abelian, and give necessary and sufficient conditions for A to have a pole at 0.

1 Introduction

In this paper we continue to study the poles of intertwining operators attached to representations induced from supercuspidal representations of maximal parabolic subgroups of quasi-split classical p-adic groups and their connection with local L-functions [1, 2, 9, 10].

To be more precise, let F be a non-Archimedean field of characteristic zero, G be a subgroup of F-rational points of a quasi-split connected reductive group $\mathbf{G}$ over F and let $P = MN$ be a maximal parabolic subgroup of G.

Let $\mathfrak{N} = \mathrm{Lie}(N)$, the Lie algebra of N. When $\mathfrak{N}$ is abelian, then it is known that $\mathfrak{N}$ is a prehomogeneous space under the action of $\mathrm{Ad}(M)$ [6, 11]. The poles of some certain intertwining operators are determined in terms of orbital integrals in [10]. Even explicit generators of these orbits have been found, together with the fact that the centralizer and twisted centralizer are actually equal when G is split [12].

Throughout this paper we assume that $\mathbf{G}$ is a quasi-split connected reductive classical group over F and P is any maximal parabolic subgroup of G. We have determined all cases when $\mathfrak{N}$ is prehomogeneous under $\mathrm{Ad}(M)$ if $\mathfrak{N}$ is non-abelian. Namely, except for two special cases, $\mathfrak{N}$ is not prehomogeneous. And in these two special cases, we have shown that the centralizers have index 2 in the twisted centralizers and the poles of standard intertwining operators have been determined.

It should be pointed out that since $\mathfrak{N}$ can be graded as $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ by α, where α is the simple root that determines P. Each $\mathfrak{N}_i$, $i = 1, 2$, is a prehomogeneous space under $\mathrm{Ad}(M)$, *i.e.,* has a finite number of open orbits under $\mathrm{Ad}(M)$ by M. Sato and T. Kimura in [7]. However, it is not known whether $\mathfrak{N}$ is prehomogeneous. In fact since $\mathfrak{N}$ is reducible, it does not fall into the classification of prehomogeneous spaces in [7].

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2 Preliminaries

Let F be a non-Archimedean field of characteristic zero. Denote by $\mathcal{O}$ its ring of integers and by $\mathcal{P}$ the unique maximal ideal of $\mathcal{O}$. Let q be the number of elements in $\mathcal{O}/\mathcal{P}$ and fix a uniformizing element ϖ for which $|\varpi| = q^{-1}$, where $|\cdot|_F = |\cdot|$ denotes an absolute value for F normalized in this way.

Let $\mathbf{G}$ be a quasisplit connected reductive classical group defined over F. For an positive integer r, let

$$w_r = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in M_r(F).$$

And for any positive integer l, let

$$J_{2l} = \begin{cases} w_{2l+1} & \text{if } \mathbf{G} = SO_{2l+1}; \\ w_{2l} & \text{if } \mathbf{G} = SO_{2l}; \\ \begin{pmatrix} & w_l \\ -w_l & \end{pmatrix} & \text{if } \mathbf{G} = Sp_{2l}. \end{cases}$$

Suppose $\mathbf{G}$ is defined with respect to J_{2l}, *i.e.*, $\mathbf{G} = \{g \in \mathrm{GL}_k \,|\, {}^t g J_{2l} g = J_{2l}\}^\circ$, with the superscript indicating the connected component.

Let $\mathbf{T}$ be the maximal split torus of diagonal elements in $\mathbf{G}$, then we can take

$$\mathbf{T} = \left\{ \left. \begin{pmatrix} x_1 & & & & & & & & \\ & x_2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & x_l & & & & & \\ & & & & 1 & & & & \\ & & & & & x_l^{-1} & & & \\ & & & & & & \ddots & & \\ & & & & & & & x_2^{-1} & \\ & & & & & & & & x_1^{-1} \end{pmatrix} \right| x_i \in F^*, \quad i = 1, 2, \dots, l \right\},$$

if $\mathbf{G} = \mathrm{SO}_{2l+1}$, and otherwise,

$$\mathbf{T} = \left\{ \left. \begin{pmatrix} x_1 & & & & & & & \\ & x_2 & & & & & & \\ & & \ddots & & & & & \\ & & & x_l & & & & \\ & & & & x_l^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & x_2^{-1} & \\ & & & & & & & x_1^{-1} \end{pmatrix} \right| x_i \in F^*, \quad i = 1, 2, \dots, l \right\}.$$

Let $\mathbf{B} = \mathbf{TU}$ be a Borel subgroup of $\mathbf{G}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. Let Δ be the set of simple roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{U}$. Denote by $\mathbf{P} = \mathbf{MN}$ a maximal parabolic subgroup of $\mathbf{G}$ in the sense that $\mathbf{N} \subset \mathbf{U}$. Assume $\mathbf{T} \subset \mathbf{M}$ and let $\Theta = \Delta \setminus \{\alpha\}$ such that $\mathbf{M} = \mathbf{M}_\theta$. Let $\bar{\mathbf{N}}$ be the unipotent subgroup of $\mathbf{G}$ opposed to $\mathbf{N}$.

As usual, we will use $W = W(\mathbf{T})$ to denote the Weyl group of $\mathbf{T}$ in $\mathbf{G}$. Given $\widetilde{w} \in W$, we use w to denote a representative for $\widetilde{w}$. Particularly, let $\widetilde{w_0}$ be the longest element in W modulo the Weyl group of $\mathbf{T}$ in $\mathbf{M}$.

We will also use $G, P, M, N, \bar{N}, B, T, U$ to denote the subgroups of F-rational points of the groups $\mathbf{G}, \mathbf{P}, \mathbf{M}, \mathbf{N}, \bar{\mathbf{N}}, \mathbf{B}, \mathbf{T}, \mathbf{U}$, respectively. Let Φ be the set of roots of G, and let Φ^+ be the positives ones. Let $\sum(\Theta)$ be the subset of Φ that are the linear combinations of the elements from Θ and $\sum^+(\Theta)$ be the subset consisting of its positive elements.

Let $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of G. For any $g \in G$, We will use $\mathrm{Int}(g)$ to denote the inner morphism of G induced by g, *i.e.,* for any $u \in G$, $\mathrm{Int}(g) \circ u = gug^{-1}$. We will use $\mathrm{Ad}(g)$ to denote the adjoint action on $\mathfrak{g}$ induced from $\mathrm{Int}(g)$.

Let $\mathfrak{N} = \mathrm{Lie}(N)$, the Lie algebra of N. Then $\mathfrak{N}$ can be graded by α as $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$, *i.e.,* for any $t \in \{\text{center of } M\}$, and for any $\mathfrak{n}_1 \in \mathfrak{N}_1, \mathfrak{n}_2 \in \mathfrak{N}_2$,

$$\mathrm{Ad}(t) \circ \mathfrak{n}_1 = \alpha(t)\mathfrak{n}_1 \qquad \mathrm{Ad}(t) \circ \mathfrak{n}_2 = 2\alpha(t)\mathfrak{n}_2.$$

M acts on $\mathfrak{N}$ by adjoint action, in particular, each $\mathfrak{N}_i$, $i = 1, 2$, is invariant under $\mathrm{Ad}(M)$. Notice $\mathfrak{N}_2$ is the center of $\mathfrak{N}$. Suppose $N_i = \exp(\mathfrak{N}_i), i = 1, 2$, then $N = N_1N_2$ with N_2 being the center of N.

Suppose $\Delta = \{\alpha_i \mid i = 1, 2, \dots, l\}$. Let e_i $(1 \le i \le l) \in \mathrm{Hom}(T, F^*)$ such that $e_i(T) = x_i$, then $\alpha_i = e_i - e_{i+1}, i = 1, 2 \cdots l - 1$, and

$$\alpha_l = \begin{cases} e_l, & \text{if } G = \mathrm{SO}_{2l+1}(F); \\ e_{l-1} + e_l, & \text{if } G = \mathrm{SO}_{2l}(F); \\ 2e_l, & \text{if } G = \mathrm{Sp}_{2l}(F). \end{cases}$$

Suppose $\alpha = \alpha_n = e_n - e_{n+1}$, then $M \cong \mathrm{GL}_n(F) \times \mathrm{SO}_{2m+1}(F), \mathrm{GL}_n(F) \times \mathrm{SO}_{2m}(F)$ or $\mathrm{GL}_n(F) \times \mathrm{Sp}_{2m}(F)$, depending on whether $\mathbf{G}$ is of type B_l, D_l, or C_l, respectively. For convenience of notation, we set $G' = \mathrm{GL}_n(F)$ and

$$G_m = \begin{cases} \mathrm{SO}_{2m+1}(F), & \text{if } G = \mathrm{SO}_{2l+1}(F); \\ \mathrm{SO}_{2m}(F), & \text{if } G = \mathrm{SO}_{2l}(F); \\ \mathrm{Sp}_{2m}(F), & \text{if } G = \mathrm{Sp}_{2l}(F). \end{cases}$$

3 Non-Prehomogeneity

For any $Y \in M_n(F)$, we set $\varepsilon(Y) = w_n\, {}^tY w_n^{-1}$. Then $\varepsilon(\varepsilon(Y)) = Y$ since $w_n^{-1} = w_n$. We define an action ε of G' on $M_n(F)$ by $\varepsilon(g) \circ A = gA\varepsilon(g)$, $\forall g \in G', A \in M_n(F)$. And we call the group $G'_{\varepsilon,A} = \{g \in G' | \varepsilon(g) \circ A = A\}$ the ε- twisted centralizer of A in G'.

Definition 3.1 For any $A \in M_n(F)$, we say that A is *ε-symmetric* if $\varepsilon(A) = A$ and *skew-ε-symmetric* if $\varepsilon(A) = -A$. Denote by $M_n^{\varepsilon}(F)$ the subspace of $M_n(F)$ consisting of ε-symmetric elements, and by $M_n^{s\varepsilon}(F)$ the subspace of $M_n(F)$ consisting of skew-ε-symmetric elements.

Lemma 3.2 *$M_n(F) = M_n^{\varepsilon}(F) \oplus M_n^{s\varepsilon}(F)$, and both $M_n^{\varepsilon}(F)$ and $M_n^{s\varepsilon}(F)$ are closed under $\varepsilon(G')$.*

Proof Straightforward. ■

Lemma 3.3 *Let $n \in N$ and suppose that*

$$n = \begin{pmatrix} I_n & X & Y \\ 0 & I_k & X' \\ 0 & 0 & I_n \end{pmatrix}.$$

Then we have

$$X' = \begin{cases} -J_{2m}\,{}^tXw_n, & \text{if } G \text{ is orthogonal;} \\ J_{2m}\,{}^tXw_n, & \text{if } G \text{ is symplectic.} \end{cases}$$

And

$$XX' = \begin{cases} Y + \varepsilon(Y), & \text{if } G \text{ is orthogonal;} \\ Y - \varepsilon(Y), & \text{if } G \text{ is symplectic.} \end{cases}$$

In particular, if $n \in N_2$, then $X = 0$ and $Y \in M_n^{s\varepsilon}(F)$(or $M_n^{\varepsilon}(F)$) if G is orthogonal (or symplectic respectively). If $n \in N_1$, then $Y \in M_n^{\varepsilon}(F)$(or $M_n^{s\varepsilon}(F)$) if G is orthogonal (or symplectic respectively).

Proof The first part is a counterpart of [1, Lemma 2.1], the rest is straightforward. ■

Lemma 3.4 *Use $n(X,Y)$ to denote n in Lemma 3.3. For any $n(X,Y) \in N$, write*

$$A = \begin{cases} \frac{Y+\varepsilon(Y)}{2}, & \text{if } G \text{ is orthogonal;} \\ \frac{Y-\varepsilon(Y)}{2}, & \text{if } G \text{ is symplectic.} \end{cases}$$

And

$$B = \begin{cases} \frac{Y-\varepsilon(Y)}{2}, & \text{if } G \text{ is orthogonal;} \\ \frac{Y+\varepsilon(Y)}{2}, & \text{if } G \text{ is symplectic.} \end{cases}$$

Then $Y = A + B$ with Y being decomposed as in Lemma 3.2.

Let $n_1 = n(X,A), n_2 = n(0,B)$. Then $n_i \in N_i, i = 1,2$, and $n = n_1n_2$. Moreover, for any $B \in M_n^{s\varepsilon}(F)$ (or $M_n^{\varepsilon}(F)$, according to whether G is orthogonal or symplectic, respectively), $n(0,B) \in N_2 \subset N$.

Proof Straightforward. ■

Let $M_n^s(F) = \{A \mid A \in M_n(F), A = {}^tA\}$ be the subspace of n-dimensional symmetric matrices, and $M_n^{ss}(F) = \{A \mid A \in M_n(F), A = -{}^tA\}$ be the subspace of n-dimensional skew-symmetric matrices. Then it is clear that $M_n(F) = M_n^s(F) \oplus M_n^{ss}(F)$.

Define a group action δ of G' on $M_n(F)$ as $\delta(g) \circ A = gA^tg, \forall g \in G', A \in M_n(F)$. Then we have the following.

Lemma 3.5 *$M_n^s(F)$ is a prehomogeneous space under δ.*

Proof Let $\mathrm{GL}_n^s(F) = \mathrm{GL}_n(F) \cap M_n^s(F)$, then $\mathrm{GL}_n^s(F)$ is a dense open subset of $M_n^s(F)$.

For any $A \in \mathrm{GL}_n^s(F)$, it is a basic fact in linear algebra that there is $g \in G'$, such that $gA^tg = \mathrm{diag}(a_1, a_2, \ldots, \alpha_n)$ for some $a_i \neq 0, i = 1, 2, \ldots, n$. Choose a complete set of representatives $S = \{\varepsilon_i, i = 1, 2, \ldots, \kappa\}$ of $F^*/(F^*)^2$, where $\kappa = \mathrm{card}(F^*/(F^*)^2)$. Suppose $a_i = t_i^2\varepsilon_i$ with $\varepsilon_i \in S$, let $g_1 = \mathrm{diag}(t_1^{-1}, t_2^{-1}, \ldots, t_n^{-1})$. Then $\delta(g_1g) \circ A = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. So $\mathrm{GL}_n^s(F)$ has only finite number of generators under $\delta(G')$, *i.e.*, $M_n^s(F)$ is a prehomogeneous space. ■

Corollary 3.6 *$M_n^\varepsilon(F)$ is a prehomogeneous space under $\varepsilon(G')$.*

Proof There is an isomorphism $f\colon M_n^\varepsilon(F) \longrightarrow M_n^s(F)$ defined by $f(A) = Aw_n, \forall A \in M_n^\varepsilon(F)$. If we notice the fact that $\varepsilon(g) = f \circ \delta(g) \circ f^{-1}, \forall g \in G'$, then the proof is trivial. Moreover, for any $A \in M_n^\varepsilon(F) \cap G'$, there is $g \in G'$, such that

$$\varepsilon(g) \circ A = \begin{pmatrix} & & & \varepsilon_1 \\ & & \varepsilon_2 & \\ & \iddots & & \\ \varepsilon_n & & & \end{pmatrix}, \tag{3.1}$$

with $\varepsilon_i \in S,\ i = 1, 2, \ldots, n$. ■

Lemma 3.7 *$M_n^{ss}(F)$ is a prehomogeneous space under $\delta(G')$. More precisely, suppose*

$$B = \begin{pmatrix} 0 & b_{1,2} & b_{1,3} & \cdots & b_{1,n} \\ -b_{1,2} & 0 & b_{2,3} & \cdots & b_{2,n} \\ -b_{1,3} & -b_{2,3} & 0 & \cdots & b_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{1,n} & -b_{2,n} & -b_{3,n} & \cdots & 0 \end{pmatrix}$$

is an arbitrary element in $M_n^{ss}(F)$, then there is $g \in G'$, such that

$$\delta(g) \circ B = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & \ddots & \end{pmatrix}. \tag{3.2}$$

Moreover, such g fixes the vector $(0, \ldots, 0, 1)^t$ by left multiplication, and consequently $\delta(g)$ will fix the element $E_{n,n}$.

Proof Let $r_n = 2[n/2]$, where $[n/2]$ is the maximal integer that is no greater than $n/2$. Let $\bar{M}_n^{ss}(F) = \{A | A \in M_n^{ss}(F), \mathrm{rank}(A) = r_n\}$, then $\bar{M}_n^{ss}(F)$ is a dense open subset of $M_n^{ss}(F)$.

If $n = 1$, then the lemma is trivial.

If $n = 2$, let $g = \mathrm{diag}(1, b_{1,2}^{-1})$ if $B \neq 0$. Then g will satisfy the lemma.

Suppose the lemma is true for all $k \leq n - 1$. Let $k = n$.

We can always assume $b_{1,2} \neq 0$. Otherwise, we first assume that there is one $i, 3 \leq i \leq n$, such that $b_{1,i} \neq 0$. Let $K_{2,i} = I_n + E_{2,i}$, where for any pair of positive integers $\{i, j\}$, $E_{i,j}$ is an elementary matrix in $M_n(F)$, whose $\{i, j\}$'s entry equals to 1,

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all other entries are 0. Then the $\{1,2\}'$s entry of $K_{2,i}B^tK_{2,i}$ is $b_{2,i}$, which is not 0. On the other hand, if such i does not exist, then it will fall into the induction hypothesis.

Now let

$$K_i = I_n - \frac{b_{1,i}}{b_{1,2}}E_{i,2}, \quad i = 3,\dots,n, \text{ and } h_1 = \prod_{i=3}^{n} K_i.$$

Then

$$B_1 = \delta(h_1)\circ B = \begin{pmatrix} 0 & b_{1,2} & 0 & \cdots & 0 \\ -b_{1,2} & 0 & b_{2,3} & \cdots & b'_{2,n} \\ 0 & -b_{2,3} & 0 & \cdots & b'_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -b'_{2,n} & -b'_{3,n} & \cdots & 0 \end{pmatrix}.$$

Let $P_i = I_n + \frac{b'_{2,i}}{b_{1,2}}E_{i,1}$, $3 < i \leq n$, and set

$$h' = \prod_{i=3}^{n} P_i, \quad h'' = \operatorname{diag}(b_{1,2}^{-1}, 1, \cdots, 1), \quad h_2 = h''h'.$$

Then

$$B_2 = \delta(h_2)\circ B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & b''_{3,4} & \cdots & b''_{3,n} \\ 0 & 0 & -b''_{3,4} & 0 & \cdots & b''_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -b''_{3,n} & -b''_{4,n} & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & B' \end{pmatrix},$$

with $B' \in M^{ss}_{n-2}(F)$.

By induction hypothesis, there is $g_1 \in \mathrm{GL}_{n-2}(F)$, such that $\delta(g_1)\circ B'$ satisfies equation (3.2) when $k = n-2$.

Let $h_3 = \operatorname{diag}(I_2, g_1)$, $g = h_3h_2h_1$, then $\delta(g)\circ B$ satisfies equation (3.2). Therefore there is only one generator of $\bar{M}^{ss}_n(F)$ under $\delta(G')$ which automatically implies that $M^{ss}_n(F)$ is a prehomogeneous space under $\delta(G')$. The property that $g(0,\dots,0,1)^t = (0,\dots,0,1)^t$ is obvious from the construction of g, thus, $\delta(g)\circ E_{n,n} = E_{n,n}$. ■

Corollary 3.8 *$M^{s\varepsilon}_n(F)$ is prehomogeneous under $\varepsilon(G')$. Moreover, for any $B \in M^{s\varepsilon}_n(F)$ there is $g \in G'$ such that*

$$\varepsilon(g)\circ B = \begin{pmatrix} & & & & 1 & 0 \\ & & & & 0 & -1 \\ & & 1 & 0 & & \\ & & 0 & -1 & & \\ & \iddots & & & & \end{pmatrix}. \tag{3.3}$$

In addition, such g fixes $(0,\dots,0,1)^t$ by left multiplication, and consequently, $\varepsilon(g)$ fixes $E_{n,1}$.

Proof This is a direct result of Lemma 3.7, and the proof is similar to Corollary 3.6. ■

Let $\mathrm{GL}_n^{s\varepsilon}(F) = \mathrm{GL}_n(F) \cap M_n^{s\varepsilon}(F)$, then $\mathrm{GL}_n^{s\varepsilon}(F)$ is a dense open subset of $M_n^{s\varepsilon}(F)$ when n is even and is an empty set when n is odd. For this reason and the purpose of further use, we let B_n be the matrix which has a form as the right side of equation (3.3) with rank r_n. Then from Lemma 3.7 and Corollary 3.8, B_n is a generator of the unique dense open orbit of $M_n^{s\varepsilon}(F)$ under $\varepsilon(G')$. We then define:

$$E_n = \begin{cases} B_n, & \text{if } n \text{ is even;} \\ B_n + E_{n,1}, & \text{if } n \text{ is odd.} \end{cases}$$

We can define a map f from $M_{n\times k}(F)$ to $M_n^{\varepsilon}(F)$(or $M_n^{s\varepsilon}(F)$) by $f(X) = XX'$. Notice f is a polynomial function in terms of the entries of X.

Lemma 3.9 *If $n \leq m$, then f is surjective. In particular, if $n \geq 2$ and $m \geq 1$, then for almost all X, rank$(XX') \geq 2$.*

Proof Suppose first that G is orthogonal, let $A = (a_{i,j})_{n\times n}$ be an arbitrary element in $M_n^{\varepsilon}(F)$. Let $X = (I_n, 0, X_1)$ where $X_1 = -A/2$. Then

$$X' = \begin{pmatrix} \varepsilon(X_1) \\ 0 \\ I_n \end{pmatrix},$$

and $XX' = -(X_1 + \varepsilon(X_1)) = A$ as desired.

If G is symplectic, then the proof is similar. The rest of the lemma is straightforward. ■

For any $m = (g, h, \varepsilon(g^{-1})) \in M$, where $g \in G'$ and $h \in G_m$, we have $Int(m) \circ n(X,Y) = n(gXh^{-1}, gY\varepsilon(g))$, (see also [1,2]). Moreover, if we decompose $M_n(F)$ (as Y is concerned) into subspaces as in Lemma 3.2, then both $M_n^{\varepsilon}(F)$ and $M_n^{s\varepsilon}(F)$ are invariant under $\mathrm{Int}(M)$.

Lemma 3.10 *There is an open dense subset* O *in N, such that for any* $n(X,Y) \in \mathrm{O}$, $\det(Y) \neq 0$.

Proof Write $Y = A + B$ as in Lemma 3.4. By Lemmas 3.3 and 3.4, $\det(Y)$ is a polynomial function in terms of the entries of X and B. So we only need to show that $\det(Y) \not\equiv 0$.

If G is symplectic, let $X = 0$ and $B = Y \in \mathrm{GL}_n^{\varepsilon}(F)$. If G is orthogonal and n is even, choose $X = 0$ and $B = Y = E_n$. In both cases, $\det(Y) \neq 0$.

If G is orthogonal and n is odd, let $B = B_n$, where B_n is defined as before. And let

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix} \in M_{n\times(2m+1)}(F).$$

Then $Y = B_n + E_{n,1} = E_n$, and obviously $\det(Y) \neq 0$.

Thus, the subset of N satisfying $\det(Y) = 0$ is a closed subset (in Zariski topology). ■

Remark. The above lemma can also be used to prove the fact that up to a closed subset of N, $w_0^{-1}n(X,Y) \in P\bar{N}$ by applying Lemma 2.2 in [1] where $\bar{N}$ is the unipotent subgroup opposite to N.

Theorem 3.11 *If $n > 1$ and $m \neq 0$, then N does not have a finite number of open orbits under* $\mathrm{Int}(M)$, i.e., $\mathfrak{N}$ *is not a prehomogenous space under* $\mathrm{Ad}(M)$.

Proof Suppose $\mathrm{O} = \bigcup \mathrm{O}_i$ is a dense open subset of N where each O_i is an orbit of N under $\mathrm{Int}(M)$. Let $n(X_i, Y_i)$ be a representative of O_i under $\mathrm{Int}(M)$. Write $Y_i = A_i + B_i$ as in Lemma 3.4. By Lemma 3.4 and Corollaries 3.6 and 3.8, we can always assume B_i has a same form as the right side of equation (3.1) or (3.3) depending on whether G is symplectic or orthogonal, respectively. If G is orthogonal, we can even fix B_i as B_n by Corollary 3.8.

Suppose $n(X,Y) \in \mathrm{O}$ with $Y = A + B$ being decomposed as in Lemma 3.4, then there is an i such that $n(X,Y) \in \mathrm{O}_i$. Thus, there exists an $m = (g, h, \varepsilon(g^{-1})) \in M$, such that $\mathrm{Int}(m) \circ n(X,Y) = n(X_i, Y_i)$. Consequently, $gY\varepsilon(g) = Y_i$, $gB\varepsilon(g) = B_i$, by the uniqueness of the decomposition in Lemma 3.2.

Therefore, if G is symplectic or if G is orthogonal and n is even, then

$$\frac{\det(B)}{\det(Y)} = \frac{\det(B_i)}{\det(Y_i)}, \tag{3.4}$$

since by Lemma 3.10, we can always assume that both $\det(Y)$ and $\det(Y_i)$ are nonzero. The left side of equation (3.4) is a rational function in terms of the entries of X and B. By Corollaries 3.6 and 3.8 and Lemma 3.9, it is obviously nonconstant. Therefore, the set of $n(X,Y)$ satisfying equation (3.4) is only a closed subset of N, a contradiction!

If G is orthogonal and n is odd, let $B' = g^{-1}E_{n,1}\varepsilon(g)^{-1}$. Then $g(B+B')\varepsilon(g) = E_n$, and consequently, we will have:

$$\frac{\det(B+B')}{\det(Y)} = \frac{\det(E_n)}{\det(Y_i)}.$$

By the proof of Lemma 3.7 and Corollary 3.8, the entries of g are rational functions of that of B, so are the entries of B'. Now the same argument of the above paragraph applies which will lead to a contradiction. ■

Remark. We use E_n and $B + B'$ because the determinants of both B and B_i are 0 when G is orthogonal and n is odd.

4 Cases When $\mathfrak{N}$ is Prehomogeneous

By Theorem 3.11, $\mathfrak{N}$ has a finite number of open orbits under $\mathrm{Ad}(M)$ only when $n = 1$ or $m = 0$. Since the prehomogeneity of $\mathfrak{N}$ has been studied in [6, 7, 11] when $\mathfrak{N}$ is abelian, we will only study the prehomogeneity when $\mathfrak{N}$ is non-abelian.

When $m = 0$, the only case that $\mathfrak{N}$ is non-abelian is $G = \mathrm{SO}_{2l+1}(F)$. While if $n = 1$, the unique case that $\mathfrak{N}$ is non-abelian is $G = \mathrm{Sp}_{2l}(F)$.

Theorem 4.1 *If $G = \mathrm{SO}_{2l+1}(F)$ and $m = 0$, then N is a prehomogeneous space under* $\mathrm{Int}(M)$.

Proof Suppose $n(X,Y)\in N$, in this case $M=G'$ and

$$X=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\in M_{n\times 1}(F).$$

By restricting to a dense open subset, we can assume that $X\neq 0$, then there is $g\in G'$ such that $gX=(0,\dots,0,1)^t=X_1$. Therefore, $X'\varepsilon(g)=(gX)'=(-1,0,\dots,0)$, and consequently, $gXX'\varepsilon(g)=-E_{1,n}$.

Write $Y=A+B$ as in Lemma 3.4, let $B'=\varepsilon(g)\circ B$. By Corollary 3.8, there is $g'\in G'$ such that $\varepsilon(g')\circ B'=B_n$ by restricting B to a dense open subset of $M_n^{s\varepsilon}(F)$. Moreover, $g'X_1=X_1$ and $\varepsilon(g')$ fixes $E_{n,1}$. Therefore $\mathrm{Int}(g'g)\circ n(X,Y)=n(X_1,B_n-\frac{1}{2}E_{n,1})$, in other words, there is only one dense open orbit of N under $\mathrm{Int}(M)$. ■

Theorem 4.2 *If $G=\mathrm{Sp}_{2l}(F)$ and $n=1$, then N is a prehomogeneous space under* $\mathrm{Int}(M)$.

Proof Suppose $n(X,Y)\in N$, then $X\in M_{1\times(2m-2)}(F)$ and $XX'=0$. Also in this case $M=\mathrm{GL}_1\times\mathrm{Sp}_{2m}(F)$, $G'=\mathrm{GL}_1=F^*$, and $Y\in M_1(F)=F$.

Assume $a=Y\neq 0$, this assumption will apply to a dense open subset of N. Write $a=b^2\varepsilon_i$ for a suitable $\varepsilon_i\in S$, let $g_1=(b^{-1},I_{2m},b)$. Then $\mathrm{Int}(g_1)\circ n(X,Y)=n(X_1,\varepsilon_i)$, where $X_1=b^{-1}X$.

Suppose $X_1=(x_1,\dots,x_m,x_{m+1},\dots,x_{2m})$. We can assume $x_1\neq 0$, by doing so, it will only amount to a closed subset of N. Let $X_1'=(x_1,\dots,x_m)$, then there exists a $g\in\mathrm{GL}_m(F)$ such that $X_1'g=(1,0,\dots,0)\in M_{1\times m}(F)$. Let

$$h'=\mathrm{diag}(g,w_m{}^tg^{-1}w_m^{-1})\in Sp_{2m}(F),\quad h_1=\mathrm{diag}(1,h',1)\in M,\quad \text{and } X_2=X_1h'.$$

Then $\mathrm{Int}(h_1^{-1})\circ n(X_1,\varepsilon_i)=n(X_2,\varepsilon_i)$, where

$$X_2=(1,0,\dots,0,x'_{m+1},\dots,x'_{2m})\in F$$

for some suitable $x'_{m+1},\dots,x'_{2m}\in F$.

Let

$$Q=\begin{pmatrix}-x'_{m+1}&\cdots&-x'_{2m-1}&-x'_{2m}\\0&\cdots&0&-x'_{2m-1}\\\vdots&\ddots&\vdots&\vdots\\0&\cdots&0&-x'_{m+1}\end{pmatrix}\in M_n^{s\varepsilon}(F),$$

and

$$h"=\begin{pmatrix}I_m&Q\\0&I_m\end{pmatrix}\in\mathrm{Sp}_{2m}(F).$$

Then $X_1h'h"=X_2h"=(1,0,\dots,0)\in M_{1\times 2m}(F)$.

Denote $E_1=(1,0,\dots,0)\in M_{1\times 2m}(F)$, let $h_2=\mathrm{diag}(1,h",1)$ and $h=h_2^{-1}h_1^{-1}g_1$, then $\mathrm{Int}(h)\circ n(X,Y)=n(E_1,\varepsilon_i)$. Therefore, there are only finitely many generators for a dense open subset of N under $\mathrm{Int}(M)$. *i.e.*, $\mathfrak{N}$ is a prehomogeneous space under $\mathrm{Ad}(M)$. In particular, the number of open orbits is $\mathrm{card}(S)$. ■

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5 Centralizers and Twisted Centralizers on Prehomogeneous Cases

We now suppose $G = \mathrm{SO}_{2l+1}(F)$, $\alpha = e_l$; or $G = \mathrm{Sp}_{2l}(F)$, and $\alpha = e_1 - e_2$. Then

$$M = \begin{cases} \mathrm{GL}_l(F) \times 1, & \text{if } G \text{ is orthogonal;} \\ \mathrm{GL}_1(F) \times \mathrm{Sp}_{2l-2}(F), & \text{if } G \text{ is symplectic.} \end{cases}$$

And

$$G_m = \begin{cases} 1, & \text{if } G \text{ is orthogonal;} \\ \mathrm{Sp}_{2l-2}(F), & \text{if } G \text{ is symplectic,} \end{cases}$$

by definition.

By Theorems 4.1 and 4.2, N is prehomogeneous under $\mathrm{Int}(M)$.

We choose w_0 as follows:

$$w_0 = \begin{cases} \begin{pmatrix} 0 & 0 & I_n \\ 0 & 1 & 0 \\ I_n & 0 & 0 \end{pmatrix}, & \text{if } G \text{ is orthogonal;} \\ \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_{2l-2} & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \text{if } G \text{ is symplectic.} \end{cases}$$

Lemma 5.1 *Suppose $G = \mathrm{SO}_{2l+1}(F)$ and $n = n(X, Y) \in N$. Then $w_0^{-1} n \in P\bar{N}$ if and only if $Y \in \mathrm{GL}_n(F)$, in which case*

$$(5.1) \qquad w_0^{-1} n = \begin{pmatrix} \varepsilon(Y^{-1}) & -Y^{-1}X & I_n \\ 0 & 1 - X'Y^{-1}X & X' \\ 0 & 0 & Y \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ (Y^{-1}X)' & 1 & 0 \\ Y^{-1} & Y^{-1}X & I_n \end{pmatrix},$$

with $X'Y^{-1}X = 0$.

Proof This is [1, Lemma 2.2]. The proof is straightforward. ■

Lemma 5.2 *Suppose $G = \mathrm{Sp}_{2l}(F)$ and $n = n(X, Y) \in N$. Then $w_0^{-1} n \in P\bar{N}$ if and only if $Y \in \mathrm{GL}_n(F)$, in which case*

$$(5.2) \qquad w_0^{-1} n = \begin{pmatrix} -\varepsilon(Y^{-1}) & -Y^{-1}X & I_n \\ 0 & I_{2m} - X'Y^{-1}X & X' \\ 0 & 0 & -Y \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ (Y^{-1}X)' & I_{2m} & 0 \\ Y^{-1} & Y^{-1}X & I_n \end{pmatrix},$$

and $I_{2m} - X'Y^{-1}X \in \mathrm{Sp}_{2m}(F)$.

Proof This is also [1, Lemma 2.2], but since we chose a different J_{2l}, the right side of equation (5.2) is a little bit different from the expression there. ■

Write equations (5.1) and (5.2) as $w_0^{-1} n_i = m_i n_i n_i^-$, where m_i, n_i, n_i^- belong to M, N and $\bar{N}$, respectively. Define

$$M_{n_i} = \mathrm{Cent}_M(n_i) = \{m \in M \mid \mathrm{Int}(m) \circ n_i = n_i\}$$

as the centralizer of n_i in M, and

$$M^t_{m_i} = \mathrm{Cent}^t_{m_i} = \{m \in M | w_0(m)m_i m^{-1} = m_i\}$$

as the twisted (by means of w_0) centralizer of m_i in M. Then $M_{n_i} \subset M^t_{m_i}$ by [10, Lemma 2.1].

Theorem 5.3 *Suppose G, M, α as above, then for any $n_i \in \mathrm{O}$, where O has the same meaning as in Theorem* 4.1 *or* 4.2, $|M^t_{m_i}/M_{n_i}| = 2$.

Proof We only need to prove the lemma for any generator of each orbit, since any two elements in a same orbit are $\varepsilon(G')$ conjugate to each other, their centralizers and twisted centralizers are therefore $\varepsilon(G')$ conjugate to each other.

First suppose G is orthogonal, then by Theorem 4.1, there is only one orbit of N under $\mathrm{Int}(M)$. We can also choose a representative of this orbit as $n = n(X_1, B_n - \frac{1}{2}E_{n,1})$, where $B_n, E_{n,1}$ and $X_1 = (0, \dots, 0, 1)^t \in M_{n\times 1}(F)$ are as in Theorem 4.1. Then in this case $m_i = B_n - \frac{1}{2}E_{n,1}$ if we identify $(\varepsilon(G'^{-1}), 1, G')$ with G'.

Suppose $g \in M^t_{m_i} \subset G'$ such that $\varepsilon(g) \circ m_i = m_i$. Then by Lemma 3.2, $\varepsilon(g) \circ B_n = B_n$ and $\varepsilon(g) \circ (-E_{n,1}) = \varepsilon(g) \circ (X_1 X_1') = (gX_1)(gX_1)' = -E_{n,1}$. Thus $gX_1 = \pm X_1$. If $gX_1 = X_1$, then $g \in M_{n_i}$; if $gX_1 = -X_1$, then $-g \in M_{n_i}$. Therefore $|M^t_{m_i}/M_{n_i}| = 2$.

Now suppose G is symplectic, then by Theorem 4.2, there are finitely many open orbits of N under $\mathrm{Int}(M)$. The generator of each orbit can be chosen as $n_i = n(E_1, \varepsilon_i)$ with $\varepsilon_i \in S$. If $m = (k, h, k^{-1}) \in M^t_{m_i}$, with $k \in F^*$ and $h \in \mathrm{Sp}_{2l-2}(F)$. Then $\mathrm{Int}(m) \circ m_i = m_i$, where $m_i = (-\varepsilon_i, I_{2l-2} + E_{2l-2,1}, -\varepsilon_i)$ is determined by Lemma 5.2.

Thus $k^2\varepsilon_i = \varepsilon_i$ and $(kE_1h)'(kE_1h) = E_{2l-2,1} = E_1'E_1$. Therefore, $kE_1h = \pm E_1$. If $kE_1h = E_1$, then $m \in M_{n_i}$; if $kE_1h = -E_1$, then $(-1, I_{2l-2}, -1)m \in M_{n_i}$. Whence, $|M^t_{m_i}/M_{n_i}| = 2$. ■

6 Poles of Intertwining Operators on Prehomogeneous Cases

For a connected reductive p-adic group H, we use ${}^\circ\mathcal{E}(H)$ to denote the collection of equivalence classes of unitarizable irreducible admissible supercuspidal representations of H.

Let $(\tau', V') \in {}^\circ\mathcal{E}(\mathrm{GL}_n(F))$ and $(\tau, V) \in {}^\circ\mathcal{E}(G_m)$, then $\tau' \otimes \tau$ is a unitary supercuspidal representation of M. Let

$$I(s, \tau' \otimes \tau) = \mathrm{Ind}^G_{MN}((\tau' \otimes |\det(\ )|^s) \otimes \tau \otimes 1_N).$$

We will use $\mathbf{V}(s, \tau' \otimes \tau)$ to denote the space of $I(s, \tau' \otimes \tau)$. In order to understand the reducibility of $I(\tau' \otimes \tau) = I(0, \tau' \otimes \tau)$, one must determine the poles of the standard intertwining operator

$$A(s, \tau' \otimes \tau, w_0) f(g) = \int_N f(w_0^{-1} n g) dn$$

associated to $\tau' \otimes \tau$ (cf. [1, 3, 9, 10]), where $f \in \mathbf{V}(s, \tau' \otimes \tau)$. By Bruhat's theorem (cf. [4]) we may assume that $w_0(\tau' \otimes \tau) \simeq \tau' \otimes \tau$, which is equivalent to assuming $\tau' \simeq \widetilde{\tau'}$ [13].

Denote by $\bar{N}$ the unipotent radical opposed to N. Let

$$\mathbf{V}(s, \tau' \otimes \tau)_0 = \{h \in \mathbf{V}(s, \tau' \otimes \tau) \mid \operatorname{supp}(h) \subset \bar{N} \text{ modulo } P\}.$$

By a lemma of Rallis (cf. [9]), it is enough to compute the poles that arise when $A(s, \tau' \otimes \tau, w_0)$ is applied to functions in $\mathbf{V}(s, \tau' \otimes \tau)_0$ and evaluated at the identity.

Let ${}^L G' = \mathrm{GL}_n(\mathbb{C})$ be the L- group of G', r be the adjoint action of ${}^L G'$ on the Lie algebra ${}^L\mathfrak{n}$ of ${}^L N$, the L-group of N. Let ρ_n be the standard representation of $\mathrm{GL}_n(\mathbb{C})$, then $\rho_n \otimes \rho_n = \Lambda \rho_n^2 \oplus \mathrm{Sym}^2(\rho_n)$. Let SO_n^* be any of the quasi-split orthogonal groups which has $\mathrm{SO}_n(\mathbb{C})$ as the connected component of its L- group if n is even.

6.1 G is Orthogonal

We will still consider the case when $G = \mathrm{SO}_{2l+1}(F)$ and $\alpha = e_l$. Notice in this case $n = l$, $M = \mathrm{GL}_n$ and $G_m = 1$.

We let $(\tau', V') \in {}^\circ\mathcal{E}(\mathrm{GL}_n(F))$ and

$$I(s, \tau') = \mathrm{Ind}_{MN}^{G}((\tau' \otimes |\det(\ )|^s) \otimes 1_N).$$

In this special case, we will use $\mathbf{V}(s, \tau'), I(\tau'), A(s, \tau', w_0), \mathbf{V}(s, \tau')_0$ to denote the general settings $\mathbf{V}(s, \tau' \otimes \tau), I(s, \tau' \otimes \tau), A(s, \tau' \otimes \tau, w_0), \mathbf{V}(s, \tau' \otimes \tau)_0$ as defined at the beginning of this section, respectively.

Let $h \in \mathbf{V}(s, \tau')_0$. Fix open compact subsets $L \subset M_n(F)$ and $L' \subset M_{n\times 1}(F)$. We assume that for some $v' \in V'$, h satisfies:

$$h \begin{pmatrix} I_n & 0 & 0 \\ (Y^{-1}X)' & 1 & 0 \\ Y^{-1} & Y^{-1}X & I_n \end{pmatrix} = \xi_L(Y^{-1})\xi_{L'}(Y^{-1}X)(v'),$$

where ξ_L and $\xi_{L'}$ are the characteristic functions of L and L', respectively. Let $\widetilde{\mathbf{V}}'$ be the dual spaces of $\mathbf{V}'$. Choose $\widetilde{v}' \in \widetilde{\mathbf{V}}'$ and let $\psi_{\tau'}$ be the matrix coefficient of τ' given by pair $(v', \widetilde{v}')$. Then, from Lemma 5.1, $\langle \widetilde{v}', A(s, \tau', w_0)h(e)\rangle$ is equal to

$$\int_{(X,Y)} \psi_{\tau'}(Y)|\det(Y)|^{-s-\langle\rho,\tilde{\alpha}\rangle}\xi(X,Y)d(X,Y), \tag{6.1}$$

where the integral is over the collection of F-rational solutions (X, Y) satisfying Lemmas 3.3 and 5.1. Here

$$\rho = \frac{1}{2}\sum_{\beta\in\Phi^+\setminus\sum^+(\Theta)} \beta, \quad \xi(X,Y) = \xi_L(Y^{-1})\xi_{L'}(Y^{-1}X), \quad \widetilde{\alpha} = \langle\rho,\alpha\rangle^{-1}\rho,$$

and $d(X, Y)$ is a choice of Haar measure on N.

By Theorem 4.1, there is only one orbit O of N under $\mathrm{Int}(G')$. For any $n(X, Y) \in$ O, define $d^*(X, Y) = |\det(Y)|^{-\langle\rho,\tilde{\alpha}\rangle}d(X, Y)$, then $d^*(X, Y)$ is an invariant measure on O (see [1]). Therefore, the integral in (6.1) will be changed to:

$$\int_{(X,Y)} \psi_{\tau'}(Y)|\det(Y)|^{-s}\xi(X,Y)d^*(X,Y). \tag{6.2}$$

Moreover, the representative of this orbit can be chosen as $n(X_1, B_n)$, where $X_1 = (1, 0, \ldots, 0)^t \in M_{n\times 1}(F)$ and B_n as the right side of equation (3.3). Hence, the unique dense open subset O can be expressed as $n(gX_1, g(B_n - \frac{1}{2}E_{n,1})\varepsilon(g))$ as g runs through G'. Thus, $d^*(X, Y)$ induces an invariant measure on G'/M_{n_i}. Furthermore, by [10, Lemma 2.3], it also induces an invariant measure on the quotient $G'/M^t_{m_i}$ since $M^t_{m_i}/M_{n_i} = 2$ by Theorem 5.3. Therefore, if we let $Y_1 = B_n - \frac{1}{2}E_{n,1}$, then equation (6.2) can be expressed as:

$$(6.3) \qquad 2\int_{G'/G'_{\varepsilon,Y_1}} \psi_{\tau'}(gY_1\varepsilon(g))|\det(gY_1\varepsilon(g))|^{-s}\xi(gX_1, gY_1\varepsilon(g))d\dot{g},$$

where $M^t_{m_i} = G'_{\varepsilon,Y_1}$ by definition.

Let ω' be the central character of τ'. Since we are assuming that τ' is self-dual, ω'^2 is trivial. We then can choose $f \in C_c^\infty(G')$ such that

$$\psi_{\tau'}(g') = \int_{Z(G')} f(zg')\omega'(z^{-1})d^\times z.$$

Substitute the above equation into (6.3), then the expression will be:

$$(6.4) \quad 2\int_{G'/G'_{\varepsilon,Y_1}}\int_{Z(G')} f(zgY_1\varepsilon(g))\omega'(z^{-1})d^\times z|\det(gY_1\varepsilon(g))|^{-s}\xi(gX_1, gY_1\varepsilon(g))d\dot{g}.$$

By making a substitution $gz \to g$, we can rewrite expression (6.4) as:

$$(6.5) \qquad 2\sum_{\gamma\in S}\omega'(\gamma)\int_{G'/G'_{\varepsilon,Y_1}}\int_{Z(G')} f(\gamma gY_1\varepsilon(g))|\det z|^{-2s}|\det(gY_1\varepsilon(g))|^{-s}$$
$$\xi_L(z^{-2}\varepsilon(g)^{-1}Y_1^{-1}g^{-1})\xi_{L'}(z^{-1}\varepsilon(g)^{-1}Y_1^{-1}X_1)d^\times z d\dot{g}.$$

Now we have the following.

Lemma 6.1 *The intertwining operator $A(s, \tau', w_0)$ is convergent for $s > 0$ and has a pole at $s = 0$ if and only if*

$$(6.6) \qquad \sum_{\gamma\in S}\omega'(\gamma)\int_{G'/G'_{\varepsilon,Y_1}} f(\gamma gY_1\varepsilon(g)^{-1})d\dot{g} \neq 0.$$

Proof This has been proved in [9], and a more general result has also been established in [1]. Here we will use the finiteness of orbits to give a shorter proof.

We can use a similar argument as that of [1, Lemma 4.5] to prove our lemma. Namely, the integrand inside (6.5) is nonzero only when

$$gY_1\varepsilon(g) \in \gamma^{-1}\operatorname{supp}(f) \cap z^2\operatorname{supp}(\xi_L)^{-1} = \mathbf{C},$$

where $\operatorname{supp}(\xi_L)^{-1}$ is the subset of $\bar{N}$ consisting of the inverse elements of $\operatorname{supp}(\xi_L)$. Thus, g must belong to a compact subset of $G'/G'_{\varepsilon,Y_1}$ and $z^{-2} \in \operatorname{supp}(\xi_L)\cdot\mathbf{C}$. Consequently, $|z|$ must be bounded from below.

Therefore, there exists μ such that when $|z|_F > \mu$, the order of the integrals in (6.5) can be interchanged. By the fact that $f, \xi_L, \xi_{L'}$ are all bounded, the conclusion of the lemma follows immediately. ■

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Moreover, Shahidi in [9] has shown that the orbital integrals appearing in equation (6.6) are all equal, *i.e.*, for any $\gamma \in S$,

$$\int_{G'/G'_{\varepsilon,Y_1}} f(\gamma g Y_1 \varepsilon(g)^{-1}) d\dot{g} = \int_{G'/G'_{\varepsilon,Y_1}} f(g Y_1 \varepsilon(g)^{-1}) d\dot{g}.$$

We thus obtain the following.

Theorem 6.2 *The intertwining operator $A(s, \tau', w_0)$ has a pole at $s = 0$ or equivalently $I(\tau')$ is irreducible if and only if $\omega' = 1$ and*

$$\int_{G'/G'_{\varepsilon,Y_1}} f(g Y_1 \varepsilon(g)^{-1}) d\dot{g} \neq 0. \tag{6.7}$$

In that case:

(a) *if n is odd, then $A(s, \tau', w_0)$ has a pole at $s = 0$;*
(b) *if n is even, then $A(s, \tau', w_0)$ has a pole at $s = 0$ if and only if τ' comes from $\mathrm{SO}_n^*(F)$.*

Proof If ω' is nontrivial, then equation (6.6) is zero. Part (a) is [9, Proposition 3.10], and part (b) is Corollary 10.6 from the same paper. We give here a shorter proof for part (b).

By Theorem 4.1, there is only one orbit of N under $\mathrm{Int}(M)$. Moreover, by the proof of Theorem 4.1, we can choose $n_i(kX_1, B_n - \frac{1}{2}k^2 E_{n,1})$ as a generator of this orbit for any $k \in F^*$. Let $Y_i = B_n - \frac{1}{2}k^2 E_{n,1}$, then (6.7) will be changed to:

$$\int_{G'/G'_{\varepsilon,Y_i}} f(g Y_i \varepsilon(g)^{-1}) d\dot{g} \neq 0.$$

Because n is even, both B_n and Y_i belong to G'. Since $f \in C_c^\infty(G')$, it is clear that $gY_i\varepsilon(g)^{-1} \in \mathrm{supp}(f)$ if and only if $\bar{g}$ belongs to a compact set $\mathbb{C}_i$ of $G'/G'_{\varepsilon,Y_i}$, where $\bar{g}$ is the representative of g in $G'/G'_{\varepsilon,Y_i}$. Moreover, when $|k|$ is small enough, these $\mathbb{C}_i$ will be independent of k. We will use $\mathbb{C}$ to denote such uniform $\mathbb{C}_i$.

For each $\bar{g} \in \mathbb{C}$, there is a neighborhood $\mathrm{O}(g)$ of g such that for any $g' \in \mathrm{O}(g)$, there is a positive number u_g, such that $f(g'B_n\varepsilon(g')^{-1}) = f(g'Y_i\varepsilon(g')^{-1})$ when $|k| < u_g$. By the compactness of $\mathbb{C}$, we can choose k small enough such that for any $gY_i\varepsilon(g)^{-1} \in \mathrm{supp}(f)$, $f(gB_n\varepsilon(g)^{-1}) = f(gY_i\varepsilon(g)^{-1})$. Therefore, the determining condition (6.7) will be changed to:

$$\int_{G'/G'_{\varepsilon,B_n}} f(g B_n \varepsilon(g)^{-1}) d\dot{g} \neq 0.$$

But this is the determining condition of the same intertwining operators for SO_n^* if we take $M = \mathrm{GL}_n(F)$ there. Thus, $A(s, \tau', w_0)$ has a pole at $s = 0$ if and only if τ' comes from $\mathrm{SO}_n^*(F)$ by means of the definition in [9]. ■

6.2 G is Symplectic

We now consider the case $G = \mathrm{Sp}_{2l}(F)$ and $\alpha = e_1 - e_2$. Notice $M = \mathrm{GL}_1(F) \times \mathrm{Sp}_{2l-2}(F) = F^* \times G_m$.

Let χ be a character of $F^* = \mathrm{GL}_1(F)$ to the unit circle in the complex plane and V' be the space of χ. Since we may assume χ is self-dual, $\chi^2 = 1$. Let $(\tau, V) \in {}^\circ\mathcal{E}(G_m)$ and

$$I(s, \chi \otimes \tau) = \mathrm{Ind}_{MN}^G((\chi \otimes |\cdot|^s) \otimes \tau \otimes 1_N).$$

Let $h \in \mathbf{V}(s, \chi \otimes \tau)_0$. Fix open compact subsets $L \subset F$ and $L' \subset M_{1\times 2m}(F)$. We assume that for some $v' \in V'$, $v \in V$, h satisfies:

$$h \begin{pmatrix} I_n & 0 & 0 \\ (Y^{-1}X)' & I_{2m} & 0 \\ Y^{-1} & Y^{-1}X & I_n \end{pmatrix} = \xi_L(Y^{-1})\xi_{L'}(Y^{-1}X)(v' \otimes v),$$

where ξ_L and $\xi_{L'}$ are the characteristic functions of L and L', respectively. Let $\widetilde{\mathbf{V}}', \widetilde{\mathbf{V}}$ be the dual spaces of $\mathbf{V}'$ and $\mathbf{V}$, respectively. Choose $\widetilde{v}' \in \widetilde{\mathbf{V}}'$ and $\widetilde{v} \in \widetilde{\mathbf{V}}$, let ψ_χ and f_τ be the matrix coefficient of χ and τ given by pairs $(v', \widetilde{v}')$ and $(v, \widetilde{v})$, respectively. Then from Lemma 5.2, $\langle \widetilde{v}' \otimes \widetilde{v}, A(s, \chi \otimes \tau, w_0)h(e)\rangle$ is equal to

$$\int_{(X,Y)} \psi_\chi(-Y) f_\tau(I_{2m} - X'Y^{-1}X)|Y|^{-s-\langle\rho,\tilde{\alpha}\rangle}\xi(X,Y)d(X,Y),$$

which is proportional to

$$\int_{(X,Y)} \chi(Y) f_\tau(I_{2m} - X'Y^{-1}X)|Y|^{-s-\langle\rho,\tilde{\alpha}\rangle}\xi(X,Y)d(X,Y), \tag{6.8}$$

where the integral is over the collection of F-rational solutions (X, Y) satisfying Lemmas 3.3 and 5.2. Here $\rho, \xi(X, Y), d(X, Y)$ have a same meaning as in Subsection 6.1.

By Theorem 4.2, there are only a finite number of open orbits O of N under $\mathrm{Int}(G')$. For any $n(X, Y) \in$ O, define $d^*(X, Y) = |Y|^{-\langle\rho,\tilde{\alpha}\rangle}d(X, Y)$, then $d^*(X, Y)$ is an invariant measure on O (cf [1]). Therefore, the integral in (6.8) will be changed to:

$$\int_{(X,Y)} \chi(Y) f_\tau(I_{2m} - X'Y^{-1}X)|Y|^{-s}\xi(X,Y)d^*(X,Y). \tag{6.9}$$

Moreover, the representative of each orbit can be chosen as $n(E_1, \varepsilon_i)$, where $X_1 = (1, 0, \dots, 0) \in M_{1\times n}(F)$ and $\varepsilon_i \in S$. Hence, each open subset of O can be expressed as $n(gX_1h, g^2\varepsilon_i)$ as g and h run through G' and G_m, respectively. Thus, $d^*(X, Y)$ induces an invariant measure on G'/M_{n_i}. Furthermore, by the same reason as before, it also induces an invariant measure $d\dot{m}$ on the quotient $M/M^t_{m_i}$ since $M^t_{m_i}/M_{n_i} = 2$ by Theorem 5.3. Therefore, if we let $Z_i = I_{2m} - \varepsilon_i^{-1}E_{2m,1}$, then $M^t_{m_i} = \{\pm 1\} \times C(Z_i)$ where $C(Z_i)$ is the centralizer of Z_i in G_m. Then equation (6.9) can be expressed as:

$$2\int_{F^*/\{\pm 1\}} \int_{G_m/C(Z_i)} \sum_{\varepsilon_i \in S} \chi(g^2\varepsilon_i) f_\tau(hZ_ih^{-1})|g^2\varepsilon_i|^{-s}\xi(gE_1, g^2\varepsilon_i)d\dot{h}d\dot{g},$$

where $d\dot{g}, d\dot{h}$ are invariant measures on $G'/\{\pm 1\}$ and $G_m/C(Z_i)$, respectively, induced from $d\dot{m}$.

Then we have the following.

Lemma 6.3 *The intertwining operator $A(s,\chi\otimes\tau,w_0)$ is convergent for $s>0$ and has a pole at $s=0$ if and only if*

$$\int_{G_m/C(Z_i)}\sum_{\varepsilon_i\in S}\chi(\varepsilon_i)f_\tau(hZ_ih^{-1})d\dot{h}\neq 0. \tag{6.10}$$

Proof The proof is similar to that of Lemma 6.1. Actually, it can be regarded as an improvement of the results in [1, 10] in these two special cases. ■

Since $G_m=\mathrm{Sp}_{2m}(F)$, we will fix $T, e_i, i=1,2,\dots,m$ as in Section 2. Let $\beta=e_1-e_2$ and choose a maximal parabolic subgroup $P=MN$ with $M=M_\beta$. Then $M=\mathrm{GL}_1(F)\times\mathrm{Sp}_{2m-2}(F)=M_1\times M_2$. Let $T_1=\{\mathrm{diag}(t_1,1,\dots,1,1,\cdots,1,t_1^{-1})|t_1\in F^*\}\cong F^*$ be a torus in M.

For each $i, 1\le i\le m$, we choose a root vector of $-2e_i$ as: $\mathfrak{g}_{-2e_i}=E_{2l+1-i,i}$. Let $U_{-2e_i}(x)=\exp(x\mathfrak{g}_{-2e_i})$ be the unipotent subgroup of G_m attached to $-2e_i$. Then $Z_i=U_{-2e_1}(-\varepsilon_i)$. Since $\bar{N}PN$ is a dense subset of G_m, (6.10) can be changed to:

$$\int_{\bar{N}PN/\bar{N}PN\cap C(Z_i)}\sum_{\varepsilon_i\in S}\chi(\varepsilon_i)f_\tau(hZ_ih^{-1})d\dot{h}\neq 0. \tag{6.11}$$

But it can be easily shown that $\bar{N}PN\cap C(Z_i)=\bar{N}M_2$, thus, (6.11) is equivalent to:

$$\int_{M_1N}\sum_{\varepsilon_i\in S}\chi(\varepsilon_i)f_\tau(hZ_ih^{-1})d\dot{h}\neq 0. \tag{6.12}$$

We state our main result in this case as follows.

Theorem 6.4 *The intertwining operator $A(s,\chi\otimes\tau,w_0)$ has a pole at $s=0$; equivalently, $I(\chi\otimes\tau)$ is irreducible if $\chi=1$.*

Proof For any fixed $v\in\mathbf{V}$, $\widetilde{v}\in\widetilde{\mathbf{V}}$, let $K_{v,\widetilde{v}}$ be a minimal compact subgroup of U_{-2e_1} such that $K_0=\mathrm{supp}(f_\tau)\cap U_{-2e_1}\subset K_{v,\widetilde{v}}$. Let $\phi\colon\mathbf{V}\longrightarrow\mathbf{V}$ be defined by

$$\phi(v_1)=\mathrm{vol}(K_{v,\widetilde{v}})^{-1}\int_{K_{v,\widetilde{v}}}\tau(k)v_1dk,\quad\forall\, v_1\in\mathbf{V}.$$

Then $\mathbf{V}=\mathbf{V}^{K_{v,\widetilde{v}}}\oplus\mathrm{Ker}\,\phi$, with $\mathrm{Ker}\,\phi$ being the orthogonal complement of $\mathbf{V}^{K_{v,\widetilde{v}}}$ and ϕ is a projection from $\mathbf{V}$ to $\mathbf{V}^{K_{v,\widetilde{v}}}$. We will use $v_1^{K_{v,\widetilde{v}}}$ to denote $\phi(v_1)$.

Since $(\tau,V)\in{}^\circ\mathcal{E}(G_m)$, $\widetilde{\mathbf{V}}$ can be identified with $\mathbf{V}$ through the Hermitian inner product $\langle\cdot,\cdot\rangle$. Let $\widetilde{\tau}$ be the contragredient representation of τ on $\widetilde{\mathbf{V}}$, then $\widetilde{\tau}(g)=\tau(g)$ for all $g\in G_m$ under the above identification. The left side of inequality (6.12) will be changed to

$$\begin{aligned}\int_{M_1N}\sum_{\varepsilon_i\in S}f_\tau(hZ_ih^{-1})d\dot{h}&=\int_N\int_{F^*}\sum_{\varepsilon_i\in S}\langle\tau(u\cdot U_{-2e_1}(-\varepsilon_it^2)\cdot u^{-1})v,\widetilde{v}\rangle d\dot{t}d\dot{u}\\&=\int_N\int_{F^*}\sum_{\varepsilon_i\in S}\langle\tau(U_{-2e_1}(-\varepsilon_it^2))\tau(u^{-1})v,\tau(u^{-1})\widetilde{v}\rangle d\dot{t}d\dot{u},\end{aligned}$$

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where $d\dot{u}, d\dot{t}$ are the restriction measures of $d\dot{h}$ on N, M_1, respectively.

For any $u \in N$, let $v_u = (\tau(u^{-1})v)^{K_{v,\widetilde{v}}}$ and $\widetilde{v}_u = (\tau(u^{-1})\widetilde{v})^{K_{v,\widetilde{v}}}$. Then

$$\begin{aligned}\int_{F^*}\sum_{\varepsilon_i\in S}\langle\tau(U_{-2e_1}(-\varepsilon_i t^2))\tau(u^{-1})v, \tau(u^{-1})\widetilde{v}\rangle d\dot{t} &= \mathrm{vol}(K_0)\langle v_u, \tau(u^{-1})v\rangle \\ &= \mathrm{vol}(K_0)\langle v_u, \widetilde{v}_u\rangle.\end{aligned}$$

In particular, if we choose $\widetilde{v} = v$, then the right side of the above equation is non-negative. We can also choose such v that $v^{K_{v,\widetilde{v}}} \neq 0$, then if u belongs to a small neighborhood of 1, $\tau(u^{-1})v = v$. Thus $\langle v_u, \widetilde{v}_u\rangle > 0$.

Therefore, for some $v \in \mathbf{V}$ and $\widetilde{v} \in \widetilde{\mathbf{V}}$, the left side of (6.10) is non-zero and $A(s, \chi\otimes\tau, w_0)$ has a pole at $s = 0$. ■

Remark. If $\sigma = \chi\otimes|\det(\cdot)|^{s_1}$ is a self-dual representation of M_1, then by the results in [8], $A(s, \sigma\otimes\tau, w_0)$ has a pole at $s = s_1$ if and only if $A(s, \chi\otimes\tau, w_0)$ has a pole at $s = 0$. For this reason, we have simplified our assumption on χ.

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