Adv. Appl. Prob. **16**, 929–932 (1984)
Printed in N. Ireland

BOUNDS FOR THE VARIANCE OF THE BUSY PERIOD OF THE $M/G/\infty$ QUEUE

M. F. RAMALHOTO,* *Instituto Superior Técnico, Lisbon*

Abstract

Some bounds for the variance of the busy period of an $M/G/\infty$ queue are calculated as functions of parameters of the service-time distribution function. For any type of service-time distribution function, upper and lower bounds are evaluated in terms of the intensity of traffic and the coefficient of variation of the service time. Other lower and upper bounds are derived when the service time is a NBUE, DFR or IMRL random variable. The variance of the busy period is also related to the variance of the number of busy periods that are initiated in $(0, t]$ by renewal arguments.

LOWER AND UPPER BOUNDS; STOCHASTIC ORDER RELATIONS; VARIANCE OF NUMBER OF BUSY PERIODS

The busy-period distribution of the $M/M/\infty$ queue has been studied by Shanbhag (1966) and Conolly (1971). Shanbhag (1966), p. 278, presents a formula for the Laplace transform of the joint probability and density function of N and BP for an $M/G/\infty$ queue with group arrival, where N is the total number of customers served during a busy period and BP represents the duration in time of the busy period. Conolly (1971), by direct arguments in the time domain, obtains a formula for the joint probability and density function of N and BP. Another way of deriving the Laplace–Stieltjes transform of the distribution function of BP is to consider that $Z = \mathrm{BP} + \mathrm{IP}$, where IP represents the idle period of the infinite-server queue and Z the interval of time between two consecutive busy periods. In an $M/G/\infty$ queue it is known that $E[Z] = \exp(\lambda\alpha)/\lambda$ and

$$E[Z^2] = \lambda^{-1} 2 \exp(2\lambda\alpha) \int_0^\infty \left[\exp\left(-\lambda \int_0^t [1 - G(x)]\,dx\right) - \exp(-\lambda\alpha)\right] dt + 2\exp(\lambda\alpha)/\lambda^2$$

(see Takács (1962), p. 211, Theorem 3); here $G(.)$ represents the distribution function of the service time (a positive random variable (r.v.)); $\alpha = \int_0^\infty x\,dG(x) < \infty$; and $\lambda > 0$ is the parameter of the Poisson input process.

The r.v. BP and IP are stochastically independent and the r.v. IP is exponentially distributed with parameter $\lambda > 0$. Combining all these results with $\int_0^t [(1 - G(x)]\,dx = \alpha - \int_t^\infty [1 - G(x)]\,dx$, it follows that $\mathrm{Var}[\mathrm{BP}] = \lambda^{-1} h_1(\rho, G^*(.)) + \lambda^{-2} h_2(\rho)$, where $\rho = \lambda\alpha$; $G^*(t) = \int_0^t [1 - G(x)]\,dx/\alpha$; $h_1(\rho, G^*(.)) = 2\exp(\rho) \int_0^\infty [\exp(\rho[1 - G^*(t)]) - 1]\,dt$ and $h_2(\rho) = \exp(\rho)[2 - \exp(\rho) - \exp(-\rho)]$. For further details see Ramalhoto (1983).

Received 12 April 1983; revision received 15 August 1984.

* Postal address: Department of Mathematics, Instituto Superior Técnico, Av. Rovisco Pais, 1000 Lisboa, Portugal.

Research partially supported by the Centro de Estatística e Aplicações do INIC under the Applied Stochastic Processes Research Project.

https://doi.org/10.2307/1427350 Published online by Cambridge University Press

1. Lower and upper bounds of the Var [BP] **as functions of** ρ **and** γ_S^2

For any service-time distribution function the following result is valid.

Proposition 1. $f_1(\lambda^{-2}, \rho, \gamma_S^2) \leqq \mathrm{Var}\,[\mathrm{BP}] \leqq f_2(\lambda^{-2}, \rho, \gamma_S^2)$, where

$$f_1(\lambda^{-2}, \rho, \gamma_S^2) = \max\{\lambda^{-2}\exp(\rho)[\rho^2(\gamma_S^2+1)+2-\exp(\rho)-\exp(-\rho)], 0\};$$

$$f_2(\lambda^{-2}, \rho, \gamma_S^2) = \lambda^{-2}\exp(\rho)[\rho(\exp(\rho)-1)(\gamma_S^2+1)+2-\exp(\rho)-\exp(-\rho)];$$

and γ_S^2 is the coefficient of variation of the service time.

Proof. For any $t \in \mathbb{R}^+$, $a(t, G(.)) = \lambda \int_t^\infty [1-G(x)]\,dx$ is non-negative so $\exp(a(t, G(.)) \geqq 1 + a(t, G(.))$. Thus

$$\begin{aligned} h_1(\lambda, \alpha, G(.) &\geqq 2\exp(\lambda\alpha)\int_0^\infty a(t, G(.))\,dt \\ &= 2\exp(\lambda\alpha)\lambda\int_0^\infty \left(\int_t^\infty [1-G(x)]\,dx\right) dt \\ &= \lambda\exp(\lambda\alpha)(\sigma_S^2+\alpha^2), \end{aligned}$$

from which the lower bound follows. Let $b(t, G(.)) = \alpha^{-1}\int_t^\infty [1-G(x)]\,dx$. If $\alpha \neq 0$ and $\alpha \neq \infty$ then for any $t \in \mathbb{R}^+$, $0 \leqq b(t, G(.)) \leqq 1$ and $[b(t, G(.))]^n \leqq b(t, G(.))$ for $n = 1, 2, \cdots$. Therefore, $\exp(\rho b(t, G(.))) - 1 \leqq b(t, G(.))[\exp(\rho)-1]$. And,

$$\mathrm{Var}\,[\mathrm{BP}] \leqq ((\exp(\rho)-1)/\rho)\exp(\rho)(\sigma_S^2+\alpha^2) + (\exp(\rho)/\lambda^2)[2-\exp(\rho)-\exp(\rho)].$$

2. Relations of stochastic ordering and the Var [BP]

For a service time NBUE the following upper bound is derived solely in terms of the intensity of traffic.

Proposition 2. If the service time is a NBUE (new better than used in expectation) r.v. of mean value α, then

$$\mathrm{Var}\,[\mathrm{BP}] \leqq \lambda^{-2}\left[2\rho\exp(\rho)\sum_{n=1}^{\infty}\rho^n/(nn!) + h_2(\rho)\right]$$

(the right-hand side of this inequality is the variance of the busy period when the service time is exponential of parameter $1/\alpha$). The inequality is reversed if the service time is a NWUE (W for worse) r.v.

Proof. For the distribution function of a r.v. NBUE of mean value $\alpha, G(.)$, $\int_b^\infty [1-G(x)]\,dx \leqq \int_b^\infty \exp(-x/\alpha)\,dx$ for all $b \geqq 0$ (see, for instance, Ross (1983), p. 273); the inequality is reversed for NWUE. The result follows directly from the above inequality.

For a service time DFR (decreasing failure rate) a tighter lower bound than that in Proposition 1 is the following.

Proposition 3. If the service time is a DFR r.v. then

$$\mathrm{Var}\,[\mathrm{BP}] \geqq \lambda^{-2}\left[2\exp(\rho)\rho\sum_{n=1}^{\infty}(\rho^n/nn!)\exp((-n/2)(\gamma_S^2-1)) + h_2(\rho)\right].$$

Proof. For a DFR r.v., $1-G(x) \geqq \exp(-x/\alpha - \gamma_S^2/2 + \tfrac{1}{2})$, (see Ross (1983), p. 265), from which the result follows.

A non-negative r.v. with distribution function $G(.)$ is defined to have an IMRL (increasing mean residual life) if $\alpha = E[S] < \infty$ and $E[S-t \mid S>t]$ is increasing (monotone non-decreasing) in $t \geqq 0$. Brown (1981) showed that for an IMRL r.v. this is equivalent to the distribution function $G^*(x) = \alpha^{-1}\int_0^x [1-G(y)]\,dy$ being that of a DFR r.v. Then as pointed out in the last paragraph, $1-G^*(x) \geqq \exp([-x/m_1 - m_2/2(m_1)^2+1]) = \exp(-2x\alpha/\mu_2 - (2\alpha/3\mu_2^2)\mu_3+1)$, where m_r denotes the rth moment about the origin of the distribution function $G^*(.)$, and $m_r = \mu_{r+1}/[(r+1)\alpha]$, where μ_r are the corresponding moments of $G(.)$, see Cox (1962), p. 64.

For a service time IMRL a lower bound in terms of the intensity of traffic and the first three central moments of the service time is given in the following proposition.

Proposition 4. If the service time is an IMRL r.v. then

$$\operatorname{Var}[\mathrm{BP}] \geqq \rho^{-1}\exp(\rho)\mu_2 \sum_{n=1}^{\infty} (\rho^n/nn!)\exp(-n((2\alpha\mu_3/3\mu_2^2)-1)) + \lambda^{-2}h_2(\rho)$$

where $\alpha = E[S]$, $\mu_2 = E[S^2]$ and $\mu_3 = E[S^3]$.

Proof.

$$\begin{aligned}\operatorname{Var}[\mathrm{BP}] &= 2\lambda^{-1}\exp(\rho)\int_0^\infty [\exp(\rho[1-G^*(t)])-1]\,dt + \lambda^{-2}h_2(\rho)\\ &\geqq 2\lambda^{-1}\exp(\rho)\int_0^\infty [\exp(\rho\exp(-2\alpha\mu_3/3\mu_2^2+1)\\ &\quad \times\exp(-(2\alpha/\mu_2)t)) - 1]\,dt + \lambda^{-2}h_2(\rho)\\ &= 2\lambda^{-1}\exp(\rho)\sum_{n=1}^{\infty}[\rho^n\exp(n((-2\alpha\mu_3/3\mu_2^2)+1))]/n\\ &\quad \times\int_0^\infty \exp(-(2n\alpha/\mu_2)t)\,dt + \lambda^{-2}h_2(\rho),\end{aligned}$$

from which the result follows.

3. Relationship between the Var [BP] and the Var [$N(t)$]

Let $N(t)$ be the number of busy periods initiated in $(0, t]$. Brown and Solomon (1974) have evaluated the moments of $N(t)$ for the $M/G/\infty$ queue. (The mean value of $N(t)$ had already been evaluated by Takács (1954) for the type II counter.) In a type II counter with the input forming a renewal process and an arbitrary impulse distribution function, the number $N(t)$ of recorded particles in $(0, t]$, i.e. the $N(t)$ in the $GI/G/\infty$ queue, is represented by a renewal process (Cox and Isham (1980), p. 102). Thus $\operatorname{Var}[N(t)] \simeq (\operatorname{Var}[Z]/E[Z]^3)t$, for t large enough. In the $M/G/\infty$ queue, $\operatorname{Var}[N(t)] \simeq [(1/\lambda^2 + \operatorname{Var}[\mathrm{BP}])/(\exp(\lambda\alpha)/\lambda)^3]t$, for t large enough. Therefore, formulas for Var [BP] lead to formulas for Var [$N(t)$] and vice versa. From renewal theory for t large enough, $\operatorname{Var}[N(t)]/E[N(t)] \simeq \operatorname{Var}[Z]/(E[Z])^2 = \gamma_Z^2$, where γ_Z is the coefficient of variation of Z in a $GI/G/\infty$ queue. In the $M/G/\infty$ queue $E[Z] = \lambda^{-1}\exp(\lambda\alpha)$ and $\operatorname{Var}[Z] = \operatorname{Var}[\mathrm{BP}] + \lambda^{-2}$. By Proposition 1, $c(\rho, \gamma_S^2) \leqq \gamma_Z^2 \leqq d(\rho, \gamma_S^2)$ where

$$c(\rho, \gamma_S^2) = \exp(-2\rho)\lambda^2\,\mathrm{Max}\,\{\lambda^{-2}\exp(\rho)[\rho^2(\gamma_S^2+1)+2-\exp(\rho)-\exp(-\rho)], 0\} + \exp(-2\rho),$$

and

$$d(\rho, \gamma_S^2) = \exp(-\rho)[\rho(\exp(\rho)-1)(\gamma_S^2+1) - (\exp(\rho)-2)].$$

Acknowledgement

The author is grateful to the referee for the suggestion of Section 3.

References

BROWN, M. (1981) Further monotonicity properties for specialized renewal processes. *Ann. Prob.* **9,** 891–895.

BROWN, M. AND SOLOMON, H. (1974) Some results for secondary processes generated by a Poisson process. *Stoch. Proc. Appl.* **2,** 337–348.

CONOLLY, B. W. (1971) The busy period for the infinite capacity service system $M/G/\infty$. In *Studii di Probabilità, Statistica e Ricerca Operativa in Onore di G. Pompilj*, Istituto di Calcolo delle Probabilità, Università di Roma, Oderisi Gubbio, Roma, 128–130.

COX, D. R. (1962) *Renewal Theory*. Methuen, London.

COX, D. R. AND ISHAM, V. (1980) *Point Processes.* Chapman and Hall, London.

RAMALHOTO, M. F. (1983) A note on the variance of the busy period of the $M/G/\infty$ system. Technical Report, Linha-3, Processos Estocásticos Aplicados, do CEAUL do INIC.

ROSS, S. (1983) *Stochastic Processes.* Wiley, New York.

SHANBHAG, D. N. (1966) On infinite server queues with batch arrivals. *J. Appl. Prob.* **3,** 274–279.

TAKÁCS, L. (1954) On secondary processes generated by a Poisson process and their applications in physics. *Acta Math. Acad. Sci. Hungar.* **5,** 203–236.

TAKÁCS, L. (1962) *An Introduction to Queueing Theory.* Oxford University Press, New York.

https://doi.org/10.2307/1427350 Published online by Cambridge University Press