ON S_2-GROUPS AND GROUPS OF MOEBIUS TRANSFORMATIONS

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Let PGL(2, F) denote the group of all Moebius transformations

$$z \to \frac{az+b}{cz+d} \qquad (ad - bc \neq 0)$$

over a field F. The object of this paper is to prove the following theorem.

THEOREM 1. *G is an S_2-group and the centre of G is trivial if and only if G is isomorphic to a group* PGL(2, F), char $F \neq 2$.

This theorem was proved for finite groups in **(1)**. The present paper extends the result to infinite groups and also improves the method of proof used in that paper. Many of the theorems given there were proved for infinite groups and are used here with appropriate references.

For the definition of the terms S_1 and S_2, see **(1; 2)**. a^x and H^x stand for $x^{-1}ax$ and $x^{-1}Hx$ respectively. $G - H$ is the set of elements of G not in H. $Z(H)$ is the centre of the group H and $N(H)$ its normalizer in G. By $C(h)$ we denote the centralizer of h.

The proof rests on the following theorem.

THEOREM 2. *If H is an S_2-subgroup of G, the centre of G is trivial, and $x \in G - H$, then $H^x \neq H$, $H \cap H^x$ is a non-normal S_1-subgroup of H and $H - (H \cap H^x)$ contains an involution.*

Proof. By **(1,** Theorem 9, Corollary 1**)** $N(H) = H$ so that $H^x \neq H$. Suppose $h \in H - H^x, h_1 \in H$, and $h^{h_1} \in H - H^x$.

(i) Let $h^2 \neq 1$. Then, by the property S_2, there exists exactly one

$$k \in H^x \ni h^k = h^{h_1}.$$

Then $kh_1^{-1} \in C(h) \subseteq H$ by **(1,** Lemma 1**)**. Therefore $k \in H \cap H^x$.

(ii) Let $h^2 = 1$. Then, by the property S_2, H^x contains an involution b with the property $h^b = h$.

If $b \in H$, then, by **(1,** Theorem 11**)** $C(b) \cap H = H \cap H^x$, which is a contradiction. Hence $b \notin H$.

Since $b \notin H$, $b \notin N(H)$ so that $H \neq H^b$. Assume $h \in A = H \cap H^b$; then by **(1,** Theorem 5**)** $C(h) = A \cup bA$, where the union is disjoint.

Received August 15, 1966.

Now, by the property S_2, there exist exactly two $k_1, k_2 \in H^x$ with the property $h^{k_1} = h^{k_2} = h^{h_1}$. Then $k_1 h_1^{-1}, k_2 h_1^{-1} \in C(h) = A \cup bA$.

If both $k_1 h_1^{-1}, k_2 h_1^{-1} \in bA$, then $k_1 k_2^{-1} \in A \subseteq H$. But $k_1 k_2^{-1} \neq 1$ and $k_1 k_2^{-1} \in H^x$ so that $k_1 k_2^{-1} = b$. Thus $b \in A$, which is a contradiction. Hence either $k_1 h_1^{-1} \in A$ or $k_2 h_1^{-1} \in A$, say $k_1 h_1^{-1} \in A$.

Then $k_1 \in H \cap H^x$. But $k_2 \notin H$, for otherwise $b = k_1 k_2^{-1} \in H$.

Thus $H \cap H^x$ is an S_1-subgroup of H.

Suppose $H \cap H^x \lhd H$. Now, by **(1**, Theorem 11**)** $H \cap H^x$ contains exactly one involution, say y, so that $y \in Z(H)$, which is impossible as $H \cap H^x$ is an S_1-subgroup of H.

It remains to show that $H - (H \cap H^x)$ contains an involution. If not, then y is the only involution of H so that, again, $y \in Z(H)$.

This proves Theorem 2.

We now proceed to the proof of Theorem 1. It has already been noted **(1)** that if G is isomorphic to PGL(2, F), char $F \neq 2$, then G has the property S_2 and has trivial centre.

Suppose that G has an S_2-subgroup H and $x \notin H$. Then it follows from Theorem 2 **(1**, Theorem 11; **2**, Theorems 1 and 2**)** that H is isomorphic to a group S(2, F), where S(2, F) stands for the group of similarity transformations $z \to az + b$ $(a \neq 0)$ over the field F. It now follows that the representation of G on the left cosets of H is triply transitive and only the identity fixes three cosets. Applying the method of Zassenhaus **(**see **1** or **5)**, it follows that G is isomorphic to a group PGL(2, F). The characteristic of F is not 2 since the groups PGL(2, F), char $F = 2$, do not have the property S_2 **(2)**.

This completes the proof of Theorem 1 and this theorem, together with **(1**, Theorem 8**)**, gives a complete description of all S_2-groups.

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https://doi.org/10.4153/CJM-1968-046-8 Published online by Cambridge University Press