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THE EXISTENCE OF QUADRATIC DIFFERENTIALS IN SIMPLY CONNECTED REGIONS OF THE COMPLEX PLANE

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1. Introduction. The general coefficient theorem [**2**] and the extended general coefficient theorem [**3**] state that the existence of certain quadratic differentials is a sufficient condition for a function to be a solution of certain extremum problems. The purpose of this paper is to show that in the case of simply connected regions this condition is also necessary.

We shall do this by a variational method of the Schiffer-Golusin-type. The main difficulty is, that the class of admissible functions for the general coefficient theorem is restricted and we must therefore have a method of variation with restrictions. We will basically use the method of Lagrange multipliers but in a somewhat unusual manner and therefore we shall give complete proofs. We believe that this method, which we will develop in §§ 2 and 3 is also interesting for its own sake.

2. The space of variations. In this paragraph we compile a list of the variations which we will use later. Let f be a fixed function, which maps the unit circle Δ conformally onto a region $\Omega \subset \mathbf{C}$, such that $f(0) = 0$.

(1) Our first type of variation is the following:
Let $w_0 \in \Omega$. Then the function

$$\phi(w) = w + \frac{tAw}{w - w_0} \qquad (A \in \mathbf{C})$$

maps, for small t, the complement of Ω in a 1-1 manner onto a continuum, the complement of which we will call Ω_t. We denote by $f_t(z)$ a function, which maps the unit circle onto Ω_t such that $f_t(0) = 0$. It is determined up to rotation of the unit circle around 0. It is known [**1**] that f_t can be normed in a way, that it depends real-analytically on t and has an asymptotic representation:

$$f_t(z) = f(z) + t\frac{wA}{w - w_0} - tzf'(z)\left[\frac{w_0A}{z_0f'(z_0)^2(z - z_0)} - \frac{\bar{w}_0\bar{A}z}{z_0f'(z_0)^2(1 + \bar{z}_0z)}\right] + o(t)$$
$$(w = f(z);\, z_0 = f^{-1}(w_0)).$$

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(2) If w_0 is an exterior point of Ω, then $w + twA/(w - w_0)$ maps Ω for small t conformally onto a region Ω_t. The function

$$f_t(z) = f(z) + \frac{tAf(z)}{f(z) - w_0}$$

gives therefore a second type of variation.

Beside those two types we will use the following elementary types of variation:

(3) $f_t(z) = f(z) + Atf(z)$.

(4) $f_\phi(z) = f(ze^{i\phi}) = f(z) + i\phi zf'(z) + o(t)$.

(5) The next type of variation we need is different from the others because we can admit $t \geqq 0$ only. We apply at a boundary point z_0 a small radial cut. We can map Δ conformally in such a way onto this region, that 0 corresponds to 0 and z_0 to the end of the cut. To find an asymptotic development of this mapping which we will call μ_t, we consider the Koebe-function $k(z) = z/(1 - z)^2$, which maps Δ conformally onto $\mathbf{C}\backslash(-\infty, -\frac{1}{4}]$. We have for $z_0 = -1$: $\mu_t(z) = k^{-1}((1 - t)k(z))$ and therefore

$$\left.\frac{\partial \mu_t}{\partial t}\right|_{t=0} = -\frac{k(z)}{k'(z)} = -z\frac{1-z}{1+z}.$$

In the case that $z_0 \neq -1$ we must replace $\mu_t(z)$ by $-z_0\mu_t(-z/z_0)$ and get $\mu_t = z + tz(z + z_0)/(z - z_0) + o(t)$. Of course μ_t depends real analytically on t. Our variation is now

$$f_t(z) = f(\mu_t(z)) = f(z) + tf'(z)z\frac{z + z_0}{z - z_0} + o(t).$$

We can use linear combinations with real coefficients of the variations of the types (1)–(4). The space of these linear combinations has a natural basis. We denote by e_{w_0}, respectively e_{w_0}', the two variations of type (1) (if $w_0 \in \Omega$) or of type (2) (if $w_0(c\Omega)^0$) which belong to $A = 1$ respectively $A = i$. We denote further with e_s respectively e_s' the two variations $f_t = f + ft$ and $f_t = f + itf$, and with e_ϕ the variation $f_\phi = f(e^{i\phi})$. So the space X_0 has a representation

$$X_0 = \bigoplus_{w \notin \partial\Omega} \mathbf{R}_w^{\,2} \oplus \mathbf{R}^3.$$

Besides that, we can use linear combinations with non-negative coefficients of variations of the type (5). We denote $\bigoplus_{z \in \partial \mathbf{R}} R_z$ by X_1, with $\{e_z\}$ its natural basis, and by X_+ the subset of X_1 of vectors which have no negative coefficients.

To each $x \in X_0 + X_+$ which is close enough to the origin we can define a varied function f_x, e.g. in the following way: First we apply the variations of the types (1)–(3) simultaneously, then the variations of type (5) also simultaneously and at last the variation e_ϕ. It is easy to check that f_x depends real analytically on x in each finite dimensional subspace, and that the variations we have calculated are just the partial derivatives.

3. Variations with restrictions. Let $N_0, N_1 \ldots, N_n$ be real functions on $X_0 + X_+$ with $N_i(0) = 0$, such that for each finite dimensional subspace E of $X_0 + X_1$, the restrictions of the N_i to $E \cap X_0 + X_+$ are continuously differentiable in a neighbourhood of 0. We call x admissible if

$$N_1(x) = \ldots = N_n(x) = 0.$$

We denote by dN_i the linear functions on $X_0 + X_1$ which have on each vector of the basis the value of the derivative in the direction of this vector. For an arbitrary vector $u \in X_0 + X_+$ we then have

$$dN_i(u) = \lim_{t \to 0+} \frac{N_i(tu)}{t}.$$

The N_i can be put together to a mapping $N : X_0 + X_+ \to \mathbf{R}^{n+1}$ and in the same manner the dN_i to a mapping $dN : X_0 + X_1 \to \mathbf{R}^{n+1}$. The following theorem gives necessary conditions that $N_0(x) \leqq 0$ for all admissible x.

THEOREM 1. *If $N_0(x) \leqq 0$ for all admissible x then there are real numbers $\lambda_0\lambda_1 \ldots \lambda_n$ not all equal to 0 such that for all $x \in X_0$*

$$\sum_{i=0}^{n} \lambda_i dN_i(x) = 0, \tag{1}$$

and for all $x \in X_+$

$$\sum_{i=0}^{n} \lambda_i dN_i(x) \leqq 0. \tag{2}$$

We shall need a Lemma which is slight generalization of a well-known theorem on inverse functions. The proof is word for word the same as in [**4**] and we will therefore omit it.

LEMMA 1. *Let $\mathcal{N}$ be a function, which maps the ball $U_r = \{x \in \mathbf{R}^{n+1} | \|x\| \leqq r\}$ into $\mathbf{R}^{n+1}$ such that $\mathcal{N}(0) = 0$. If the restriction of $\mathcal{N}$ to every line-segment in U_r is differentiable except at finitely many points and if there is an $s < 1$ such that for each unit vector h $\|d_h\mathcal{N} - h\| \leqq 1$ holds (where d_h denotes the derivative in direction of h), then $\mathcal{N}$ maps U_r onto a neighbourhood of 0.*

We now prove:

LEMMA 2. *If $N(x) \leqq 0$ for all admissible x then $dN(X_0 + X_+) \neq \mathbf{R}^{n+1}$.*

Proof. We denote by $\{e_i\}$ the natural basis of $\mathbf{R}^{n+1}$. If $dN(X_0 + X_+)$ is all of $\mathbf{R}^{n+1}$ there would be u_i^+ and u_i^- in $X_0 + X_+$ such that $dN(u_i^+) = e_i$ and $dN(u_i^-) = -e_i$. To a vector $r = (r_0, \ldots, r_n) \in \mathbf{R}^{n+1}$ we define $v(r) = \sum |r_i| u_i$, where

$$u_i = \begin{cases} u_i^+ \text{ if } r_i \geqq 0 \\ u_i^- \text{ if } r_i < 0. \end{cases}$$

Then v maps $\mathbf{R}^{n+1}$ into $X_0 + X_+$. The composite Function $\mathcal{N} = Nov$ maps $\mathbf{R}^{n+1}$ into $\mathbf{R}^{n+1}$ such that $\mathcal{N}(0) = 0$. It is continuous and except at the coordinate hyperplanes (where the normal derivative need not to be continuous) continuously differentiable.

At the origin we have:

$$d_{e_i}\mathcal{N} = \lim_{t \to 0+} \frac{N(tu_i^+)}{t} = e_i \quad \text{and} \quad d_{-e_i}\mathcal{N} = \lim_{t \to 0+} \frac{N(tu_i^-)}{t} = -e_i.$$

It follows that there is a neighbourhood of the origin in which the assumptions of Lemma 1 hold. Therefore there is an $\epsilon_0 > 0$ such that for each $0 \leqq \epsilon < \epsilon_0$ ϵe_n has an inverse image s_ϵ. But this means that $v(s_\epsilon)$ is an admissible variation with $N_0(v(s_\epsilon)) = \epsilon \geqq 0$ which contradicts the assumption. This completes the proof of Lemma 2.

$X_0 + X_+$ is a convex cone and therefore $dN(X_0 + X_1)$ is also a convex cone and since it is not the whole space, it has a supporting hyperplane at 0. Let the equation of this hyper-plane be $\sum_{i=0}^{n} \lambda_i y_i = 0$ and be normalized such that for $x \in X_0 + X_+$ we have $\sum \lambda_i dN_i(x) \leqq 0$.

The vector space $dN(x_0)$ lies, of course, on the hyperplane and therefore for $x \in X_0$ we have: $\sum \lambda_i dN_i(x) = 0$; this completes the proof of Theorem 1.

4. The Extremum problem. In order to state our problem we need first a concept of homotopy of two functions.

Definition. Let $B' \subset B$ be two finite subsets of the complex plane. We say that two univalent analytic functions $f_1 : \Delta \to \mathbf{C}$, $f_2 : \Delta \to \mathbf{C}$ with $B' \in f_i(\Delta)$ are admissibly homotopic with respect to (B', B) if there is a continuous function $h(z, t)$ $(z \in \Delta, 0 \leqq t \leqq 1)$ such that:

(1) $h(z, 0) = f_1(z)$, $h(z, 1) = f_2(z)$.

(2) If $w_0 \in B$, then $h(z, t) = w_0$ if and only if $z = f_1^{-1}(w_0)$.

(3) If $w_0 = f(z_0) \in B'$, then

$$\lim_{z \to z_0} (\arg[f_2(z) - w_0] - \arg[f_1(z) - w_0]) = 0$$

where arg is continued continuously along the deformation path $h(z, t)$.

Remarks. The limit in (3) always exists and is equal to

$$\arg(f_1'(z_0)/f_2'(z_0)) + 2k\pi.$$

(3) means therefore that $\arg f_1(z_0) = \arg f_2'(z_0)$ and that $k = 0$. (Compare [**2**].)

One checks easily, that this is an equivalence relation and that the equivalence classes are closed with respect to locally uniform convergence. We shall call such an equivalence class a homotopy class.

$C = B \backslash f(\Delta)$ is the same for all functions of a homotopy-class. We further denote $D = B \backslash C$ and $E = D \backslash B'$.

Consider a given homotopy class H with respect to (B', B) and for each $b \in B'$ a system of complex numbers b_i $(i = 1 \dots r_b)$. We consider now the class of functions of H which also satisfy $f^{(i)}(z_b) = b_i$ $(z_b = f^{-1}(b))$. We want to maximize among these functions the real part of a functional $L(f)$ which we shall assume to be a linear combination of the form:

$$\operatorname{Re} L(f) = \operatorname{Re} \sum_{b \in E} d_b \log f'(z_b) + \operatorname{Re} \sum_{b \in B'} \sum_{i=r_b+1}^{2r_b} d_b{}^i f^{(i)}(z_b) \qquad d_b{}^i, d_b \in \mathbf{C}.$$

We assume here, that $\operatorname{Re} d_b \log f'(z_b)$ can be defined consistently for all functions which are analytic in a neighbourhood of D. We can further assume that $f(0) = 0 \in D$.

In order to apply Theorem 1 we have to formulate our side conditions with the help of functions on $X_0 + X_+$. For each $c \in C$ we define a linear function $N_c : X_0 + X_1 \to \mathbf{C}$ by its values on the basis vectors:

$$\begin{aligned} N_c(e_{w_0}) &= c/(c - w_0), N_c(e_{w_0}') = ic/(c - w_0); \\ N_c(e_s) &= c, \qquad\qquad N_e(e_s') = ic; \\ \mathrm{N}_c(e_{z_0}) &= N_c(e_\phi) = 0. \end{aligned}$$

If one writes down the variation which is defined by x one sees easily that if f omits c and $N_c(x) = 0$ then also f_x omits c. It should be observed that N_c must be split into real and imaginary parts if Theorem 1 is to be applied.

The second restriction is that the $f^{(i)}(z_b)$ $(i = 0, \dots, z_b, b \in D; r_b = 0$ if $b \in E)$ stay fixed. We define $N_b{}^i(x) = f_x{}^{(i)}(z_b) - f^{(i)}(z_b)$, which is a differentiable function on $X_0 + X_+$. Also, $N_b{}^i$ must be split in real and imaginary parts. We let $N_0(x) = L(f_x) - L(f)$, which is also a differentiable function on $X_0 + X_+$.

According to Theorem 1 there are λ_0, $v_b{}^i$, $\mu_b{}^i$, v_c and μ_c such that

$$\lambda_0 d \operatorname{Re} N_0(x) + \sum_{b \in D} \sum_{i=r_b+1}^{2r_b} (v_b{}^i d \operatorname{Re} N_b{}^i(x) + \mu_b{}^i d \operatorname{Im} N_b{}^i(x)) \\ + \sum_{c \in C} (v_c \operatorname{Re} N_c(x) + \mu_c \operatorname{Im} N_c(x)) \begin{cases} = 0, x \in X_0 \\ \leqq 0, x \in X_+. \end{cases}$$

We can combine the terms in brackets by setting $\lambda_b{}^i = v_b{}^i - i\mu_b{}^i$ and $\lambda_c = v_c - i\mu_c$ and we have:

$$\text{(3)} \quad \operatorname{Re}\left\{\lambda_0 dN_0(x) + \sum_{b \in D} \sum_{i=r_b+1}^{2r_b} \lambda_b{}^i dN_b{}^i(x) + \sum_{c \in C} \lambda_c N_c(x)\right\} \begin{cases} = 0, x \in X_0 \\ \leqq 0, x \in X_+, \end{cases}$$

where λ_0 is real, and not all λ_i vanish.

We set $A = a + ia'$ and we have according to (1):

$$\lambda_b{}^i dN_b{}^i(ae_{w_0} + a'e_{w_0}') \\ = \frac{d^i}{dz^i}\left[\frac{\lambda_b{}^i A f(z)}{f(z) - w_0} - \lambda_b{}^i \frac{Azf'(z)w_0}{z_0 f'(z_0)^2 (z - z_0)} + \lambda_b{}^i \frac{\overline{Aw_0} z^2 f'(z)}{\overline{z_0 f'(z_0)^2}(1 - \bar{z}_0 z)}\right]_{z=z_b}.$$

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We are interested in the real part only and can therefore change the last term to its complex conjugate to obtain:

$$\operatorname{Re} \lambda_b^i \, dN_b^i \, (ae_{w_0} + a'e_{w_0}')$$
$$= \operatorname{Re} A\left\{\lambda_b^i \frac{d^i}{dz^i} \frac{f(z)}{f(z) - w_0} - \frac{w_0}{z_0 f'(z_0)^2} \frac{d^i}{dz^i} \frac{zf(z)\lambda_b^i}{z - z_0} - \frac{w_0 \overline{\lambda_b^i}}{z_0 f'(z_0)^2} \overline{\frac{d^i}{dz^i} \frac{z^2 f'(z)}{1 - \bar{z}_0 z}}\right\}.$$

For $\lambda_0 \operatorname{Re} dN_0(ae_{w_0} + a'e'_{w_0})$, we obtain by the same reasoning as above:

$$\lambda_0 \operatorname{Re} dN_0(ae_{w_0} + a'e_{w_0}') = \operatorname{Re} A\left\{\lambda_0 dL\left(\frac{f(z)}{f(z) - w_0}\right)\right.$$
$$\left. - \frac{\lambda_0 w_0}{z_0 f'(z_0)^2}\left[dL\left(\frac{zf'(z)}{z - z_0}\right) - \overline{dL\left(\frac{z^2 f'(z)}{1 - z_0 z}\right)}\right]\right\},$$

where

$$dL(g(z)) = \sum_{b \in E} \frac{d_b g'(z_b)}{f'(z_b)} + \sum_{b \in B'} \sum_{i=r_b+1}^{2r_b} f^{(i)}(z_b) \cdot d_b^{\,i}.$$

We now set:

$$Q_b^{\,i}(w_0) = \lambda_b^{\,i} \frac{d^i}{dz^i}\left(\frac{f(z)}{f(z) - w_0}\right)\bigg|_{z=z_b},$$

$$R_b^{\,i}(z_0) = z_0 \lambda_b^{\,i} \frac{d^i}{dz^i} \frac{zf'(z)}{z - z_0}\bigg|_{z=z_b} - z_0 \overline{\lambda_b^{\,i}} \, \overline{\frac{d^i}{dz^i} \frac{z^2 f'(z)}{1 - \bar{z}_0 z}}\bigg|_{z=z_b},$$

$$Q_0(w_0) = \lambda_0 dL\left(\frac{f(z)}{f(z) - w_0}\right),$$

$$R_0(z_0) = z_0 \lambda_0 dL\left(\frac{zf'(z)}{z - z_0}\right) - z_0 \lambda_0 \overline{dL\left(\frac{z^2 f'(z)}{1 - \bar{z}_0 z}\right)}.$$

(3) now yields for $x = ae_{w_0} + a'_a e_{w_0}' \in X_0$

$$\operatorname{Re} A\left\{\left[Q_0(w_0) - \frac{R_0(z_0) w_0}{z_0^2 f'(z_0)}\right] + \sum_{b \in D} \sum_{i=0}^{r_b}\left[Q_b^{\,i}(w_0) - \frac{R_b^{\,i}(z_0) w_0}{z_0^2 f'(z_0)^2}\right]\right.$$
$$\left. + \sum_{c \in C} \frac{c\lambda_c}{c - w_0}\right\} = 0$$

and since A is arbitrary we get after dividing by w_0

$$(4) \quad \frac{1}{w_0}\left[Q_0(w_0) + \sum_{b \in D} \sum_{i=0}^{r_b} Q_b^{\,i}(w_0) + \sum_{c \in C} \frac{c\lambda_c}{c - w_0}\right]$$
$$= \frac{1}{z_0^2 f'(z_0)^2}\left[R_0(z_0) + \sum_{b \in B} \sum_{i=0}^{r_b} R_b^{\,i}(z_0)\right].$$

We will call the left hand side of this equation $Q(w_0)$ and the term in square brackets on the right hand side $R(z_0)$. They are meromorphic functions of w_0 respectively z_0. $R(z_0)$ has no poles on the boundary, and $Q(w_0)$ has simple poles at the points of C.

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Since the Q_b^i are analytic except at b, where they have a pole of order $i + 1$, the poles of Q at a point b are of the following form:

$$Q(w_0) = \frac{\lambda_0}{w_0} dL\left(\frac{f(z)}{f(z) - w_0}\right) + \frac{r(w_0)}{(w_0 - b)^{r_b+1}}$$

where $r(w_0)$ is analytic at b.

Next we prove that $Q(w)dw^2$ has at most a simple pole at infinity. All the Q_b^i and Q_0 have simple zeros at infinity; therefore $Q(w_0)$ has a double zero. We want to prove that also the term with $1/w_0^2$ vanishes. We have, close to infinity

$$\frac{f(z)}{f(z) - w_0} = -\sum_{k=1}^{\infty} \left(\frac{f(z)}{w_0}\right)^k \quad \text{and} \quad \frac{c}{c - w_0} = -\sum_{k=1}^{\infty} \left(\frac{c}{w_0}\right)^k$$

and therefore by definition:

$$(5) \quad Q(w_0) = \frac{1}{w_0}\left[\frac{1}{w_0}\left(\lambda_0 dL(f(z)) + \sum_b \sum_i \lambda_b^i f^{(i)}(z_b) + \sum_{c \in C} c\lambda_c\right) + \frac{1}{w_0^2}(\ldots)\right].$$

In order to show that the first term is zero we apply (3) to $ae_s + a'e'_s$ and obtain:

$$\operatorname{Re} A\left[\lambda_0 dLf(z)) + \sum_b \sum_i \lambda_b^i f^{(i)}(z_b) + \sum_{c \in C} c\lambda_c\right] = 0,$$

and since A is arbitrary:

$$\lambda_0 dL(f(z)) + \sum_b \sum_i \lambda_b^i f^{(i)}(z_b) + \sum_{c \in C} c\lambda_c = 0;$$

but this is, according to (5) just the term with $1/w_0^2$ in the development of $Q(w_0)$ around infinity. So $Q(w)^2dw^2$ has at most a simple pole at infinity.

To show, that $R(z_0)$ is real on the boundary we need some auxiliary calculation. We set $z_0 = 1/\bar{z}_0$ and have

$$\begin{aligned} 2i \operatorname{Im} R_0(z_0) &= \lambda_0 dL\left(\frac{1}{\bar{z}_0}\frac{zf'(z)}{z - z_0}\right) - \overline{\lambda_0 dL\left(\frac{\bar{z}_0 z^2 f'(z)}{1 - \bar{z}_0 z}\right)} \\ &\quad - \overline{\lambda_0 dL\left(\frac{1}{\bar{z}_0}\frac{zf'(z)}{z - z_0}\right)} + \lambda_0 dL\left(\frac{\bar{z}_0 z^2 f'(z)}{1 - \bar{z}_0 z}\right) \\ &= \lambda_0 dL\left(\frac{zf'(z)}{z\bar{z}_0 - 1} + \frac{\bar{z}_0 z^2 f'(z)}{1 - \bar{z}_0 z}\right) - \overline{\lambda_0 dL\left(\frac{\bar{z}_0 z^2 f'(z)}{1 - \bar{z}_0 z} + \frac{zf'(z)}{z\bar{z}_0 - 1}\right)} \\ &= 2i \operatorname{Im} dL(zf'(z))\lambda_0. \end{aligned}$$

In the same way we derive for R_b^i on the boundary:

$$2i \operatorname{Im} R_b^i = 2i \operatorname{Im} \frac{d^i}{dz^i} zf'(z)\lambda_i^b \Big|_{z=z_b}.$$

Therefore we have also on the boundary:

$$\operatorname{Im} R(z) = -\operatorname{Re}(dL(izf'(z))\lambda_0 + \sum_b \sum_i \frac{d^i}{dz^i} izf'(z)\lambda_b{}^i).$$

The right side of the above is just the left side of (3) applied to e_ϕ and is therefore zero. So $R(z)$ is real on the boundary.

We now apply (3) to $e_{z_0}(|z_0| = 1)$ and obtain:

$$(6) \qquad \operatorname{Re}\left\{\lambda_0 dL\left(zf'(z)\frac{z+z_0}{z-z_0}\right) + \sum_b \sum_i \lambda_b{}^i \frac{d^i}{dz^i}\left(zf'(z)\frac{z+z_0}{z-z_0}\right)\right\} \leqq 0,$$

and since for $|z_0| = 1$

$$\frac{z-z_0}{z-z_0} = \frac{z_0}{z-z_0} - \frac{\bar{z}_0 z}{1-\bar{z}_0 z},$$

the left hand side of (6) is precisely $\operatorname{Re} R(z_0)$. We can write (4) in the form $R(z)(dz/z)^2 = Q(w)dw^2$, and since $(dz/z)^2 < 0$ on the boundary we have $Q(w)dw^2 \geqq 0$ on the boundary.

$Q(w)$ does not vanish identically because not all the $\lambda_b{}^i = 0$. We will use this fact to show that $f(\Delta)$ has no exterior points. In fact let w_0 be an exterior point. Applying (3) to $ae_{w_0} + a'e_{w_0}{}'$ we obtain:

$$\operatorname{Re} A\left\{\lambda_0 dL\left(\frac{f(z)}{f(z)-w_0}\right) + \sum_b \sum_i \frac{d^i}{dz^i}\left(\frac{f(z)}{f(z)-w_0}\right)\lambda_b{}^i + \sum_{c\in C}\frac{c\lambda_c}{c-w_0}\right\},$$

and since A is arbitrary the expression in brackets is zero. But this expression is just $Q(w_0)$; therefore $Q(w_0)$ would be identically zero if there were exterior points, which is not possible.

We have proven the following theorem.

THEOREM 2. *If f is an extremal function, then $f(\Delta)$ has no exterior points and there is a rational function $Q(w)$ with the following properties:*

(i) *$Q(w)$ has at most simple poles at the points $c \in C$.*

(ii) *At the points $b \in D$, Q has poles of the form:*

$$Q(w) = \frac{\lambda_0}{w_0} dL \frac{f(z)}{f(z)-w} + \frac{r(w_0)}{(w_0-b)r_b + 1}$$

where $r(w_0)$ is analytic at b.

(iii) *$Q(w)dw^2 \geqq 0$ on the boundary of $f(\Delta)$.*

We use the extended general coefficient Theorem [3] to prove:

THEOREM 3. *If $f(\Delta)$ has no exterior points and if there is a $Q(w)$ in $f(\Delta)$ satisfying* (i)–(iii), *then f is an extremal function; it is maximal if $\lambda_0 > 0$, minimal if $\lambda_0 < 0$ and the only admissible function if $\lambda_0 = 0$.*

Proof. If g is any other function then gof^{-1} has around each point of D an expansion:

$$gof^{-1}(w) = \sum_{i=0}^{\infty} \alpha_i^b (w-b).$$

We now express $L(g)$ in terms of α_i^b and get:

$$L(g) = \sum_{b \in E} d_b[\log \alpha_1^b + \log f'(z_b)] + \sum_{b \in B'} \sum_{k=0}^{2r_b} \gamma_i^b \alpha_i^b$$

where the γ_i^b are certain complex numbers.

If g is also an admissible function, then $\alpha_1^b = 1, \alpha_2^b = \ldots, = \alpha_{r_b}^b = 0$, and therefore

$$L(g) = \sum_{b \in E} d_b \log \alpha_1^b + \sum_{b \in B'} \sum_{k=r_b+1}^{2r_b} \gamma_i^b \alpha_i^b + L(f). \tag{7}$$

On the other hand we have:

$$\frac{w}{w-w_0} = \frac{b+(w-b)}{(w-b)-(w_0-b)} = -\left[\frac{1}{w_0-b} + w_0 \sum_{i=1}^{\infty} \frac{(w-b)^k}{(w_0-b)^{k+1}}\right],$$

and therefore

$$dL\left(\frac{w}{w-w_0}\right) = -w_0\left[\sum_{b \in E} \frac{d_b}{(w_0-b)^2} + \sum_{b \in B'} \sum_{i=r_b+1}^{2r_b} \frac{\gamma_i^b}{(w_0-b)^{i+1}}\right]$$

and

$$Q(w_0) = -\sum_{k=r_b+1}^{2r_b} \frac{\gamma_k^b \lambda_0}{(w_0-b)^{k+1}} + \frac{q(w_0)}{(w_0+b)^{r_b+1}}.$$

By the extended general coefficient theorem we have

$$-\lambda_0(L(g) - L(f)) \leqq 0,$$

i.e., f is maximal if $\lambda_0 < 0$ and minimal if $\lambda_0 > 0$.

If $\lambda_0 = 0$ already, the side conditions are extremal as one sees in the same manner of the general coefficient theorem, and for each admissible function equality holds. It is easy to see then that f is the only admissible function.

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