2

Some Preliminaries

In this chapter we briefly review some basic facts in quantum mechanics, together with a few new concepts and results used in some applications later. In particular, we will first recall some aspects of what physicists usually call second quantization, which will be one of the essential tools all throughout this book. For us, second quantization is nothing but the functional settings associated with CCR and with CAR. For completeness, we will also briefly present some deformed versions of these relations, which have been considered in the literature in recent years, because of their mathematical relevance, or for their appearance in pseudo-Hermitian quantum mechanics [23–25], or even for their use in recent models [26]. This will be done in Section 2.3. Then, we will discuss some definitions and results on quantum dynamics, focusing on those aspects which are relevant for us, and discussing several aspects of the Schrödinger and Heisenberg representations, and of their dynamical contents. Next we will recall a few facts on the Heisenberg uncertainty principle and on states of a quantum system, and then, in view of their relevance in Part II of this book, we will discuss and compare two different strategies which produce a stable asymptotic behavior in the time evolution of some given observables, see Section 2.7. We will finally introduce and analyze what has been called \((H, \rho)\)-induced dynamics [5], which can be thought, roughly speaking, as a modified version of the Heisenberg dynamics in presence of what is called a rule, whose effect cannot be described in terms of any Hamiltonian operator. This is the content of Sections 2.8 and 2.9, where we will also discuss how a suitable \((H, \rho)\)-induced dynamics can produce some stabilization in the large time limit.

2.1 The Bosonic Number Operator

Let \(\mathcal{H}\) be a Hilbert space and \(B(\mathcal{H})\) the set of all the bounded operators on \(\mathcal{H}\). \(B(\mathcal{H})\) is a so-called \(C^\ast\)-algebra, that is an algebra with involution which is complete under a norm, \(|\langle \cdot |\cdot \rangle|\), satisfying the \(C^\ast\)-property: \(|A^\ast A| = |A|^2\), for all
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A ∈ B(ℋ). As a matter of fact, B(ℋ) is usually seen as a concrete realization (or, in mathematical terms, a representation) of an abstract C*-algebra. Let ℳ be our physical system and ℬ the set of all the operators useful for a complete description of ℳ, which includes the observables of ℳ, i.e., those quantities which are measured in a concrete experiment. Let ℋ be the Hilbert space where ℳ is defined. For simplicity, it would be convenient to assume that ℬ is a C*-algebra by itself, possibly coinciding with the original set B(ℋ), or, at least, with some closed subset of B(ℋ). However, this is not always possible in our concrete applications. This is because of the crucial role of some unbounded operators within our scheme: unbounded operators do not belong to any C*-algebra. However, if X is such an operator, and if it is self-adjoint (as the observables of a quantum system are usually assumed to be), then U(t) = e^{iXt} is unitary and, therefore, bounded. In particular we have ||e^{iXt}|| = 1, for all t ∈ ℝ, and X can be recovered from U(t) by taking its time derivative in t = 0 and multiplying the result by −i. For this reason, C*-algebras and their subsets are relevant even when unbounded operators appear. However, if we want to define X properly, the first problem we meet is the definition of the domain of X, D(X), i.e., the set of vectors of ℋ on which X can act, returning vectors which are again in ℋ. If X is bounded, then D(X) coincides with ℋ. Otherwise D(X) is a proper subset of ℋ. It is crucial that D(X) is not too small. In fact, X is well defined only if D(X) is dense in ℋ [27].

Remarks: (1) In this book the mathematical aspects related to unbounded operators will not play an essential role, even when they appear in the game. The reason is the following: In some models, like in [9], even if the observables of the system ℳ are unbounded, an integral of motion I exists for the model. This means, as we will show in Section 2.4, that a (self-adjoint) operator I exists which commutes with the Hamiltonian of ℳ. When this happens, not all the (infinite-dimensional) Hilbert space ℋ attached to ℳ is available: the only vectors of the orthonormal (o.n.) basis F_φ = {φ_n}, n = (n_1, n_2, ...), n_j ≥ 0, of ℋ which are needed in the description of ℳ are those labeled by quantum numbers which are compatible with the existence of I, and these are only a finite number. This implies that, even if dim(ℋ) = ∞, the dynamics of ℳ is restricted to an effective Hilbert space ℋ_{eff}, finite-dimensional, which is the span of those φ_n whose n satisfies the selection rule induced by I. Therefore, the observables of ℳ become finite matrices in ℋ_{eff}, so that they are all bounded. We refer the reader interested in mathematical aspects of unbounded operators and of unbounded operator algebras to [27–30], where it is discussed how an algebraic structure can be constructed also for unbounded operators, at least under some suitable mathematical assumptions.

(2) It is also convenient to notice that, in most of our computations, we will be interested in computing mean values of the form ⟨f, ABf⟩, with A and B possibly unbounded, and with f ∈ ℋ with a very special property: f belongs (always!) to the
domain of $B$, $D(B)$, and is such that $Bf \in D(A)$, the domain of $A$. Hence, even if $AB$ does not in general make sense, $ABf$ is perfectly defined in $\mathcal{H}$. Hence, $|\langle f, ABf \rangle| < \infty$, for all such operators $A$ and $B$, and for all these vectors $f$.

A special role in our analysis is played by the CCR: we say that a set of operators $\{a_l, a_l^\dagger, l = 1, 2, \ldots, L\}$, acting on the Hilbert space $\mathcal{H}$, satisfies the CCR, if the following hold:

$$ [a_l, a_n^\dagger] = \delta_{ln} \mathbb{I}, \quad [a_l, a_n] = [a_l^\dagger, a_n^\dagger] = 0, \quad (2.1) $$

for all $l, n = 1, 2, \ldots, L$, $\mathbb{I}$ being the identity operator on $\mathcal{H}$. These operators, which are widely analyzed in any textbook in quantum mechanics, see [2,31] for instance, are those which are used to describe $L$ different modes of bosons. From these operators we can construct $\hat{n}_l = a_l a_l^\dagger$ and $\hat{N} = \sum_{l=1}^{L} \hat{n}_l$ which are both self-adjoint: $\hat{n}_l = (\hat{n}_l)^\dagger$ and $\hat{N} = (\hat{N})^\dagger$. In particular $\hat{n}_l$ is the number operator for the $l$-th mode, while $\hat{N}$ is the number operator for the system $S$ described by our operators.

**Remark:** The equality $\hat{n}_l = (\hat{n}_l)^\dagger$ above can be seen as the result of a simple, but formal, computation: $\hat{n}_l^\dagger = (a_l^\dagger a_l)^\dagger = a_l^\dagger (a_l^\dagger)^\dagger = a_l^\dagger a_l = \hat{n}_l$. To be more rigorous, we should do much more than this. In particular, we should also consider the domains of $\hat{n}_l$ and of $\hat{n}_l^\dagger$, since these could be different.\(^1\) This is particularly relevant here, since each $\hat{n}_l$ is unbounded. However, we will be satisfied with this formal result, since these mathematical aspects are not particular relevant for what we will discuss in the following.

An o.n. basis of $\mathcal{H}$ can be constructed as follows: we introduce the vacuum of the theory, that is, a vector $\varphi_0$ which is annihilated by all the operators $a_l$: $a_l \varphi_0 = 0$ for all $l = 1, 2, \ldots, L$. Then we act on $\varphi_0$ with the operators $a_l^\dagger$ and with their powers,

$$ \varphi_{n_1, n_2, \ldots, n_L} := \frac{1}{\sqrt{n_1! \cdots n_L!}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots (a_L^\dagger)^{n_L} \varphi_0, \quad (2.2) $$

$n_l = 0, 1, 2, \ldots$, for all $l$, and we normalize the vectors obtained in this way. The set of the $\varphi_{n_1, n_2, \ldots, n_L}$’s in (2.2) forms a complete and o.n. set in $\mathcal{H}$, and they are eigenstates of both $\hat{n}_l$ and $\hat{N}$:

$$ \hat{n}_l \varphi_{n_1, n_2, \ldots, n_L} = n_l \varphi_{n_1, n_2, \ldots, n_L} $$

\(^1\) Given an operator $X$ acting on $\mathcal{H}$ and with domain $D(X)$, its adjoint $X^\dagger$ is defined by introducing first its domain $D(X^\dagger)$ as follows:

$$ D(X^\dagger) = \{ g \in \mathcal{H} : \exists f \in D(X) : \langle f, g \rangle = \langle Xf, g \rangle, \ \forall f \in D(X) \}. $$

Then we put $X^\dagger g = g_X$, for all $g \in D(X^\dagger)$. $X$ is self-adjoint if $\langle Xf, g \rangle = \langle f, Xg \rangle$, for all $f, g \in D(X)$, and if $D(X) = D(X^\dagger)$. 

and
\[ \hat{N} \varphi_{n_1,n_2,\ldots,n_L} = N \varphi_{n_1,n_2,\ldots,n_L}, \]
where \( N = \sum_{l=1}^{L} n_l \). Hence, \( n_l \) and \( N \) are eigenvalues of \( \hat{n}_l \) and \( \hat{N} \) respectively. Moreover, using the CCR we deduce that
\[ \hat{n}_l \left( a_l \varphi_{n_1,n_2,\ldots,n_L} \right) = (n_l - 1) \left( a_l \varphi_{n_1,n_2,\ldots,n_L} \right), \]
for \( n_l \geq 1 \) while, if \( n_l = 0 \), \( a_l \) annihilates the vector, and
\[ \hat{n}_l \left( a_l^\dagger \varphi_{n_1,n_2,\ldots,n_L} \right) = (n_l + 1) \left( a_l^\dagger \varphi_{n_1,n_2,\ldots,n_L} \right), \]
for all \( l \) and for all \( n_l \). For these reasons the following interpretation is given in the literature: if the \( L \) different modes of bosons of \( S \) are described by the vector \( \varphi_{n_1,n_2,\ldots,n_L} \), this means that \( n_1 \) bosons are in the first mode, \( n_2 \) in the second mode and so on. The operator \( \hat{n}_l \) acts on \( \varphi_{n_1,n_2,\ldots,n_L} \) and returns \( n_l \), which is exactly the number of bosons in the \( l \)-th mode. The operator \( \hat{N} \) counts the total number of bosons. Moreover, the operator \( a_l \) destroys a boson in the \( l \)-th mode, if there is at least one. Otherwise \( a_l \) simply destroys the state. Its adjoint, \( a_l^\dagger \), creates a boson in the same mode. This is why in the physical literature \( a_l \) and \( a_l^\dagger \) are usually called the \textit{annihilation} and the \textit{creation} operators.

The vector \( \varphi_{n_1,n_2,\ldots,n_L} \) in Equation (2.2) defines a \textit{vector (or number) state} over the set \( \mathcal{A} \) as
\[ \omega_{n_1,n_2,\ldots,n_L}(X) = \langle \varphi_{n_1,n_2,\ldots,n_L}, X \varphi_{n_1,n_2,\ldots,n_L} \rangle, \quad (2.3) \]
where \( \langle , \rangle \) is the scalar product in the Hilbert space \( \mathcal{H} \), and \( X \in \mathcal{A} \). These states will be used to \textit{project} from quantum to classical dynamics and to fix the initial conditions of the system under consideration, in a way which will be clarified later on. Something more concerning states will be discussed later in this chapter.

As observed previously, the operators \( a_l, a_l^\dagger, \hat{n}_l \) and \( \hat{N} \) are all unbounded. This can be easily understood since, for instance,
\[ \| \hat{n}_l \| = \sup_{0 \neq \varphi \in \mathcal{H}} \frac{\| \hat{n}_l \varphi \|}{\| \varphi \|} \geq \sup_{\{ n_j \geq 0, j=1,2,\ldots,L \}} \| \hat{n}_l \varphi_{n_1,n_2,\ldots,n_L} \| = \sup_{\{ n_j \geq 0, j=1,2,\ldots,L \}} n_l = \infty, \]
and it is clearly related, as already observed, to the fact that \( \mathcal{H} \) is infinite-dimensional. We have already pointed out that unbounded operators have severe domain problems, since they cannot be defined in all of \( \mathcal{H} \) [27], but only on a dense subset of \( \mathcal{H} \). In fact, an operator \( X \) acting on \( \mathcal{H} \) is bounded if, and only if, \( D(X) = \mathcal{H} \). We want to stress once more that this will not be a major problem for us: in fact, first of all, each vector \( \varphi_{n_1,n_2,\ldots,n_L} \) belongs to the domains of all the operators
which are relevant for us, and the linear span of the set $\mathcal{F}_\psi = \{ \varphi_{n_1,n_2,\ldots,n_L}, n_j \geq 0 \}$ is left stable under the action of all these operators. Secondly, as already discussed, it may happen that the infinite-dimensional Hilbert space $\mathcal{H}$ is replaced by an effective Hilbert space, $\mathcal{H}_{\text{eff}}$, which becomes *dynamically finite-dimensional* because of the existence of some conserved quantity and because of the initial conditions, which impose some constraints on the accessible levels to the agents of the system [9].

### 2.2 The Fermionic Number Operator

Given a set of operators $\{b_l, b_l^\dagger, l = 1, 2, \ldots, L\}$ acting on a certain Hilbert space $\mathcal{H}_F$, we say that they satisfy the CAR if the conditions

$$\{b_l, b_n^\dagger\} = \delta_{l,n} \mathbb{I}, \quad \{b_l, b_n\} = \{b_l^\dagger, b_n^\dagger\} = 0 \quad (2.4)$$

hold true for all $l, n = 1, 2, \ldots, L$. Here, $\{x, y\} := xy + yx$ is the *anticommutator* of $x$ and $y$ and $\mathbb{I}$ is now the identity operator on $\mathcal{H}_F$. These operators, which are considered in many textbooks on quantum mechanics, see, for instance [2, 31], are those which are used to describe different modes of fermions. As for bosons, from these operators we can construct $\hat{n}_l = b_l^\dagger b_l$ and $\hat{N} = \sum_{l=1}^L \hat{n}_l$, which are both self-adjoint. In particular, $\hat{n}_\ell$ is the *number operator* for the $\ell$-th mode, while $\hat{N}$ is the *global number operator* for $S$. Compared with bosonic operators, the operators introduced here satisfy a very important feature: if we try to square them (or to rise to higher powers), we simply get zero: for instance, (2.4) implies that $b_l^2 = 0$. This is of course related to the fact that fermions satisfy the Pauli exclusion principle [31], while bosons do not.

The Hilbert space of our system is constructed as for bosons: we introduce the *vacuum* of the theory, that is, a vector $\Phi_0$ which is annihilated by all the operators $b_l$: $b_l \Phi_0 = 0$ for all $l = 1, 2, \ldots, L$. Then we act on $\Phi_0$ with the operators $(b_l^\dagger)^{n_l}$:

$$\Phi_{n_1,n_2,\ldots,n_L} := (b_1^\dagger)^{n_1} (b_2^\dagger)^{n_2} \cdots (b_L^\dagger)^{n_L} \Phi_0, \quad (2.5)$$

$n_l = 0, 1$, for all $l$. Of course, we do not consider higher powers of the $b_l^\dagger$’s, since these powers would simply destroy the vector. This explains why no normalization appears. In fact, for all allowed values of the $n_l$’s, the normalization constant $\sqrt{n_1! n_2! \cdots n_L!}$ in (2.2) is equal to one. These vectors form an o.n. set which spans all of $\mathcal{H}_F$, and they are eigenstates of both $\hat{n}_l$ and $\hat{N}$, similar to what we have seen for CCR:

$$\hat{n}_l \Phi_{n_1,n_2,\ldots,n_L} = n_l \Phi_{n_1,n_2,\ldots,n_L}$$

and

$$\hat{N} \Phi_{n_1,n_2,\ldots,n_L} = N \Phi_{n_1,n_2,\ldots,n_L} ,$$
where $N = \sum_{l=1}^{L} n_l$. A major difference with respect to what happens for bosons is that the eigenvalues of $\hat{n}_l$ are simply zero and one, and consequently $N$ can take any nonnegative integer value (as for bosons), but smaller or equal to $L$. Moreover, using the CAR, we deduce that

$$\hat{n}_l (b_l \Phi_{n_1, n_2, \ldots, n_L}) = \begin{cases} (n_l - 1)(b_l \Phi_{n_1, n_2, \ldots, n_L}), & n_l = 1 \\ 0, & n_l = 0, \end{cases}$$

and

$$\hat{n}_l (b_l^\dagger \Phi_{n_1, n_2, \ldots, n_L}) = \begin{cases} (n_l + 1)(b_l^\dagger \Phi_{n_1, n_2, \ldots, n_L}), & n_l = 0 \\ 0, & n_l = 1, \end{cases}$$

for all $l$. The interpretation does not differ much from that for bosons, and then $b_l$ and $b_l^\dagger$ are again respectively called the annihilation and the creation operators. However, in some sense, $b_l^\dagger$ is also an annihilation operator since, acting on a state with $n_l = 1$, it destroys that state: we are trying to put together two identical fermions, and this operation is forbidden by the Pauli exclusion principle.

Of course, $\mathcal{H}_F$ has a finite dimension. In particular, for just one mode of fermions, $\dim(\mathcal{H}_F) = 2$, while $\dim(\mathcal{H}_F) = 4$ if $L = 2$. This also implies that, contrary to what happens for bosons, all the fermionic operators are bounded and can be represented by finite-dimensional matrices, independently of the existence of any integral of motion.

As for bosons, the vector $\Phi_{n_1, n_2, \ldots, n_L}$ in (2.5) defines a vector (or number) state over the algebra $\mathcal{A}$ of the operators over $\mathcal{H}_F$ as

$$\omega_{n_1, n_2, \ldots, n_L}(X) = \langle \Phi_{n_1, n_2, \ldots, n_L}, X \Phi_{n_1, n_2, \ldots, n_L} \rangle,$$

where $\langle , \rangle$ is the scalar product in $\mathcal{H}_F$, and $A \in \mathcal{A}$. Again, these states will be used to project from quantum to classical dynamics and to fix the initial conditions of the considered system. This will be clarified in the next chapters, when discussing concrete applications of both CCR and CAR.

### 2.3 Other Possibilities

The ones considered so far are not the only ladder operators arising when dealing with quantum mechanical systems. Many other possibilities have been introduced along the years in the literature, in many different contexts and for many different situations. Each one of these alternatives has some particularly interesting mathematical features and turns out to be useful in certain concrete applications. In the rest of this section we will briefly review some of them.
2.3 Other Possibilities

2.3.1 Quons

One such alternative is based on the following (one-mode) \( q \)-mutation relation:

\[
[A, A^\dagger]_q := AA^\dagger + qA^\dagger A = 1,
\]

where \( q \) is a real number between plus and minus one, \( A \) is a given operator acting on some Hilbert space \( \mathcal{H} \), and \( A^\dagger \) is its adjoint. When \( q = 1 \) the \( q \)-mutation relation produces CAR (when supplemented with the condition \( A^2 = 0 \)), while CCR are recovered when \( q = -1 \). If \( q \in ]-1, 1[ \), Equation (2.7) describes particles which are neither bosons nor fermions. Many details on quons are discussed in [32–37]. Particularly relevant for us is the fact that \( A \) behaves as a lowering operator, while \( A^\dagger \) is a raising operator, as discussed in the remainder of this section.

In [37] it is proved that the eigenstates of \( \hat{N}_0 = A^\dagger A \) are analogous to the bosonic ones, except that for the normalization. A simple concrete realization of Equation (2.7) can be deduced as follows: let \( \mathcal{F}_c = \{ e_k, k = 0, 1, 2, \ldots \} \) be the canonical o.n. basis in \( \mathcal{H} = l^2(\mathbb{N}_0) \), the set of all the square summable sequences, with all zero entries except in the \((k + 1)\)-th position, which is equal to one: \( \langle e_k, e_m \rangle = \delta_{k,m} \). If we take

\[
A = \begin{pmatrix}
0 & \beta_0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \beta_1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \beta_2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \beta_3 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \beta_4 & \cdots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix},
\]

(2.8)

it follows that (2.7) is satisfied if \( \beta_0^2 = 1 \) and \( \beta_n^2 = 1 + q\beta_{n-1}^2 \), \( n \geq 1 \). Then \( \beta_n^2 \) coincides with \( n + 1 \) if \( q = 1 \), and with \( \frac{1 - q^{n+1}}{1 - q} \) if \( q \in ]-1, 1[ \). It is convenient to fix \( \beta_n > 0 \) for all \( n \geq 0 \). Moreover, it is clear that \( A e_0 = 0 \), and \( A^\dagger \) behaves, as stated, as a raising operator since from Equation (2.8) we deduce

\[
e_{n+1} = \frac{1}{\beta_n} A^\dagger e_n = \frac{1}{\beta_n} \left( A^\dagger \right)^{n+1} e_0,
\]

(2.9)

for all \( n \geq 0 \). Here we have introduced the notation \( \beta_n! := \beta_n \beta_{n-1} \cdots \beta_2 \beta_1 \). In the literature this quantity is sometimes called the \( q \)-factorial. Of course, from (2.9) it follows that \( A^\dagger e_n = \beta_n e_{n+1} \). Using the matricial form for \( A \) in (2.8) it is also easy to check that \( A \) acts as a lowering operator on \( \mathcal{F}_c \): \( A e_m = \beta_{m-1} e_{m-1} \), for all \( m \in \mathbb{N}_0 \), where we have also introduced \( \beta_{-1} = 0 \), to ensure that \( A e_0 = 0 \).

Then, calling \( \hat{N}_0 = A^\dagger A \), we have

\[
\hat{N}_0 e_m = \beta_{m-1}^2 e_m,
\]

(2.10)
for all \( m \in \mathbb{N}_0 \). The operator \( \hat{N} \), formally defined in [37] as 
\[
\hat{N} = \frac{1}{\log(q)} \log(\mathbb{I} - \hat{N}_0(1 - q)) \quad \text{for } q > 0,
\]

satisfies the eigenvalue equation \( \hat{N}e_m = me_m \), for all \( m \in \mathbb{N}_0 \). For this reason \( \hat{N} \) is called the number operator for the quons, while \( \hat{N}_0 \) is not. Notice that, however, they share the same eigenvectors.

It should be stressed that the one in Equation (2.8) is not the only possible way to represent an operator \( A \) satisfying Equation (2.7). For instance, in [34], the authors adopt the following representation of \( A \) and \( A^\dagger \) in \( L^2(\mathbb{R}) \):

\[
A = \frac{e^{-2i\alpha x} - e^{i\alpha \frac{d}{dx}}e^{-i\alpha x}}{-i \sqrt{1 - e^{-2\alpha^2}}} , \quad A^\dagger = \frac{e^{2i\alpha x} - e^{i\alpha \frac{d}{dx}}e^{i\alpha x}}{i \sqrt{1 - e^{-2\alpha^2}}} ,
\]

where \( \alpha = \sqrt{-\frac{\log(q)}{2}} \) or, which is the same, \( q = e^{-2\alpha^2} \). In this case, since \( \alpha \) is assumed to belong to the set \([0, \infty)\), \( q \) ranges in the interval \([0, 1]\).

For completeness we also remark that other possible \( q \)-mutation relations have also been proposed along the years. Another such possibility is

\[
[A, A^\dagger]_q = q^{-2}\hat{N},
\]

where \( \hat{N} \) is the number operator introduced before. This rule, again, gives rise to an interesting functional structure, and is applied to some physical situations, when particles do not obey the more common CCR and CAR.

### 2.3.2 Truncated Bosons

As we have seen above, in a sense quons interpolate between fermions and bosons, depending on the value of \( q \) in Equation (2.7). This is not the only way to perform a similar interpolation. A completely different method makes use of the so-called truncated bosons, discussed, for instance, in [38, 39].

We consider an operator \( B \) which obeys the following rule:

\[
[B, B^\dagger] = \mathbb{I} - L K,
\]

in which \( L = 2, 3, 4, \ldots \) is a fixed natural number, while \( K \) is a self-adjoint projection operator, \( K = K^2 = K^\dagger \), satisfying the equality \( KB = 0 \). The presence of the term \( L K \) in Equation (2.12) makes it possible to find a representation of \( K \) and \( B \) in terms of \( L \times L \) matrices, which would not be possible in absence of such a correction. In fact, in this case, we would recover the CCR, which does not admit any finite-dimensional representation. Here, on the other hand, \( K, B \) and \( B^\dagger \) act on an \( L \)-th-dimensional Hilbert space, which we call \( \mathcal{H}_L \).

**Remark:** Assume, contrary to what was just stated, that the CCR \( [a, a^\dagger] = \mathbb{I} \) can be represented in a finite-dimensional Hilbert space \( \mathcal{H}_0 \). This implies that \( a, a^\dagger \) and
2.3 Other Possibilities

$I$ are finite matrices, with finite trace $tr$. But any trace satisfies, in particular, the following properties: $tr(A + B) = tr(A) + tr(B)$ and $tr(AB) = tr(BA)$, for all operators $A$ and $B$ on which the trace is defined.\(^2\) Hence we get $tr([a, a^\dagger]) = tr(aa^\dagger) - tr(a^\dagger a) = 0$, while $tr(I) = M$, with $M = dim(\mathcal{H}_0)$. This is clearly impossible, and shows that CCR cannot live in any finite-dimensional Hilbert space. On the other hand, this contradiction does not appear if we use (2.12). In fact, again we have $tr([B, B^\dagger]) = 0$, but we also find that $tr(I - LK) = tr(I) - Ltr(K) = L - L = 0$. This is because, see [38, 39], $tr(K) = 1$.

In [39] it is shown that the matrices for $B$ and $B^\dagger$ are essentially the truncated versions of the analogous, infinite-dimensional matrices for the bosonic annihilation and creation operators. In [39] it is also discussed how to construct an o.n. basis of eigenvectors of the self-adjoint operator $H_0 = \frac{1}{2} (Q^2_0 + P^2_0)$, where $Q_0 = \frac{B + B^\dagger}{\sqrt{2i}}$ and $P_0 = \frac{B - B^\dagger}{\sqrt{2i}}$ are the truncated position and momentum operators. These vectors turn out to be eigenvectors of both $H_0$ and $K$, and their explicit construction is strongly based on the fact that $H_0$ is a positive operator, other than being self-adjoint. Hence they are labeled by two quantum numbers. That’s why the operators $B$ and $B^\dagger$ still behave as ladder operators, but in a slightly more elaborated way [38, 39]. We refer to these papers for more details on this particular aspect. Here we just want to stress that, if $L = 2$, $B$ and $B^\dagger$ are two-by-two matrices, as for the fermionic case, while the size of the matrices representing $B$ and $B^\dagger$ increases with $L$, and in particular $B$ becomes an infinite matrix when $L$ diverges. In this sense, we can say that, roughly speaking, the truncated bosons in Equation (2.12) interpolate between fermions and bosons.

2.3.3 Pseudo-Versions of These Rules

All the commutation rules considered so far, from the CCR in Equation (2.1) to the CAR in Equation (2.4), passing for the $q$-mutator in Equation (2.7) and for the rule in Equation (2.12), can be deformed further in such a way that the raising operator, which is usually the adjoint of the lowering operator, is something different. For instance, $[a, a^\dagger] = I$ can be replaced by $[A, B] = I$, where $B \neq A^\dagger$. This, on some particular occasions, proved to be useful, since it turns out that many models originally proposed in the so-called PT or pseudo-Hermitian quantum mechanics [24, 25] can be rewritten in terms of these operators. We refer to [40] for a rather rich discussion on the so-called $\mathcal{D}$-pseudo-bosons, whose mathematics is somehow complicated due to the essential appearances of unbounded operators.

\(^2\) In this case, for all matrices $A$ and $B$ on $\mathcal{H}_0$. 

Deformed versions of quons and of truncated bosons have been proposed and analyzed quite recently [33, 41]. The relevant rules in these cases are, respectively,

\[ [A, B]_q = \mathbb{I}, \quad [C, D] = \mathbb{I} - L K, \]

where, in particular, \( B \neq A^\dagger \) and \( D \neq C^\dagger \), while \( q, L \) are as before, and \( K \) is again an orthogonal projector, \( K = K^\dagger = K^2 \), with \( KC = DK = 0 \). Our interest in these kind of operators in the context of this book is related to the fact that they can be used, in principle, in connection with Hamiltonians which are manifestly not self-adjoint. This is important since we can produce, in this way, some sort of irreversible dynamics which cannot be easily deduced with the original (i.e., canonical) forms of the commutation relations proposed before.

In the rest of this section we just focus on a few aspects of the simplest deformation, among those cited here. In particular, we will discuss what pseudo-fermions are, just to give the flavor of the kind of mathematics needed in their description.

The starting point in this section is a modification of the CAR (for a single mode) in (2.4), which is replaced here by the following rules:

\[ \{a, b\} = 1, \quad \{a, a\} = 0, \quad \{b, b\} = 0, \] (2.13)

where the interesting situation is when \( b \neq a^\dagger \). Since \( \det(a) = \det(b) = 0 \), \( a \) and \( b \) are not invertible. Therefore two nonzero vectors, \( \varphi_0 \) and \( \Psi_0 \), exist in \( \mathcal{H} = \mathbb{C}^2 \), the Hilbert space of our system, such that \( a \varphi_0 = 0 \) and \( b^\dagger \Psi_0 = 0 \). In general, \( \varphi_0 \neq \Psi_0 \).

We introduce further the following nonzero vectors

\[ \varphi_1 := b \varphi_0, \quad \Psi_1 = a^\dagger \Psi_0, \] (2.14)

as well as the non-self-adjoint operators

\[ N = ba, \quad N^\dagger = a^\dagger b^\dagger. \] (2.15)

The reason why \( \varphi_1 \) and \( \Psi_1 \) are surely nonzero vectors is because they satisfy, among other properties, the following equality: \( \langle \varphi_1, \Psi_1 \rangle = 1 \), see Equation (2.19). This would not be possible if one of these two vectors were zero. Of course, it makes no sense to consider \( b^n \varphi_0 \) or \( a^{\dagger n} \Psi_0 \) for \( n \geq 2 \), since all these vectors are automatically zero, due to Equation (2.13). This is analogous to what happens for ordinary fermions. Let now introduce the self-adjoint operators \( S_\varphi \) and \( S_\Psi \) via their action on a generic \( f \in \mathcal{H} \):

\[ S_\varphi f = \sum_{n=0}^{1} \langle \varphi_n, f \rangle \varphi_n, \quad S_\Psi f = \sum_{n=0}^{1} \langle \Psi_n, f \rangle \Psi_n. \] (2.16)
2.3 Other Possibilities

It is very easy to get the following results:

1. 
   \[ a \varphi_1 = \varphi_0, \quad b^\dagger \Psi_1 = \Psi_0. \]  
   \hspace{1cm} (2.17)

2. 
   \[ N \varphi_n = n \varphi_n, \quad N^\dagger \Psi_n = n \Psi_n, \]  
   \hspace{1cm} (2.18)

   for \( n = 0, 1 \).

3. If the normalizations of \( \varphi_0 \) and \( \Psi_0 \) are chosen in such a way that \( \langle \varphi_0, \Psi_0 \rangle = 1 \), then
   \[ \langle \varphi_k, \Psi_n \rangle = \delta_{k,n}, \]  
   \hspace{1cm} (2.19)

   for \( k, n = 0, 1 \).

4. \( S_\varphi \) and \( S_\Psi \) are bounded, strictly positive, self-adjoint and invertible. They satisfy
   \[ \| S_\varphi \| \leq \| \varphi_0 \|^2 + \| \varphi_1 \|^2, \quad \| S_\Psi \| \leq \| \Psi_0 \|^2 + \| \Psi_1 \|^2, \]  
   \hspace{1cm} (2.20)

   \[ S_\varphi \Psi_n = \varphi_n, \quad S_\Psi \varphi_n = \Psi_n, \]  
   \hspace{1cm} (2.21)

   for \( n = 0, 1 \), as well as \( S_\varphi = S_\Psi^{-1} \) and the following intertwining relations:
   \[ S_\Psi N = N^\dagger S_\varphi, \quad S_\varphi N^\dagger = NS_\Psi. \]  
   \hspace{1cm} (2.22)

The above formulas show that (i) \( N \) and \( N^\dagger \) behave as (non-Hermitian\(^3\)) fermionic number operators, having (real) eigenvalues 0 and 1; (ii) their related eigenvectors are respectively the vectors in \( \mathcal{F}_\varphi = \{ \varphi_0, \varphi_1 \} \) and \( \mathcal{F}_\Psi = \{ \Psi_0, \Psi_1 \} \); (iii) \( a \) and \( b^\dagger \) are lowering operators for \( \mathcal{F}_\varphi \) and \( \mathcal{F}_\Psi \) respectively; (iv) \( b \) and \( a^\dagger \) are raising operators for \( \mathcal{F}_\varphi \) and \( \mathcal{F}_\Psi \) respectively; (v) the two sets \( \mathcal{F}_\varphi \) and \( \mathcal{F}_\Psi \) are biorthonormal; (vi) the very well-behaved operators \( S_\varphi \) and \( S_\Psi \) map \( \mathcal{F}_\varphi \) into \( \mathcal{F}_\Psi \) and vice versa; (vii) \( S_\varphi \) and \( S_\Psi \) intertwine between the operators \( N \) and \( N^\dagger \), which are manifestly not self-adjoint.

As already claimed, \( a \) and \( b \), as well as similar ladder operators, can be used, quite efficiently, in the analysis of quantum systems driven by Hamiltonians which are not self-adjoint, as those appearing in connection with gain and loss systems. We refer to [40] for more details.

Remark: A rather general way to construct ladder operators uses, as its main ingredient, an o.n. basis \( \mathcal{F}_\epsilon = \{ \epsilon_n, \ n \geq 0 \} \) of the Hilbert space \( \tilde{\mathcal{H}} \) where the system is defined, and a sequence of nonnegative numbers: \( 0 \leq \epsilon_0 < \epsilon_1 < \epsilon_2 < \ldots \). Then we can define an operator \( c \) as follows:

\[ cf = \sum_{k=0}^{\infty} \epsilon_k \langle e_{k+1}, f \rangle e_k, \]

\(^3\) In this book Hermitian and self-adjoint will be used as synonyms.
for all \( f \) in the domain of \( c, D(c) \), i.e., for all those \( f \in \mathcal{H} \) for which this series is convergent. It is clear that \( c \) is a lowering operator and that \( e_k \in D(c) \) for all \( k: c e_k = \epsilon_{k-1} e_{k-1} \) for \( k \geq 1 \) with \( c e_0 = 0 \), while its adjoint, \( c^\dagger \), turns out to be a raising operator. Moreover, if \( \epsilon_k = k \), \( c \) and \( c^\dagger \) are just the bosonic annihilation and creation operators considered in Section 2.1, and a simple computation shows that \( [c, c^\dagger] = 1 \). However, if \( \epsilon_k \neq k \), \( c \) and \( c^\dagger \) are operators of some different nature. They are introduced in some applications of quantum optics, for instance in connection with the so-called nonlinear coherent states.

### 2.3.4 Simple Ladder Operators for Finite-Dimensional Hilbert Spaces

Ladder operators for finite-dimensional Hilbert spaces already appeared before, for fermions, in Section 2.2, for truncated bosons, in Section 2.3.2 and for their pseudo-versions, in Section 2.3.3. Fermions and pseudo-fermions live in two-dimensional Hilbert spaces, while truncated bosons and truncated pseudo-bosons can be defined in any finite-dimensional Hilbert space, with dimension equal or greater than two. The way in which these latter are introduced is not so simple, and one may wonder if other kind of ladder operators for finite-dimensional Hilbert spaces can be constructed in some simpler way. The answer is affirmative and this alternative strategy will be relevant, in particular in Chapter 7, where similar operators will be used in the construction of a model for tumor cell proliferation.

For concreteness, to show how our construction works, we will restrict to a five-dimensional Hilbert space \( \mathcal{H}_5 = \mathbb{C}^5 \). The same approach can be easily extended to \( \mathcal{H}_N \), for all finite \( N = 2, 3, 4, \ldots \).

Let \( \mathcal{E}_5 = \{ e_j, j = 0, 1, 2, 3, 4 \} \) be the canonical o.n. basis of \( \mathcal{H}_5 \). Each \( e_j \) is a five-dimensional vector with all zero entries, except the \((j + 1)\)-th entry, which is one. We define an operator \( b^\dagger \) via its action on the \( e_j \)'s:

\[
\begin{align*}
  b^\dagger e_0 &= e_1, & b^\dagger e_1 &= \sqrt{2} e_2, & b^\dagger e_2 &= \sqrt{3} e_3, & b^\dagger e_3 &= \sqrt{4} e_4, & b^\dagger e_4 &= 0.
\end{align*}
\]

It is clear that \( b^\dagger \) behaves as a sort of fermionic raising operator, destroying the upper level. The matrix expression for \( b^\dagger \) in the basis \( \mathcal{E}_5 \) is the following:

\[
\begin{pmatrix}
  0 & 0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 & 0 \\
  0 & \sqrt{2} & 0 & 0 & 0 \\
  0 & 0 & \sqrt{3} & 0 & 0 \\
  0 & 0 & 0 & \sqrt{4} & 0
\end{pmatrix}
\]
and the adjoint is

\[
b = \begin{pmatrix}
  0 & 1 & 0 & 0 & 0 \\
  0 & 0 & \sqrt{2} & 0 & 0 \\
  0 & 0 & 0 & \sqrt{3} & 0 \\
  0 & 0 & 0 & 0 & \sqrt{4} \\
  0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

These operators cannot satisfy the canonical commutation relation \([b, b^\dagger] = I_5\), \(I_5\) being the identity operator in \(H_5\), since this would be possible only in an infinite-dimensional Hilbert space. In fact, straightforward computations show that

\[
[b, b^\dagger] = I_5 - 5P_4,
\]

where \(P_4\) is the projection operator on \(e_4\): \(P_4 f = \langle e_4, f \rangle e_4\), for all \(f \in H_5\). Notice that \(I_5 - 5P_4\) is the following diagonal matrix: \(I_5 - 5P_4 = \text{diag}\{1, 1, 1, 1, -4\}\), which differs from the identity matrix on \(H_5\) only for the last component in its main diagonal.\(^4\) It is easy to check that \(b\) behaves as a lowering operator for \(E_5\):

\[be_0 = 0, \quad be_1 = e_0, \quad be_2 = \sqrt{2} e_1, \quad be_3 = \sqrt{3} e_2, \quad be_4 = \sqrt{4} e_3.\]  

\(\hat{N} = b^\dagger b = \text{diag}\{0, 1, 2, 3, 4\}\) is the number operator, and the following eigenvalue equation holds:

\[\hat{N} e_k = k e_k,\]

\(k = 0, 1, 2, 3, 4\). It is further easy to see that \(b^5 = (b^\dagger)^5 = 0\), which is similar to the analogous equation for fermionic ladder operators (with 5 replaced by 2).

Remarks: (1) To extend the construction to \(N \neq 5\) it is sufficient to consider the canonical o.n. basis for \(H_N, E_N\), and then use its vectors to define \(b^\dagger\) in analogy with Equation (2.23). Then \(b\) is just the adjoint of \(b^\dagger\). These are ladder operators such that \(b e_0 = 0\) and \(b^\dagger e_N = 0\). Together they produce \(\hat{N}\) as above, \(\hat{N} = b^\dagger b\).

(2) If \(N = 2\), (2.24) should be replaced by \([b, b^\dagger] = I_2 - 2P_1 = \text{diag}\{1, -1\}\). Also \(b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) and \(b^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\), which is in agreement with the well known expressions for the fermionic ladder operators.

In view of the application we are interested in, see Chapter 7, we need to construct three different families of ladder operators of this kind, one for each agent of the biological model we will describe there, and then put all these ingredients together, by working on a suitable tensor product Hilbert space. More in details, and adopting the same notation which will be used in Chapter 7, the agents of the system \(S\) we want to describe are the healthy cells, attached to the ladder operators \(h\) and \(h^\dagger\), living in a Hilbert space \(H_h\). Then we have the sick cells, described in

\(^4\) Notice that (2.24) is a particular realization of (2.12).
Some Preliminaries

terms of the operators $s$ and $s^\dagger$, which are defined on a second Hilbert space $\mathcal{H}_s$, and the medical treatment, related to $m$ and $m^\dagger$, acting on a third Hilbert space $\mathcal{H}_m$.

We call $N_{\alpha} = \dim(\mathcal{H}_\alpha)$, where $\alpha = h, s, m$, and $\mathcal{E}_\alpha = \{ e_{j}^{(\alpha)}, j = 0, 1, 2, \ldots, N_{\alpha} - 1 \}$ the o.n. basis of $\mathcal{H}_\alpha$. The operators $h$, $s$ and $m$ satisfy relations which extend those in Equations (2.23) and (2.25). First of all we have

$$h e_0^{(h)} = 0, \quad s e_0^{(s)} = 0, \quad m e_0^{(m)} = 0,$$

and then

$$e_1^{(h)} = h^\dagger e_0^{(h)}, \quad e_2^{(h)} = \frac{1}{\sqrt{2}} h^\dagger e_1^{(h)}, \ldots, e_{N_h-1}^{(h)} = \frac{1}{\sqrt{N_h - 1}} h^\dagger e_{N_h-2}^{(h)},$$

$$e_1^{(s)} = s^\dagger e_0^{(s)}, \quad e_2^{(s)} = \frac{1}{\sqrt{2}} s^\dagger e_1^{(s)}, \ldots, e_{N_s-1}^{(s)} = \frac{1}{\sqrt{N_s - 1}} s^\dagger e_{N_s-2}^{(s)},$$

$$e_1^{(m)} = m^\dagger e_0^{(m)}, \quad e_2^{(m)} = \frac{1}{\sqrt{2}} m^\dagger e_1^{(m)}, \ldots, e_{N_m-1}^{(m)} = \frac{1}{\sqrt{N_m - 1}} m^\dagger e_{N_m-2}^{(m)}.$$

Finally, we have that

$$h^\dagger e_{N_h-1}^{(h)} = 0, \quad s^\dagger e_{N_s-1}^{(s)} = 0, \quad m^\dagger e_{N_m-1}^{(m)} = 0.$$

The Hilbert space of our full system is now the tensor product $\mathcal{H} = \mathcal{H}_h \otimes \mathcal{H}_s \otimes \mathcal{H}_m$, whose dimension is clearly $N = N_h \times N_s \times N_m$. Calling $\mathbb{I}_\alpha$, the identity operator on $\mathcal{H}_\alpha$, each operator $X_h$ on $\mathcal{H}_h$ is identified with the following tensor product $\mathbb{I}_h \otimes X_s \otimes X_m$. Analogously, the operators $X_s$ and $X_m$ on $\mathcal{H}_s$ and $\mathcal{H}_m$ should be identified respectively with $\mathbb{I}_h \otimes \mathbb{I}_s \otimes X_m$ and with $\mathbb{I}_h \otimes X_s \otimes m$, both acting on $\mathcal{H}$.

Furthermore,

$$(X_h \otimes X_s \otimes X_m) (f_h \otimes f_s \otimes f_m) = (X_h f_h) \otimes (X_s f_s) \otimes (X_m f_m),$$

for all $f_h \in \mathcal{H}_h, f_s \in \mathcal{H}_s$ and $f_m \in \mathcal{H}_m$. From now on, when no confusion arises, we will just write $X_h, X_s, X_m$ instead of $X_h \otimes \mathbb{I}_s \otimes \mathbb{I}_m, \mathbb{I}_h \otimes X_s \otimes \mathbb{I}_m$ and $\mathbb{I}_h \otimes \mathbb{I}_s \otimes X_m$, and their action is obviously meant on the whole $\mathcal{H}$.

An o.n. basis for $\mathcal{H}$ is the following:

$$\mathcal{E} = \{ \varphi_{n_h, n_s, n_m} = e_{n_h}^{(h)} \otimes e_{n_s}^{(s)} \otimes e_{n_m}^{(m)}, n_{\alpha} = 0, 1, \ldots, N_{\alpha} - 1, \alpha = h, s, m \},$$

(2.27)

so that any vector $\Psi$ of $\mathcal{H}$ can be expressed as a combination of these vectors:

$$\Psi = \sum_{n_h, n_s, n_m} c_{n_h, n_s, n_m} \varphi_{n_h, n_s, n_m}.$$  

(2.28)

Here the sum is extended to all the possible values of $n_h, n_s$ and $n_m$, which of course depend on our choice of $N_h, N_s$ and $N_m$, and $c_{n_h, n_s, n_m}$ are complex scalars.
which can be deduced from $\Psi$ as follows: $c_{n_h,n_s,n_m} = \langle \varphi_{n_h,n_s,n_m}, \Psi \rangle$. Here $\langle \ldots \rangle$ is the scalar product in $\mathcal{H}$.

**Remark:** In our particular application, see Chapter 7, each vector in $\mathcal{E}$ has a clear meaning: $\varphi_{k,0,0}$, with $k > 0$, describes a situation in which the system consists only of $k$ healthy cells, with no sick cell and with no active medical treatment, whereas $\varphi_{n,3n,1}$, with $n > 0$, represents a state in which the sick cells are three times the number of the healthy ones, and a medical treatment is acting on the system.

A simple computation shows that all the ladder operators of the different agents commute. For instance, using the properties of the tensor product, we see that

$$[h,m]\varphi_{n_h,n_s,n_m} = (h e_{n_h}^{(h)} \otimes e_{n_s}^{(s)} \otimes m e_{n_m}^{(m)}) - (h e_{n_h}^{(h)} \otimes e_{n_s}^{(s)} \otimes m e_{n_m}^{(m)}) = 0.$$ 

Of course, this implies that all the operators on $\mathcal{H}_\alpha$ commute with all the operators acting on $\mathcal{H}_\beta$, if $\alpha \neq \beta$: $[X_\alpha, Y_\beta] = 0$, $\alpha, \beta = h, s, m$ and $\alpha \neq \beta$.

The number operators can be introduced as usual:

$$\hat{N}_h = h^\dagger h, \quad \hat{N}_s = s^\dagger s, \quad \hat{N}_m = m^\dagger m,$$

and they act on the elements of $\mathcal{E}$ as follows:

$$\hat{N}_h \varphi_{n_h,n_s,n_m} = n_h \varphi_{n_h,n_s,n_m}, \quad \hat{N}_s \varphi_{n_h,n_s,n_m} = n_s \varphi_{n_h,n_s,n_m}, \quad \hat{N}_m \varphi_{n_h,n_s,n_m} = n_m \varphi_{n_h,n_s,n_m}.$$ 

Then we conclude that the same general settings discussed for bosons and fermions are also recovered here in each $\mathcal{H}_N$, $N = 2, 3, 4, \ldots, N < \infty$, except, at most, for the commutation relations between ladder operators.\(^5\) We also want to stress that the procedure proposed in this section is by far simpler than that in Section 2.3.2, where we have also worked in $\mathcal{H}_N$. The operators $h$, $s$ and $m$ will be the main ingredient of our analysis in Section 7.2.

The conclusion of this long section is that bosons and fermions are not the only classes of operators which can be used in the analysis of those physical systems for which ladder operators are relevant (which are many!); not surprisingly, the use of a particular class, rather than another, is usually connected with the system we want to describe, and with the explicit interpretation we can give to these operators.

### 2.4 Dynamics for a Quantum System

So far, we have discussed some algebraic and functional properties of quantities (e.g., operators, vectors, bases) which can be useful in the analysis of quantum,
or quantum-like, systems. Now we briefly discuss how the time evolution of these
systems are usually described in the literature. We refer to [2, 3] for more
details and other possibilities.

Let $S$ be a closed quantum system. This means that $S$ does not interact with any
external reservoir (this is why, it is closed) and that its size is comparable with that
of, say, the hydrogen atom: $S$ is a microscopic system (so, it is a quantum system).

Remark: It is useful to observe that, if $S$ also interacts with another system $S_E$,
of the same size, then we can always consider the union $S_{\text{full}} = S \cup S_E$ as a larger
closed system, again microscopic, for which all the general ideas and results we are
going to describe in the next few pages apply. If the size of $S_E$ is much larger than
that of $S$, then we will say that $S$ is an open quantum system. We will discuss the
dynamics for systems like these in Section 2.7.

2.4.1 Schrödinger Representation

In this section we will briefly describe how to find the time evolution of $S$, starting
from a simple (and natural) assumption: in a closed system, the total probability
is preserved. The physical interpretation of this is clear: suppose, to be concrete,
that $S$ is a particle. It is clear that the particle, at $t = t_0$, should be somewhere in
the space. Then, recalling the probabilistic interpretation of the wave-function in
quantum mechanics, we must have $\|\Psi(t_0)\|^2 = \int_{\mathbb{R}^3} |\Psi(r, t_0)|^2 d^3r = 1$. If the particle
is not destroyed, during its time evolution (i.e., for $t \geq t_0$) it can change position in
space, but, again, it should be found somewhere. Therefore, the total probability
should stay constantly equal to one: $\|\Psi(t)\|^2 = \int_{\mathbb{R}^3} |\Psi(r, t)|^2 d^3r = 1$, for all $t \geq t_0$.
This is guaranteed if the time evolution of the particle is described by a unitary
operator, which maps $\Psi(r, t_0)$ into $\Psi(r, t)$:

$$\Psi(r, t) = U(t, t_0)\Psi(r, t_0),$$

where $U^{-1}(t, t_0) = U^\dagger(t, t_0)$. In this way, in fact, the $L^2(\mathbb{R}^3)$-norm of $\Psi$ is
preserved for all $t$: $\|\Psi(t)\| = \|\Psi(t_0)\|$. From now on, when not explicitly needed, we
will not write explicitly the dependence of $\Psi$ from other variables, since this is not
relevant for what we are discussing here. In particular, we will not write explicitly
the dependence of $\Psi$ on the space variable $r$.

Going back to the operator $U(t, t_0)$, it is clear that $U(t_0, t_0) = \mathbb{I}$, and that
$U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0)$, for all $t_0 \leq t_1 \leq t_2$. Then, if we take in particular $t_0 = t_2$,

\footnote{The relevant Hilbert space, in this context, is $L^2(\mathbb{R}^3)$, with its natural scalar product $\langle f, g \rangle = \int_{\mathbb{R}^3} \overline{f(\vec{r})}g(\vec{r})d^3\vec{r}$, for all $f, g \in L^2(\mathbb{R}^3)$ and with the norm induced by this: $\|f\|^2 = \langle f, f \rangle$. The wave function $\Psi(\vec{r}, t)$ is assumed, of course, to depend on the spatial variables $\vec{r} = (x, y, z)$ and on time, $t$. Hence the norm of $\Psi$ depends, in general, on $t$.}
we deduce that \( U^{-1}(t_1, t_0) = U(t_0, t_1) \), for all \( t_0 \) and \( t_1 \): hence the inverse of \( U(t_0, t_1) \) coincides with \( U(t_0, t_1) \) itself, but with \( t_0 \) and \( t_1 \) exchanged.

It is now easy to deduce the differential equation for \( \Psi(t) \). This is the well-known Schrödinger equation

\[
    i \frac{\partial \Psi(t)}{\partial t} = H(t) \Psi(t), \tag{2.32}
\]

where \( H(t) \) is a self-adjoint operator, the Hamiltonian of \( \mathcal{S} \), \( H(t) = \mathcal{H}^\dagger(t) \), which can be, in general, explicitly dependent on time. This point of view is known as Schrödinger representation: the wave function depends on time, while the observables, in general, do not.\(^7\) To show how Equation (2.32) follows from Equation (2.31) we compute first

\[
    \frac{\partial \Psi(t)}{\partial t} = \frac{\partial U(t, t_0)}{\partial t} \Psi(t_0) = \frac{\partial U(t, t_0)}{\partial t} U^{-1}(t, t_0) \Psi(t) = \left[ \frac{\partial U(t, t_0)}{\partial t} U^\dagger(t, t_0) \right] \Psi(t),
\]

which can be written as in Equation (2.32) defining

\[
    H(t) = i \frac{\partial U(t, t_0)}{\partial t} U^\dagger(t, t_0). \tag{2.33}
\]

The fact that \( H(t) \) is self-adjoint and that it does not depend on \( t_0 \) is not evident. However, both these statements are easy to prove. First we observe that

\[
    H^\dagger(t) = -i U(t, t_0) \frac{\partial U^\dagger(t, t_0)}{\partial t}.
\]

Now, since \( U(t, t_0)U^\dagger(t, t_0) = \mathbb{I} \), we have \( \frac{\partial U(t, t_0)U^\dagger(t, t_0)}{\partial t} = 0 \). Hence

\[
    \frac{\partial U(t, t_0)}{\partial t} U^\dagger(t, t_0) + U(t, t_0) \frac{\partial U^\dagger(t, t_0)}{\partial t} = 0.
\]

Therefore

\[
    H^\dagger(t) = -i U(t, t_0) \frac{\partial U^\dagger(t, t_0)}{\partial t} = (-1)^2 i \frac{\partial U(t, t_0)}{\partial t} U^\dagger(t, t_0) = H(t),
\]

which proves our first claim. To check now that \( H(t) \) is independent of \( t_0 \) we observe that, since \( U(t_0, t_1)U(t_1, t_0) = \mathbb{I} \) for all \( t_0 \) and \( t_1 \),

\[
    \frac{\partial U(t, t_0)}{\partial t} U^\dagger(t, t_0) = \frac{\partial U(t, t_0)}{\partial t} (U(t_0, t_1)U(t_1, t_0)) U^\dagger(t, t_0)
\]

\[
    = \frac{\partial (U(t, t_0)U(t_0, t_1))}{\partial t} (U(t_1, t_0)U(t_0, t)) = \frac{\partial U(t, t_1)}{\partial t} U^\dagger(t, t_1).
\]

\(^7\) The time dependence which may be present in this case is, for instance, the one arising because some part of \( \mathcal{S} \) is driven by external forces depending explicitly on \( t \).
Hence we have

\[ H(t) = i \frac{\partial U(t, t_0)}{\partial t} U^\dagger(t, t_0) = i \frac{\partial U(t, t_1)}{\partial t} U^\dagger(t, t_1), \]

so that our second assertion follows. Incidentally, this is the reason why in Equation (2.33) we have used \( H(t) \) rather than \( H(t, t_0) \).

**Remark:** It is interesting to show that the above result can be somehow inverted: we have shown that (2.32) follows from (2.31) and by the unitarity of \( U(t, t_0) \). It is now easy to show that Equation (2.32) implies the equality \( \| \Psi(t) \| = \| \Psi(t_0) \| \), for all \( t \geq t_0 \). In fact,

\[
\frac{d}{dt} ||\Psi(t)||^2 = \frac{d}{dt} \langle \Psi(t), \Psi(t) \rangle = \langle \Psi(t), \Psi(t) \rangle + \langle \Psi(t), \dot{\Psi}(t) \rangle = \langle -i H(t) \Psi(t), \Psi(t) \rangle
\]

\[
+ \langle \Psi(t), -i H(t) \dot{\Psi}(t) \rangle = i \langle H(t) \Psi(t), \Psi(t) \rangle - i \langle \Psi(t), H(t) \dot{\Psi}(t) \rangle = 0,
\]

since \( H(t) = H^\dagger(t) \). It is useful to stress that this result holds independently of the fact that \( H \) depends or not explicitly on time.

We have seen that \( \Psi(t) \) and \( H(t) \) can both be derived, in principle, from the operators \( U(t, t_0) \) as shown in Equations (2.31) and (2.33). Reversing the procedure, \( U(t, t_0) \) can be deduced from \( H(t) \). The explicit result, however, is deeply linked to the time dependence of \( H \), as we show now. First we rewrite Equation (2.32) as

\[
i \frac{\partial U(t, t_0)}{\partial t} \Psi(t_0) = H(t) U(t, t_0) \Psi(t_0),
\]

which should be satisfied for all possible initial states \( \Psi(t_0) \). This means that the equation for \( U(t, t_0) \) is

\[
i \frac{\partial U(t, t_0)}{\partial t} = H(t) U(t, t_0).
\]

This equation can be solved easily if \( H(t) = H \), i.e., if the Hamiltonian of \( \mathcal{S} \) does not depend explicitly on time. In this case, in fact, the solution of (2.34) is

\[
U(t, t_0) = e^{-iH(t-t_0)}.
\]

**Remark:** When we say that the operator \( e^{-iH(t-t_0)} \) is a solution of Equation (2.34) we are, in fact, a bit optimistic. The reason is the following: from one side, this operator, when replaced in (2.34), really produces an identity. Hence it is really, in mathematical terms, a solution of the differential equation. However, the technical computation of \( e^{-iH(t-t_0)} \) can be quite complicated, since we are trying to compute the exponential of some operator. Of course, if \( H = H^\dagger \), it is enough to use the spectral theorem [27], to compute this exponential, by means of the spectral family of \( H \). So, again, from a mathematical side we have no problem at all to prove the existence of \( U(t, t_0) \), which automatically turns out to be unitary, as it should. However, finding a useful expression (for computing things!) for \( e^{-iH(t-t_0)} \) can be a different story. And
2.4 Dynamics for a Quantum System

this is particularly hard for those physical systems whose dynamics can be deduced out of some (physically motivated) non-self-adjoint Hamiltonian, \( H \neq H^\dagger \), as it happens for some systems in Quantum Optics, for instance. This problem is particularly complicated if \( H \) is an operator on an infinite-dimensional Hilbert space. But it can also be a hard problem when \( H \) is a finite matrix, if its dimensionality is not small enough. This, of course, depends on the system we are analyzing.

Equation (2.34) can also be solved when \( H(t) \) depends explicitly on time, but \( H(t_1) \) commutes with \( H(t_2) \): \([H(t_1), H(t_2)] = 0\). This is the case, for instance, of a driven harmonic oscillator with Hamiltonian

\[
H = \omega a^\dagger a + \frac{1}{2} \mathbb{1} + \lambda (a + a^\dagger)V(t),
\]

where \([a, a^\dagger] = \mathbb{1}\), \( \omega \) and \( \lambda \) are real constants, and \( V(t) \) is a real valued force. In this case Equation (2.35) must be replaced by

\[
U(t, t_0) = \exp \left\{ -i \int_{t_0}^{t} H(t') dt' \right\},
\]  

(2.36)

which, in particular, returns Equation (2.35) if \( H(t) = H \). To prove that (2.36) solves (2.34) we first notice that, calling \( X(t, t_0) = -i \int_{t_0}^{t} H(t') dt' \), then \( \dot{X}(t, t_0) = -iH(t) \). Then we have

\[
[X(t, t_0), X(t', t_0)] = (-i)^2 \left[ H(t), \int_{t_0}^{t} H(t') dt' \right] = (-i)^2 \int_{t_0}^{t} [H(t), H(t')] dt' = 0.
\]

Using this commutativity it is a matter of simple computations to show that

\[
\frac{d}{dt} e^{X(t, t_0)} = \dot{X}(t, t_0) e^{X(t, t_0)} = e^{X(t, t_0)} \dot{X}(t, t_0),
\]

(2.37)

which can be written as

\[
\frac{d}{dt} e^{X(t, t_0)} = -iH(t) e^{X(t, t_0)}.
\]

Also, since \( X(t_0, t_0) = -i \int_{t_0}^{t_0} H(t') dt' = 0 \), \( e^{X(t_0, t_0)} = \mathbb{1} \), which coincides with \( U(t_0, t_0) \). Therefore \( U(t, t_0) \) satisfies the same differential equation as \( e^{X(t, t_0)} \), with the same initial condition. Hence (assuming that the Cauchy’s theorem for ordinary differential equations can be applied in our context),

\[
U(t, t_0) = e^{X(t, t_0)} = e^{-i \int_{t_0}^{t} H(t') dt'},
\]

which is exactly what we had to show, see Equation (2.36).
The situation is by far more complicated if \([H(t_1), H(t_2)] \neq 0\), for \(t_1 \neq t_2\). In this case we should introduce the operator of time ordering \(T\). It is probably more convenient to rewrite Equation (2.34) in its integral form, recalling that \(U(t_0, t_0) = 1\):

\[
U(t, t_0) = \mathbb{I} - i \int_{t_0}^{t} H(t_1) \, U(t_1, t_0) \, dt_1
\]

(2.38)

This equation gives rise to a perturbative series:

\[
U(t, t_0) = \mathbb{I} - i \int_{t_0}^{t} H(t_1) \, dt_1 + (-i)^2 \int_{t_0}^{t} \int_{t_0}^{t_1} H(t_1) H(t_2) \, dt_2 \, dt_1 + \cdots,
\]

(2.39)

where the order in which the Hamiltonian \(H(t_j)\) appears is essential, and where \(t_0 < t_1 < t_2 < \ldots < t\). It is known, see [42] for instance, that \(U(t, t_0)\) can be rewritten in the following form:

\[
U(t, t_0) = T \exp \left\{ -i \int_{t_0}^{t} H(t_1) \, dt_1 \right\},
\]

(2.40)

where \(T\) is the Dyson time-ordering operator satisfying the following:

\[
T(A(t_1)B(t_2)) = \begin{cases} 
A(t_1)B(t_2), & \text{if } t_1 < t_2 \\
B(t_2)A(t_1), & \text{if } t_2 < t_1
\end{cases}
\]

Of course, \(T\) does not modify the order of the operators \(A(t_1)\) and \(B(t_2)\) when they commute, as it was assumed for \(H(t)\) in the derivation of Equation (2.36). In fact, when this happens, Equation (2.40) coincides with Equation (2.36). In general, however, they are different and Equation (2.40) is only a formal formula which should be taken as the starting point for a perturbative expansion like the one in Equation (2.39). But this goes (much) beyond the scope of this book.

### 2.4.2 Heisenberg Representation

The possibility of changing representation in quantum mechanics is based on the fact that what is usually relevant for us is not really the time evolution of the state of \(S\), or the time evolution of its observables, but only the mean values of the observables in the state of \(S\). This is because what is usually measured in experiments are exactly these mean values, and not the operators themselves.

In what follows it is convenient to introduce a suffix to distinguish between the Schrödinger and Heisenberg representations\(^8\): in particular we use \(\Psi_S\) and \(X_S\) to indicate the state \(\Psi\) and the observable \(X\) in the Schrödinger representation, while

\(^8\) This suffix will not be used in the rest of the book to avoid useless complications in the notation.
we adopt $\Psi_H$ and $X_H$ for the same objects in the Heisenberg representation. The link between the two representations is provided by the following equality:

$$\langle \Psi_S(t), A_S \Psi_S(t) \rangle = \langle \Psi_H, A_H(t) \Psi_H \rangle, \quad (2.41)$$

where it is explicitly shown that the wave function depends on time in the Schrödinger but not in the Heisenberg representation, and that, vice versa, the observables depend on time in the Heisenberg but not in the Schrödinger representation. From Equation (2.41) and from Equation (2.31), with $\Psi$ identified with $\Psi_S$, it follows that (putting for simplicity $t_0 = 0$),

$$A_H(t) = U^\dagger(t, 0) A_S U(t, 0). \quad (2.42)$$

It is clear that, in particular, $A_H(0) = A_S$ and that $1_H = 1_S$. Moreover, $A_H(t) = A_S$ if $A_S$ commutes with $U(t, 0)$. Also, if $A$ and $B$ are two observables, calling $C_S = A_S B_S$ their product in the Schrödinger representation, then $C_H(t) = A_H(t) B_H(t)$.

**Remark:** This last result is true since $U(t, 0)$ is unitary, which is the case when $H$ (or $H(t)$) is self-adjoint. However, as already stated in Section 2.3.3, this is not always the case: in some applications it can be convenient to adopt an effective, non-self-adjoint, Hamiltonian. In this case, it is not granted that $C_H(t) = A_H(t) B_H(t)$. This is because $A_H(t) B_H(t) = U^\dagger(t, 0) A_S U(t, 0) U^\dagger(t, 0) B_S U(t, 0) \neq U^\dagger(t, 0) A_S B_S U(t, 0)$, since $U(t, 0) U^\dagger(t, 0)$ does not need to be equal to the identity operator. In other words, $U(t, 0)$ is unitary if $H$ is self-adjoint, but not in general. This is particularly evident from, say, Equation (2.35) or from Equation (2.36). However, this kind of problems will not appear often in this book, except in a few points, where we will say more on these aspects.

Going back to the relation between the Schrödinger and the Heisenberg representation, we observe that, if $C_S = [A_S, B_S]$, then $C_H(t) = [A_H(t), B_H(t)]$.

It is interesting to notice that, if $[H(t_1), H(t_2)] = 0$, then the Hamiltonian $H$ is the same in both representations, even in presence of an explicit time dependence:

$$H_H(t) = U^\dagger(t, 0) H_S(t) U(t, 0) = U^\dagger(t, 0) U(t, 0) H_S(t) = H_S(t),$$

using Equation (2.37). The reason why we are using here $H_S(t)$ is because we are also admitting the possibility that $H_S$ has an explicit time dependence, as it happens when the system $S$ is driven also by some external, time-dependent, classical field.

The next step consists in finding the differential equation for $A_H(t)$. For that we observe that, taking the adjoint of Equation (2.34), we get

$$-i \frac{\partial U^\dagger(t, t_0)}{\partial t} = U^\dagger(t, t_0) H_S(t),$$
so that, after a few simple computations,

\[ i\dot{A}_H(t) = [A_H(t), H_H(t)] + i \left( \frac{\partial A_S(t)}{\partial t} \right)_{H}, \]  

(2.43)

where

\[ \left( \frac{\partial A_S(t)}{\partial t} \right)_{H} = U^\dagger(t, t_0) \left( \frac{\partial A_S(t)}{\partial t} \right) U(t, t_0). \]

Of course, this term disappears if \( A_S(t) \) does not depend explicitly on time, i.e., if \( A_S(t) = A_S \). In this case, in particular, if \( A_S \) commutes with \( H_S \), \( [A_S, H_S] = 0 \), then, for what we have already seen, \( [A_H(t), H_H(t)] = 0 \) as well, and Equation (2.43) implies that \( A_H(t) = A_H(0) = A_S \). When this happens, the operator \( A_S \) is called a constant or integral of motion. The existence of integrals of motion for a given system \( S \) can be useful in the analysis of \( S \) and its dynamics. A particularly illuminating example of this relevance is discussed in [9].

Remark: It is well known that other quantum mechanical representations also exist, as, for instance, the interaction representation. Since this representation will not be useful for us here, we refer to specialized books in quantum mechanics for its detailed description, see [2, 3, 31] for instance.

### 2.5 Heisenberg Uncertainty Principle

Due to its relevance in ordinary quantum mechanics and in decision-making, we give here the mathematical details of the derivation of the Heisenberg uncertainty principle. This is useful in connection with what discussed in Chapter 12, where we will consider its role in the analysis of compatible and incompatible questions, a very hot topic in decision-making.

Consider two self-adjoint, possibly non-commuting, operators, \( A \) and \( B \), acting on the Hilbert space \( \mathcal{H} \): \( A = A^\dagger \), \( B = B^\dagger \), and let us assume that \( i C \) is the commutator between \( A \) and \( B \): \( [A, B] = i C \). It is easy to show that, since \( A = A^\dagger \) and \( B = B^\dagger \), the operator \( C \) must necessarily be self-adjoint as well. In fact, taking the adjoint of both sides of the equality \( [A, B] = i C \), we easily find that \( [A, B] = i C^\dagger \) as well, and the two equalities are compatible only if \( C = C^\dagger \).

Remark: Sometimes the operators \( A \) and \( B \) are unbounded. This is the case, for instance, for the position and the momentum operators. In this case the rule \( [A, B] = i C \) must be supplemented with information on the domains of the operators \( A, B \) and \( C \), but not only. In fact, in order to give a rigorous meaning to this equality we should work, for instance, on some \( \mathcal{D} \), dense in \( \mathcal{H} \), such that \( \mathcal{D} \subseteq D(AB) \cap D(BA) \cap D(C) \). If such a \( \mathcal{D} \) exists, then the commutation rule should be understood as follows: \( ABf - BAf = i Cf \), for all \( f \in \mathcal{D} \).
Let now $\varphi \in \mathcal{H}$ be a fixed vector and $<X>:=\langle \varphi, X \varphi \rangle$ the expectation value of the operator $X$ on the vector $\varphi$. If $X$ is a bounded operator, $<X>$ is well-defined for all $\varphi \in \mathcal{H}$. But, if $X$ is unbounded, $\varphi$ must be taken in the domain of $X$, $D(X)$. In particular we assume here that $\varphi \in D(A^2) \cap D(B^2)$, since this guarantees that $\varphi \in D(A) \cap D(B)$ as well.\footnote{This is because, for instance, $D(A^2) = \{ \phi \in D(A) : A\phi \in D(A) \}$. Of course, if $A$ is bounded, $A \in B(\mathcal{H})$, $D(A) = D(A^2) = \mathcal{H}$.} Let us further define $(\Delta A)^2 := \langle \varphi, (A-<A>)^2 \varphi \rangle = \langle A^2 \rangle - \langle A \rangle^2$. Analogously we have $(\Delta B)^2 = \langle B^2 \rangle - \langle B \rangle^2$. Then, since each self-adjoint operator $X = X^\dagger$ has an expectation value $<X>$ which is real, we can rewrite

$$(\Delta X)^2 = \langle \varphi, (X-<X>)^2 \varphi \rangle = \| (X-<X>) \varphi \|^2. \quad (2.44)$$

Notice that both $<X>$ and $\Delta X$ depend on $\varphi$, but we are not making explicit this dependence here, to simplify the notation. $\Delta X$ is called the uncertainty on $X$\footnote{For all $f, g \in \mathcal{H}$, the Schwartz (or Cauchy-Schwarz) inequality states that $|\langle f, g \rangle| \leq \|f\| \|g\|$.}, and measures how well we really know the value of the observable $X$. In fact, from (2.44) we conclude that $\Delta X = 0$ if and only if $(X-<X>)\varphi = 0$, which means that $\varphi$ must be an eigenstate of $X$ with eigenvalue $<X>$. On the other hand, if $\Delta X > 0$, then $\varphi$ cannot be an eigenstate of $X$. If we are measuring several times the observable $X$, then $<X>$ gives the mean value of these results, while $\Delta X$ is proportional to the width of the distribution of these results, see Figure 2.1. Of course, the smaller $\Delta X$, the better our knowledge of $X$.

Now, using the Schwartz inequality, we get

$$|\langle (A-<A>) \varphi, (B-<B>) \varphi \rangle| \leq \| (A-<A>) \varphi \| \| (B-<B>) \varphi \| = (\Delta A)(\Delta B).$$

![Figure 2.1](image-url) Black line: large $\Delta X$; Dotted line: medium $\Delta X$; Dashed line: small $\Delta X$. \footnote{This is because, for instance, $D(A^2) = \{ \phi \in D(A) : A\phi \in D(A) \}$. Of course, if $A$ is bounded, $A \in B(\mathcal{H})$, $D(A) = D(A^2) = \mathcal{H}$.}
Moreover, using the properties of $A$ and $\langle A \rangle$, and in particular the fact that $A = A^\dagger$ and that $\langle A \rangle$ is real,
\[
\langle (A - \langle A \rangle)\varphi, (B - \langle B \rangle)\varphi \rangle = \langle \varphi, (A - \langle A \rangle)(B - \langle B \rangle)\varphi \rangle.
\]
For our purposes it is now useful to write
\[
(A - \langle A \rangle)(B - \langle B \rangle) = \left(\frac{(A - \langle A \rangle)(B - \langle B \rangle) + (B - \langle B \rangle)(A - \langle A \rangle)}{2}\right) + \left(\frac{(A - \langle A \rangle)(B - \langle B \rangle) - (B - \langle B \rangle)(A - \langle A \rangle)}{2}\right) = F + i\frac{C}{2},
\]
where we have defined a new self-adjoint operator
\[
F := \frac{1}{2} [(A - \langle A \rangle)(B - \langle B \rangle) + (B - \langle B \rangle)(A - \langle A \rangle)],
\]
$F = F^\dagger$, and we have observed that
\[
(A - \langle A \rangle)(B - \langle B \rangle) - (B - \langle B \rangle)(A - \langle A \rangle) = AB - BA = iC.
\]
Summarizing, we have
\[
(\Delta A)^2(\Delta B)^2 \geq |\langle (A - \langle A \rangle)\varphi, (B - \langle B \rangle)\varphi \rangle|^2 = \left|\langle \varphi, \left( F + i\frac{C}{2} \right)\varphi \rangle \right|^2 = \left|\langle F > + i\frac{C}{2} < C > \right|^2.
\]
Taking now into account the fact that both $\langle F \rangle$ and $\langle C \rangle$ are real quantities, we conclude that
\[
(\Delta A)^2(\Delta B)^2 \geq \left|\langle F > + i\frac{C}{2} < C > \right|^2 = \langle F \rangle^2 + \frac{C^2}{4} \geq \langle C \rangle^2,
\]
so that
\[
\Delta A \Delta B \geq \frac{|\langle C \rangle|}{2}, \quad (2.45)
\]
which gives a lower bound on the product of the uncertainties on $A$ and $B$: whenever $A$ and $B$ do not commute, they cannot be known together with arbitrary precision! For instance, if we consider the position and the self-adjoint momentum operators $x$ and $p$, and we take $A = x, B = p$ and $C = \mathbb{I}$, we go back to the well-known Heisenberg inequality: the commutator becomes $[x, p] = i\mathbb{I}$, while the inequality in (2.45) takes the following well-known form: $\Delta x \Delta p \geq \frac{1}{2}$. Notice that we are working here in suitable units, taking $\hbar = 1$. Incidentally we observe that, from Equation (2.45), we get the obvious inequality $0 \geq 0$ if $\varphi$ is an eigenstate of $A$ or of $B$. Notice also that $\varphi$ can be simultaneously an eigenstate of both $A$ and $B$ if $[A, B] = 0$, i.e., if $C = 0$, but not in general. When this happens we have $\Delta A = \Delta B = 0$, which means that both $A$ and $B$ can be measured exactly. Otherwise, i.e., if $\varphi$ is an eigenstate
of, say, $A$, but not of $B$, only $A$ can be measured with no error ($\Delta A = 0$), while $B$ cannot. Still, as we have already noticed, Equation (2.45) holds in the trivial form $0 \geq 0$.

We have recently used commutativity between observables as an efficient way to describe compatible or incompatible questions. For instance, if $A$ and $B$ are two self-adjoint operators connected to two different questions (e.g., $A$ can be related to the question “is Alice happy?” and $B$ to the other question “is Alice employed?”), both with just two answers, yes or not; therefore $A$ and $B$ can be represented as two spin operators). Then, $[A, B] = 0$ can be naturally understood as if $A$ and $B$ describe two compatible questions (they do not affect each other), while these are necessarily incompatible if $[A, B] \neq 0$. Hence the commutator between $A$ and $B$ reflects the nature of the questions described by these operators. We will discuss this particular application in detail in Chapter 12.

2.6 A few Words on States

In our approach, states play an essential role, since they are used to project down the time evolution of the system $S$ we are dealing with from an operatorial to a classical level. The strategy is the following: we use operators to describe certain macroscopic systems. Let $S$ be one such system, and $X$ be an observable relevant in the analysis of $S$. Its time evolution $X(t)$ is deduced from the Hamiltonian of $S$, as discussed before, adopting, for instance, the Heisenberg representation. The Hamiltonian $H$ is written by considering all the interactions existing between the various agents of $S$, following a few minimal (and natural) rules listed in [1]. Hence, if $\Psi$ is the vector describing $S$ at time zero (and therefore also at time $t$, see Section 2.4), $x(t) = \langle \Psi, X(t)\Psi \rangle$ is interpreted as the time evolution of the classical dynamical variable represented, in our approach, by the operator $X$, when $S$ is described, at $t = 0$, by $\Psi$. In this sense $x(t)$ is the projection of the quantum-like dynamics of $X$, $X(t)$ to its original classical level.

A state of this kind, $\langle \phi, Y\phi \rangle$, for some normalized vector $\phi \in \mathcal{H}$ and some operator $Y$ on $\mathcal{H}$ is called a vector state. More in general, a state $\omega$ over a certain $^\ast$-algebra $\mathcal{A}$ is a positive linear functional which is normalized. In other words, for each element $A \in \mathcal{A}$, $\omega$ satisfies the following:

$$\omega(A^\dagger A) \geq 0, \quad \text{and} \quad \omega(1) = 1,$$

where $1$ is the identity of $\mathcal{A}$. It is known, [43], that not all the $^\ast$-algebras necessarily possess an identity. But it is also known that $1$ can be added to $\mathcal{A}$, keeping the mathematical structure essentially unchanged, see again [43]. Moreover, $\omega(A^\dagger) = \omega(A)$. These properties are clearly satisfied by any vector state. In fact,
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independently of the choice of normalized $\varphi$, we have $\langle \varphi, \mathbb{1} \varphi \rangle = \| \varphi \|^2 = 1$. Moreover, $\langle \varphi, A^\dagger A \varphi \rangle = \| A \varphi \|^2 \geq 0$. Finally, $\langle \varphi, A^\dagger \varphi \rangle = \langle A \varphi, \varphi \rangle = \langle \varphi, A \varphi \rangle$.

States over bounded operators share many nice properties: they are automatically continuous [43]; if $A_n$ is a sequence of bounded operators converging to $A$ in the uniform topology, then $\omega(A_n)$ converges to $\omega(A)$ in the topology of $C$:

$$\|A_n - A\| \to 0, \quad \implies \quad |\omega(A_n) - \omega(A)| \to 0,$$

when $n \to \infty$. A very useful inequality which each state satisfies is the Cauchy-Schwarz inequality already used, in a slightly different and simpler form, in Section 2.5: for each $A, B \in \mathcal{A}$, the following holds:

$$|\omega(A^\dagger B)|^2 \leq \omega(A^\dagger A) \omega(B^\dagger B).$$

The vector states already introduced in this chapter, see Equations (2.3) and (2.6), describe a situation in which the numbers of all the different modes of bosons or fermions are known, since the vectors we use are eigenstates of the single-mode number operators. This kind of states will be used several times in this book, but they are not the only useful states in quantum mechanics. In fact, other types of states also exist and are relevant in many physical applications. For instance, the so-called KMS states, i.e., the equilibrium states for systems with infinite degrees of freedom, are usually used to prove the existence of phase transitions or to find conditions for the existence of some thermodynamical equilibrium. Without going into the mathematical rigorous definition, see [43], a KMS state $\omega$ with inverse temperature $\beta$ satisfies the following equality, known as the KMS condition:

$$\omega(A B(i \beta)) = \omega(B A), \quad (2.46)$$

where $A$ and $B$ are general elements of $\mathcal{A}$, and $B(i \beta)$ is the time evolution of the operator $B$ computed at the complex value $i \beta$ of the time.\(^\dagger\) It is well known [44] that, when restricted to a finite size system, a KMS state is nothing but a Gibbs state which can be expressed in terms of a trace with a suitable weight involving the Hamiltonian of the system:

$$\omega(A) = \frac{tr \left( e^{-\beta H} A \right)}{tr \left( e^{-\beta H} \right)}.$$

In general, a KMS state is used to describe a thermal bath interacting with a physical system $\mathcal{S}$, if the bath has a nonzero temperature.

It is interesting to observe that, as each normalized vector defines a state, each state can be used to define a normalized vector in a certain Hilbert space. The way in which this is achieved is known as the GNS construction, where GNS stands

\(^\dagger\) More precisely, $B(i \beta)$ is an analytic extension of $B(t)$ to the complex domain [44].
2.7 More on Dynamics

This is quite important in physical applications, since different states can give rise to different, and inequivalent, representations of the same system $S$. These inequivalent representations can be related, for instance, to different thermodynamical phases of $S$.

2.7 More on Dynamics

For what we will discuss in Part II, we need to know something more on the dynamical aspects of quantum systems. In fact, for instance, when trying to describe a procedure of decision-making, we expect that the time evolution of the system produces, at least for those dynamical variables associated with the mechanism of the decision, some reasonable asymptotic value. In other words, if $d(t)$ is the function which describes the time evolution of the decision process of, say, Alice, it is quite natural to expect that, for time large enough, $d(t)$ converges to some value, $d_{\infty}$. This is what should be understood as Alice’s final decision. If $d(t)$ reaches no asymptotic value, as it is the case if Alice always oscillates between different moods, no decision is taken. Of course, one natural question (but not the only one!) is how large the time should be, i.e., what is a reasonable time needed to take a decision. There is no sharp answer to this question: in some situations the decision should be extremely fast, while in other cases there is no real need to hurry up. Some authors just fix the decision time $t_d$ from the very beginning. Hence, even if $d(t)$ is oscillating, when $t = t_d$, Alice just takes $d(t_d)$ as her final decision. Of course, this approach implies that fixing a different value for $t_d$ Alice takes a (possibly) completely different decision. Hence the result, and the whole procedure, is extremely sensitive to this choice. For this reason, we prefer to adopt here (and in many applications discussed later in the book) a different approach, based on what in quantum mechanics are called open quantum systems, in which the decision is driven by the interaction between Alice and her environment. In particular, we will see how different systems give rise to different interpretations for this environment, which could be seen as a group of people, or a set of information, or even a group of cells (or neurons).

If $S$ is a closed quantum system with time-independent, self-adjoint, Hamiltonian $H$, it is natural to suspect that only periodic or quasi-periodic effects can take place, during the time evolution of $S$. This is because the energy of $S$ is preserved, and this seems to prevent to have any damping effect. For instance, if we work in the Schrödinger representation, the time evolution $\Psi(t)$ of the wave-function of the system is simply $\Psi(t) = e^{-iHt}\Psi(0)$ and, since the operator $e^{-iHt}$ is unitary, we do not expect that $\Psi(t)$ decreases (in any reasonable sense) to zero when $t$ diverges. On the other hand, we will show that a similar decay feature is possible if
We start considering a first system, $S$, interacting with a second system, $\tilde{S}$, and we assume for the time being that both $S$ and $\tilde{S}$ are of the same size: to be concrete, we imagine here that $S$ describes a single particle whose related operators are $a$, $a^\dagger$ and $\hat{n}_a = a^\dagger a$. Analogously, $\tilde{S}$ describes a second particle whose related operators are $b$, $b^\dagger$ and $\hat{n}_b = b^\dagger b$. These operators obey the following CCR:

$$[a, a^\dagger] = [b, b^\dagger] = I,$$

while all the other commutators are assumed to be zero. In particular, $a$ and $b$ commute: $[a, b] = 0$. Hence we are dealing with two originally independent bosonic modes. A natural choice for the Hamiltonian of the interacting system $S \cup \tilde{S}$, adopting a very simple type of interaction between the particles, is the following:

$$\hat{h} = \omega_a \hat{n}_a + \omega_b \hat{n}_b + \mu (a^\dagger b + b^\dagger a),$$

where $\omega_a$, $\omega_b$, and $\mu$ must be real quantities if we are interested to have $\hat{h} = \hat{h}^\dagger$. We should recall that losing self-adjointness of $\hat{h}$ would produce, among other consequences, a non-unitary time evolution, and this is out of the scheme usually considered in ordinary quantum mechanics.$^{12}$ The Hamiltonian $\hat{h}$ contains a free part plus an interaction which is such that, if the eigenvalue of $\hat{n}_a$ increases of one unit during the time evolution, the eigenvalue of $\hat{n}_b$ must decrease, again by one unit, and vice versa. This can be easily understood since because $[\hat{h}, \hat{n}_a + \hat{n}_b] = 0$, so that $\hat{n}_a + \hat{n}_b$ is an integral of motion. The equations of motion for $a(t)$ and $b(t)$ can be deduced as in Equation (2.43) and turn out to be

$$\dot{a}(t) = i[\hat{h}, a(t)] = -i\omega_a a(t) - i\mu b(t), \quad \dot{b}(t) = i[\hat{h}, b(t)] = -i\omega_b b(t) - i\mu a(t),$$

(2.47)

whose solution can be written as $a(t) = \alpha_a(t) a + \alpha_b(t) b$ and $b(t) = \beta_a(t) a + \beta_b(t) b$. The functions $\alpha_j(t)$ and $\beta_j(t)$, $j = a, b$, are linear combinations of $e^{\lambda_j t}$, with

$$\lambda \pm = \frac{1}{2} (\omega_a + \omega_b \pm \sqrt{(\omega_a - \omega_b)^2 + 4\mu^2}).$$

Moreover, $\alpha_a(0) = \beta_b(0) = 1$ and $\alpha_b(0) = \beta_a(0) = 0$, in order to have $a(0) = a$ and $b(0) = b$. Hence we see that both $\dot{a}(t)$ and $\dot{b}(t)$, and $\dot{n}_a(t) = a^\dagger(t)a(t)$ and $\dot{n}_b(t) = b^\dagger(t)b(t)$ as a consequence, are linear combinations of oscillating functions, so that no damping, and no stable asymptotic limit, is possible for this simple model. This simple remark is important for us, because of the role that number operators like $\hat{n}_a(t)$ and $\hat{n}_b(t)$ will have in the rest of the book.

Of course the reader could be not really satisfied with this simple model because, in particular, of our specific choice of $\hat{h}$. In fact, this is useful just because it produces an analytical solution for $a(t)$ and $b(t)$. However, a similar conclusion

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$^{12}$ We should also stress that in recent years a larger and larger group of physicists and mathematicians started to be interested in what happens when the dynamics is defined in terms of some Hamiltonian which is not self-adjoint but satisfies certain symmetry properties which are physically motivated. This is what, in the literature, it is usually called pseudo-Hermitian quantum mechanics, see also Sections 2.3.3 and 2.7.2.
can still be deduced also if \( h \) is defined differently, even if \( h \) not necessarily constructed in terms of ladder operators. Suppose, to be concrete, that \( h \) is a self-adjoint \( n \times n \) matrix. This means that the system we are interested in, \( \hat{S} \), lives in a finite-dimensional Hilbert space \( \mathcal{H}_n \). This excludes, in principle, the possibility that bosonic operators play some role in the analysis of this particular system, while other ladder operators (coming from the CAR, or from truncated bosonic operators, for instance) fit into this situation. Moreover, as it was widely discussed in [9], some bosonic systems can still be efficiently described in a finite-dimensional Hilbert space, \( \mathcal{H}_{\text{eff}} \). This is what happens when some integral of motion exists which relates, and constrains, the various number operators of the model. When this happens, not all the energetic levels of \( \mathcal{H} \) can be reached by the system, and those that can give rise to \( \mathcal{H}_{\text{eff}} \) [1, 9].

Now, the time evolution of a certain observable \( D \) of \( \hat{S} \) deduced by the Hamiltonian \( h \) is, of course, \( D(t) = e^{iht}De^{-iht} \). But, since \( h = h^\dagger \), a unitary matrix \( U \) exists such that \( UhU^{-1} = h_d \), with diagonal \( h_d \). Of course, the diagonal elements of \( h_d \) are the real eigenvalues of \( h \). Then

\[
D(t) = U^{-1}e^{ih_d t}UDU^{-1}e^{-ih_d t}U,
\]

where, of course, \( e^{xh_d t} \) are diagonal matrices. Hence the mean value \( \langle D(t) \rangle \) of \( D(t) \) on any vector can only be periodic or quasi-periodic, depending on the relation between the eigenvalues of \( h \): no stable asymptotic value is possible for \( \langle D(t) \rangle \), which keeps on oscillating for all \( t \). Therefore, if we imagine that \( \langle D(t) \rangle \) describes how Alice is processing her decision, the only natural way to stop the procedure, and concretely decide something, is to introduce, from outside, an ad hoc decision time \( t_d \).

Suppose now that the system \( \hat{S} \) is replaced by an (infinitely extended) reservoir \( \mathcal{R} \), whose particles are described by an infinite set of bosonic operators \( b(k), b^\dagger(k) \) and let \( \hat{n}(k) = b^\dagger(k)b(k), k \in \mathbb{R} \), be the corresponding number operators. We assume now the following Hamiltonian for \( S_{\text{full}} = S \cup \mathcal{R} \), which extends the operator \( h \) for the two bosons proposed above:

\[
H = H_0 + \lambda H_I, \quad H_0 = \omega \hat{n}_a + \int_\mathbb{R} \omega(k)\hat{n}(k) \, dk, \quad H_I = \int_\mathbb{R} \left(ab^\dagger(k) + a^\dagger b(k)\right)f(k)dk,
\]

where \([a, a^\dagger] = \mathbb{I}, [b(k), b^\dagger(q)]] = \mathbb{I} \delta(k - q)\), while all the other commutators are zero. All the constant appearing in (2.48), as well as the regularizing function \( f(k) \), are real, so that \( H = H^\dagger \). In our simple system, \( H \) models Alice and her interactions with some environment, which can be understood as the set of her friends, parents and relatives, for instance.
Notice that an integral of motion exists also for $S_{\text{full}}$, $\hat{n}_a + \int_\mathbb{R} \hat{n}(k) \, dk$, which extends the one for $S \cup \hat{S}$, $\hat{n}_a + \hat{n}_b$. With this choice of $H$, the Heisenberg equations of motion are
\begin{align}
\dot{a}(t) &= i[H,a(t)] = -i\omega a(t) - i\lambda \int_\mathbb{R} f(k) b(k,t) \, dk, \\
\dot{b}(k,t) &= i[H,b(k,t)] = -i\omega(k)b(k,t) - i\lambda f(k) a(t),
\end{align}
which are supplemented by the initial conditions $a(0) = a$ and $b(k,0) = b(k)$. The last equation in (2.49) can be rewritten in integral form as $b(k,t) = b(k)e^{-i\omega(k)t} - i\lambda f(k) \int_0^t a(t_1)e^{-i\omega(k)(t-t_1)} \, dt_1$. We fix now, as is often done in the literature, $f(k) = 1$, and we take $\omega(k) = k$ and we replace $b(k,t)$ in the first equation in (2.49).

Then, with a change of the order of integration, and recalling that $\int_\mathbb{R} e^{-i\lambda(t-t_1)} \, dk = 2\pi \delta(t-t_1)$ and that $\int_0^t g(t_1)\delta(t-t_1) \, dt_1 = \frac{1}{2} g(t)$ for any test function $g(t)$, we conclude that
\begin{equation}
\dot{a}(t) = -(i\omega + \pi\lambda^2)a(t) - i\lambda \int_\mathbb{R} b(k) e^{-ikt} \, dk.
\end{equation}
This equation can be solved, and the solution is
\begin{equation}
a(t) = \left(a - i\lambda \int_\mathbb{R} dk \eta(k,t)b(k)\right) e^{-(i\omega + \pi\lambda^2)t},
\end{equation}
where $\eta(k,t) = \frac{1}{\rho(k)}(e^{\rho(k)t} - 1)$ and $\rho(k) = i(\omega - k) + \pi\lambda^2$. Using complex contour integration, it is possible to check that $[a(t),a^\dagger(t)] = I$ for all $t$; this means that the apparent decay of $a(t)$, described in Equation (2.51), is balanced by a contribution of the reservoir.

**Remark:** The choices $f(k) = 1$ and $\omega(k) = k$ are convenient since they allow us to get analytical solutions, see Equation (2.51). However, other choices could be adopted, and some of them will be considered in Section 2.7.1.

Let us now consider a state over $S_{\text{full}}$, $\langle X_S \otimes X_R \rangle = \langle \varphi_{n_a}, X_S \varphi_{n_a} \rangle = \langle X_R \rangle_R$, in which $\varphi_{n_a}$ is the eigenstate of the number operator $\hat{n}_a$ and $< >_R$ is a state of the reservoir, see Section 2.6. Here $X_S \otimes X_R$ is the tensor product of an operator of the system, $X_S$, and an operator of the reservoir, $X_R$. Stated differently, $\langle . \rangle_R$ is a normalized, positive linear functional over the algebra of the operators of $R$, while $= \langle \varphi_{n_a}, \varphi_{n_a} \rangle$ is a vector state over the algebra of $S$. The state $\langle . \rangle_R$ is assumed to satisfy some properties which extend to $R$ similar properties of the vector states over $S$. In particular, we require that $\langle b^\dagger(k)b(q) \rangle_R = n_b(k)\delta(k-q)$ and that $\langle b(k)b(q) \rangle_R = 0$. These are standard choices, see for instance [45], at least in the context of open quantum systems and of quantum optics in particular. Then, if for simplicity we take the function $n_b(k)$ to be constant in $k$, we get
\begin{equation}
n_a(t) := \langle \hat{n}_a(t) \rangle = \langle a^\dagger(t)a(t) \rangle = n_a e^{-2\lambda^2\pi t} + n_b \left(1 - e^{-2\lambda^2\pi t}\right),
\end{equation}
which goes to $n_b$ as $t \to \infty$. Hence, if $0 \leq n_b < n_a$, the value of $n_a(t)$ decreases with time. If, on the other hand, $n_b > n_a$, then the value of $n_a(t)$ increases for large $t$. In both cases, $n_a(t)$ admits an asymptotic value which, identifying $n_a(t)$ with the decision function $⟨D(t)⟩$ introduced before, can be interpreted as Alice’s final decision. Notice that it is essential to assume here that $R$ has an infinite number of degrees of freedom. Notice also that, in particular, if the reservoir is originally empty, $n_b = 0$, then $n_a(t) = n_a e^{-2.5 \pi t}$ decreases exponentially to zero: the system becomes empty (and Alice’s decision is zero). On the other hand, since $\hat{n}_a + \int_{\mathbb{R}} \hat{n}(k) \, dk$ is a constant of motion, the reservoir starts to be filled up.

It might be interesting to remark that the continuous reservoir considered here could be replaced by a discrete one, describing again an infinite number of particles, but labeled by a discrete index. This could be useful to provide a more natural interpretation for the reservoir in terms of agents of some kind interacting with the core of $S_{full}$, the system $S$, which in our case is just Alice. Hence, to obtain a Dirac delta distribution, which is the crucial mathematical ingredient in the derivation above, we have to replace the integral formula $\int_{\mathbb{R}} e^{-ik(t-t_1)} \, dk = 2\pi \delta(t - t_1)$ with the Poisson summation formula, which we write here as $\sum_{n \in \mathbb{Z}} e^{inx} = \frac{2\pi}{|c|} \sum_{n \in \mathbb{Z}} \delta(x - n\frac{2\pi}{c})$, for all nonzero $c \in \mathbb{R}$. Not many other differences arise when continuing the analysis of $S_{full}$.

### 2.7.1 Inhomogeneous Reservoir

In the explicit computations above it is technically important to work with a sort of homogeneous reservoir, i.e., to assume that $n_b(k) = n_b$, for all $k \in \mathbb{R}$. However, it is also natural to imagine that other reasonable (and well-motivated) choices for $n_b(k)$ also exist. This aspect was considered, for instance, in [11], where the inhomogeneity of the reservoir was used to model different sources of information reaching some agents during a procedure of decision-making. We have also used an explicitly $k$-dependent function $n_b(k)$ in [10], in connection with love affairs. Of course, due to formula $\langle b^\dagger(k)b(q)⟩_R = n_b(k) \delta(k - q)$, $n_b(k)$ must be real and nonnegative. This does not fix (at all!) the analytic expression of $n_b(k)$. Still, if we are willing to derive analytic results, it is convenient to restrict our choice of this function to some particular mathematical expressions. This is important if we want to use complex integration to perform the computations, as it is usually done in quantum optics [45]. For instance, in [11] we have taken a decreasing function of $k$. More in details, it is useful to assume that $n_b(k)$ is analytic in $k$, and that $|n_b(k)|$ does not diverge when $|k|$ diverges. In this way we can use the Jordan’s lemma and compute the integrals involved in our computations using complex methods. We fix, just as an example,

$$n_b(k) = \frac{n_b}{k^2 + \alpha^2},$$  \hspace{1cm} (2.53)
for some positive $n_b$ and for some $\alpha > 0$: this describes a situation in which not all
the parts of the bath interact with Alice (the decision maker) in the same way. We
also take $\omega_b(k) = \omega_b k$. In the computation of $\int \eta(k,t) \, dk$, which appears
when checking that $[a(t), a^\dagger(t)] = \mathbb{I}$ for all $t$, we have to consider two singularities
of the integrating functions both in the upper and lower complex semi-planes, so
the result of the integral is the sum of two residues.

The computation could be given in detail, and the analytic form of $n_a(t)$ could
be deduced for all $t$. However, if we are interested only in deducing what the final
decision is, what is relevant for us is just the large time limit of $n_a(t)$. This turns
out to be

$$n_a(\infty) := \lim_{t,\infty} n_a(t) = \frac{n_b(\alpha \omega_b^2 + \pi \lambda^2)}{\alpha \lambda^2 \omega_b^2 \left( \frac{\omega_b^2}{\omega^2} + \left( \alpha + \frac{\pi \lambda^2}{\omega_b^2} \right)^2 \right)}.$$  \hspace{1cm} (2.54)

Notice that this result makes sense only if $\alpha$ and $\lambda$ are not zero. This suggests that
the role of the complex pole in $n_b(k)$, and of the interaction between Alice and her
bath, is really essential to get some stabilization within the scheme proposed here.
We observe also that the asymptotic value of the decision function $n_a(t)$ strongly
depends on the various parameters of the model, which could be adjusted to fit
experimental data, if needed. From this point of view the choice of $n_b(k)$ in Equation (2.53) is more interesting than the one considered before, of a uniform reservoir with $n_b(k) = n_b$. In that case, in fact, $n_b$ turned out to be exactly Alice’s final
decision: this means that Alice does not really have a constructing role in her process of decision-making, since she always decides $n_b$ independently of what was her original mood. On the other hand, see Equation (2.54), if we enrich the form of the reservoir, $n_a(\infty)$ is also related to $\omega$ and to $\lambda$, which are both parameters of $H$
directly related to Alice. In particular, $\omega$ is a measure of Alice’s inertia, [1], while $\lambda$ measures the strength of the interaction between Alice and her reservoir.

Remark: Our results suggest that the asymptotic limit of $n_a(t)$ does not really de-
pend on the fact that, at $t = 0$, Alice is described by a pure state (some $\varphi_{n_a}$) or by
some linear combination of these states. It is further clear that our approach does not
give directly information on the output state, i.e., on the state after the decision is
taken. This is because we are working here in the Heisenberg representation.

2.7.1.1 Other Choices for $f(k)$ and $\omega(k)$
So far we have fixed $f(k) = 1$ and $\omega(k) = \omega_b k$. As we have already claimed, this is not
the only possible choice, in particular if we want to model possible differences
between different parts of the reservoir. Technically, what is particularly useful for
us is that Dirac’s delta distribution appears when deducing the differential equation
for $a(t)$ in Equation (2.50). And this is possible also with other choices: suppose
that the two functions $f(k)$ and $\omega(k)$ are such that

\[ \omega(k) \to \pm \infty \text{ for } k \to \pm \infty, \quad \text{and} \quad \frac{d\omega(k)}{dk} = \frac{1}{\beta} f^2(k), \]

for some real $\beta \neq 0$. These assumptions are satisfied if $f(k) = 1$ and $\omega(k) = \omega_b k$, but not only. Hence

\[ \int_{\mathbb{R}} f^2(k) e^{-i\omega(k)(t-t_1)} \, dk = \beta \int_{\mathbb{R}} e^{-i\omega(k)(t-t_1)} \, d\omega(k) = 2\pi \beta \delta(t - t_1), \]

and we can again significantly simplify the equation for $a(t)$, which now becomes

\[ \dot{a}(t) = -\left( i\omega + \beta \pi \lambda^2 \right) a(t) - i \lambda \int_{\mathbb{R}} f(k) b(k) \, e^{-i\omega(k)t} \, dk. \tag{2.55} \]

This is the equation which replaces (2.50) in this new situation. The solution can
be found as before, and we get

\[ a(t) = \left( a - i \lambda \int_{\mathbb{R}} dk \, \tilde{\eta}(k,t) b(k) f(k) \right) e^{-\left( i\omega + \beta \pi \lambda^2 \right) t}, \tag{2.56} \]

where $\tilde{\eta}(k,t) = \frac{1}{\tilde{\rho}(k)} (e^{\tilde{\rho}(k)t} - 1)$ and $\tilde{\rho}(k) = i(\omega - \omega(k)) + \beta \pi \lambda^2$. At a first sight, the situation is not particularly different from what was deduced before. However, to find $[a(t), a^\dagger(t)]$ and $n_a(t)$, we need to compute integrals, and the poles of the
integrating functions are strongly dependent on the analytic expression of $\omega(k)$: in particular, if $\omega(k)$ is not linear, the computations become rather hard. For this
reason, this strategy will not be considered further in this book, even if it could be
possibly relevant for other applications.

### 2.7.1.2 What If We Use the CAR?

In the Hamiltonian (Equation 2.48) the operators $a$ and $b(k)$ are assumed to satisfy
the CCR. However, from the point of view of decision-making, it might be more interesting
to consider the case in which the agent’s decisions are described by
fermionic operators, so that the CCR assumed for the operators in Equation (2.48)
are now replaced by the following CAR: $[a, a^\dagger] = \mathbb{I}$, with $[a, a] = 0$, and

\[ [b(k), b^\dagger(q)] = \delta(k - q) \, \mathbb{I}, \quad [b(k), b(q)] = 0. \]

Moreover, $[a^\dagger, b^\dagger(k)] = 0$, where $x^\#$ indicates either $x$ or $x^\dagger$. Assuming the same Hamiltonian (Equation 2.48), the differential equations of motion for the annihilation operators $a(t)$ and $b(k, t)$ can be deduced. They turn out to coincide with those in Equation (2.49), except that $\lambda$ must be replaced by $-\lambda$. In other words, we get

\[
\begin{align*}
\dot{a}(t) &= i[H, a(t)] = -i\omega a(t) + i\lambda \int_{\mathbb{R}} f(k) b(k, t) \, dk, \\
\dot{b}(k, t) &= i[H, b(k, t)] = -i\omega(k) b(k, t) + i\lambda f(k) a(t).
\end{align*}
\]  

\[ \tag{2.57} \]
As in the simplest bosonic case, we assume here that the function \( n_b(k) \) for the state of the reservoir is constant, \( n_b(k) \equiv n_b \geq 0 \). The CAR implies that this constant cannot exceed one, \( 0 \leq n_b \leq 1 \).

Our previous results deduced with the bosonic settings show that \( n_a(t) \) depends on \( \lambda^2 \), see (2.52). This is true even with the choice (2.53) of \( n_b(k) \), see (2.54). Therefore we do not expect any difference in the results when replacing the CCR with the CAR, since, as we have just seen, this only causes \( \lambda \) to be replaced by \( -\lambda \) in the differential equations of motion. Of course, choosing the CCR or CAR should be related to the nature of the decision we want to model: in case of a binary question (yes or not), the natural setting is probably the fermionic one. But if we imagine that infinite possible answers are possible, then we should use a bosonic version of the model. When a finite number of answers are possible, but this number is larger than two, then we could try to adopt the truncated version of bosonic operators, see Section 2.3.2. But these are just details which do not change the general strategy: decisions (of any possible kind) are driven by interactions with some (homogeneous or not) environment. We will discuss several consequences of this idea in the second part of the book.

2.7.2 Non-Self-Adjoint Hamiltonians for Damping

An alternative simple way to describe damping effects in a quantum system makes use of non-self-adjoint Hamiltonians to deduce the dynamics. In this way there is no need to consider any reservoir interacting with the system \( S \). This is, indeed, the point of view of many papers in quantum optics. Just to cite a simple example, in [46, 47] the authors considered an effective non-self-adjoint Hamiltonian describing a two-level atom interacting with an electromagnetic field. This was done in connection with pseudo-Hermitian systems, as those naturally connected with the deformed commutation rules considered in Section 2.3.3. In fact, pseudo-fermions were used later [48], in the analysis of the same system previously considered in [46, 47]. More explicitly, the starting point in the analysis of the system is the Schrödinger equation:

\[
\dot{\Psi}(t) = H_{\text{eff}} \Psi(t), \quad H_{\text{eff}} = \begin{pmatrix} -i\gamma_a & \frac{v}{\bar{v}} \\ \frac{1}{\bar{v}} & -i\gamma_b \end{pmatrix}. \tag{2.58}
\]

Here \( \gamma_a \) and \( \gamma_b \) are real quantities, related to the decay rates for the two levels, while the complex parameter \( v \) characterizes the radiation-atom interaction. We refer to [46,47], and to the references therein, for further details. Here we just want to stress that \( H_{\text{eff}} \neq H_{\text{eff}}^\dagger \) and that the analytical expression of \( \Psi(t) \) can be explicitly
deduced. The resulting time evolution describes a decay of the wave-function, essentially because $e^{-iH_{eff}t}$ is not a unitary operator, so that $\|e^{-iH_{eff}t}\Psi(0)\| \neq \|\Psi(0)\|$, in general.

Having this kind of examples in mind, it seems a natural choice to consider, for instance, the following operator:

$$\hat{h} = \omega_a \hat{n}_a + \omega_b \hat{n}_b + \mu a \dagger b,$$

(2.59)

where $a$, $a \dagger$ and $\hat{n}_a$ are the operators of a first system $S_a$, while $b$, $b \dagger$ and $\hat{n}_b$ are those of a second system, $S_b$, which in our idea should play here the role of a hyper-simplified reservoir. Here we take $\omega_a$, $\omega_b$ and $\mu$ real, but this is not sufficient, of course, to make of $\hat{h}$ a self-adjoint operator, if $\mu \neq 0$. To make it self-adjoint, we should add a contribution $\mu a \dagger b$ in its definition. However, it is exactly the lack of this term which makes $\hat{h}$ interesting for us. In fact, the interaction term $\mu a \dagger b$ in $\hat{h}$ causes the eigenvalue of the operator $\hat{n}_a$ to decrease of one unit (because of the presence of the annihilation operator $a$) and that of $\hat{n}_b$ to increase simultaneously (because of $b \dagger$), again of one unit. In other words, $\hat{h}$ seems to describe a situation in which some quanta of the system flow from $S_a$ to $S_b$. If $a$ and $b$ describe the cash of two different traders, trader $\tau_a$ is giving money to $\tau_b$. After some time, and several interactions with $\tau_b$, it is clear that $\tau_a$ ends up with no cash at all. This seems great, since it would suggest an effective procedure to have damping without making use of reservoirs with infinite degrees of freedom. Unfortunately, this very simple and apparently efficient method doesn’t work! This can be explicitly seen if we assume that, also in this case, the time evolution of any fixed operator, $x$, is given by $x(t) = e^{i\hat{h}t}xe^{-i\hat{h}t}$. This is just the same choice we have discussed for self-adjoint Hamiltonians, but the main difference here is that $e^{\pm i\hat{h}t}$ are no longer unitary operators. However, rather than a problem, this could be considered as a positive feature of this approach, since we really want to deduce damping, and damping does not really go well with unitary operators (even if they do not necessarily exclude each other, see Equation (2.52), for instance). A second technical (but not only) problem arises because, with this definition of $x(t)$, it is not true in general that $(x(t))^\dagger = (x^\dagger)(t)$, so that $x(t)$ and $x^\dagger(t)$ should be considered as independent dynamical variables. For our simple system we find

$$a(t) = a e^{-i\omega_a t}, \quad a^\dagger(t) = a^\dagger e^{i\omega_a t} + \frac{\mu}{\omega_b - \omega_a} b \left(e^{-i\omega_b t} - 1\right).$$

Hence, contrarily to what one could expect, $a^\dagger(t)$ is not simply the Hermitian conjugate of $a(t)$. Moreover, and quite relevant for us, despite our original interpretation of $\hat{h}$, it is clear from the above formulas for $a(t)$ and $a^\dagger(t)$ that the mean value of the number operator $\hat{n}_a(t) = a^\dagger(t) a(t)$ presents no damping at all.
Then we conclude that our simple minded point of view is simply wrong. One possible source of mistakes could be the fact that we have assumed here, following Equation (2.42), that the time evolution of the operator $x$ is $x(t) = e^{i\hbar t}x e^{-i\hbar t}$, even if $\hat{h} \neq \hat{h}$. This is, in fact, not so reasonable, as the following rather general argument shows: let us suppose, as in (2.58), that the Schrödinger equation which describes the evolution of our system is the usual one: $i\dot{\Phi}(t) = \hat{h} \Phi(t)$, even if $\hat{h}$ is not self-adjoint. This is quite often the starting point in many papers in quantum optics, but not only. Here $\hat{h}$ needs not being the one in Equation (2.59). Hence, if $\hat{h}$ does not depend on time, the formal solution of this equation is $\Phi(t) = e^{-i\hat{h} t}\Phi(0)$. As we have already pointed out before, what we do measure in a concrete experiment is really the mean value of some relevant observable related to the system in a state of the system. Hence we expect that the time evolution of such a mean value cannot be dependent on the representation chosen. In other words, if $x$ is such an observable, we have

$$\langle \Phi(t), x \Phi(t) \rangle = \left\langle e^{-i\hat{h} t}\Phi(0), x e^{-i\hat{h} t}\Phi(0) \right\rangle = \left\langle \Phi(0), e^{i\hat{h} t}x e^{-i\hat{h} t}\Phi(0) \right\rangle,$$

and this suggests to call $x(t)$, rather than $e^{i\hat{h} t}x e^{-i\hat{h} t}$ as we have done before, the following operator: $x(t) := e^{i\hat{h} t}x e^{-i\hat{h} t}$. It is interesting to notice that this definition cures the anomaly we have seen above, i.e., the fact that, with our previous choice, $(x(t))^\dagger \neq (x^\dagger)(t)$. In fact, with this different definition, we easily check that $(x(t))^\dagger = (x^\dagger)(t)$ for all possible $x$ and for all $t$. On the other hand, in general, it is more difficult to deduce the explicit form of the differential equations for $x(t)$: let us write $\hat{h} = H_r + iH_i$, with $H_r = \frac{1}{2}(\hat{h} + \hat{h}^\dagger) = \hat{H}_r^\dagger$ and $H_i = \frac{1}{2i}(\hat{h} - \hat{h}^\dagger) = \hat{H}_i^\dagger$. Then, the differential equation of motion for $x(t) = e^{i\hat{h} t}x e^{-i\hat{h} t}$ can be written as

$$\frac{dx(t)}{dt} = e^{i\hat{h} t} (\hat{h}^\dagger x - x\hat{h}) e^{-i\hat{h} t} = e^{i\hat{h} t} ([H_r, x] - i[H_i, x]) e^{-i\hat{h} t},$$

which involves both a commutator and an anti-commutator. This makes the situation rather more difficult, of course. There is still another reason to prefer, when possible, working with self-adjoint rather than with non-self-adjoint Hamiltonians: the reason is that, using formula $x(t) = e^{i\hat{h} t}x e^{-i\hat{h} t}$, we lose the group property of the time evolution: $(xy)(t) \neq x(t)y(t)$, for two generic operators $x$ and $y$. This has unpleasant drawbacks in the deduction of the equations of motion for the system, since this property is essential in order to get some closed system of differential equations, which is often the starting point to look for the solution of the dynamical problem.

Of course, most of these problems are just hidden if we work in the Schrödinger representation, since the validity of the Schrödinger equation is exactly the main working assumption. With this in mind, despite all these problems, sometimes
2.8 The \((H, \rho)\)-Induced Dynamics

We will use some non-self-adjoint Hamiltonians of a special type. For instance, this will be done in Chapters 4 and 6, where we will introduce first self-adjoint Hamiltonians depending on real parameters, and then we will replace some of these with complex quantities, keeping unchanged the formal expression of the Hamiltonians. These complex parameters will be shown to model quite well the existence of stress factors and positive effects acting on the system, depending on the sign of their imaginary parts. A different choice of non-self-adjoint Hamiltonian will be discussed in Chapter 7, more on the line of the Hamiltonian in (2.59), in connection with the analysis of cellular tumor growth.

2.8 The \((H, \rho)\)-Induced Dynamics

As we have already discussed before, one of the technical limitations of the approach considered in this book is that the Hamiltonian driving the time evolution of the system \(S\) is (almost) always assumed to be self-adjoint. Losing self-adjointness usually produces non-unitary time evolution. However, it may not be so for those non-self-adjoint Hamiltonians which are, for instance, similar (but not unitarily equivalent) to self-adjoint ones. In fact, in these cases, it is possible to replace the original scalar product \(\langle \ldots \rangle\) of the Hilbert space \(\mathcal{H}\) with a different one, \(\langle \ldots \rangle_{\text{new}}\), which makes of these non-self-adjoint operators new, and self-adjoint, operators. Let us show how this is possible. For that, let us consider a Hamiltonian \(H\) such that \(H \neq H^\dagger\), and let us assume that a self-adjoint, positive and invertible (but not unitary) operator \(S\) exists such that \(H = SH_0S^{-1}\), where \(H_0 = H_0^\dagger\). Then we can prove that \(H\) is self-adjoint, but with respect to the scalar product \(\langle \ldots \rangle_S\) defined as follows:

\[
\langle f, g \rangle_S := \langle S^{-2}f, g \rangle,
\]

for all \(f, g \in \mathcal{H}\). Here, to avoid useless complications, we are assuming that \(S, S^{-1}\) and \(H_0\) are bounded.\(^{14}\) Hence, \(H\) is bounded as well.

First of all, we need to check that \(\langle \ldots \rangle_S\) is really a scalar product. We just prove here that \(\langle f, f \rangle_S \geq 0\) for all \(f \in \mathcal{H}\), and that it is zero only if \(f = 0\). Since \(S^{-1}\) is self-adjoint, we have

\[
\langle f, f \rangle_S = \langle S^{-2}f, f \rangle = \langle S^{-1}f, S^{-1}f \rangle = \|S^{-1}f\|^2 \geq 0,
\]

of course. In particular, this formula shows that \(\langle f, f \rangle_S = 0\) if, and only if, \(S^{-1}f = 0\), which implies that \(f = 0\) as well.

\(^{13}\) Of course, Hamiltonians like the one in (2.59) are not of this kind.

\(^{14}\) This in particular implies that the two norms arising from \(\langle \ldots \rangle\) and \(\langle \ldots \rangle_S\), \(\|f\| = \sqrt{\langle f, f \rangle}\) and \(\|f\|_S = \sqrt{\langle f, f \rangle_S}\), are equivalent: a sequence \((f_n \in \mathcal{H})\) converges with respect to \(\|\cdot\|\) if and only if it converges with respect to \(\|\cdot\|_S\). So the two norms induce the same topology on \(\mathcal{H}\), but the conjugation is different: calling \(\langle Xf, g \rangle = \langle f, X^\dagger g \rangle\) and \(\langle Xf, g \rangle_S = \langle f, X^\# g \rangle_S\), for all \(f, g \in \mathcal{H}\), then \(X^\dagger \neq X^\#\). Of course, we are implicitly assuming that all the mean values introduced here are well defined.
Now, to prove that $H$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_S$, it is enough to check
that $\langle Hf, g \rangle_S = \langle f, Hg \rangle_S$, for all $f, g \in \mathcal{H}$. But, since $H = SH_0S^{-1}$, we first observe
that $S^{-1}H = H_0S^{-1}$ and that $H^\dagger S^{-2} = S^{-2}H$. Then, the following holds:

$$\langle Hf, g \rangle_S = \langle S^{-2}Hf, g \rangle = \langle H^\dagger S^{-2}f, g \rangle = \langle S^{-2}f, Hg \rangle = \langle f, Hg \rangle_S,$$

which is what we had to prove. This shows that, in some cases, losing self-
adjointness of $H$ is not really dramatic and does not produce results which are far
away from what one could get using self-adjoint operators: it is enough to change
the topology of the Hilbert space! But this can be done, of course, if $H$ satisfies the
condition above, i.e., if $H$ is similar to a self-adjoint operator $H_0$. This is possible
if all the eigenvalues of $H$ are real. Otherwise, the situation is much more comp-
licated but can still be interesting for us. In fact, this will be the relevant case in
Chapters 4 and 6.

From what we have discussed so far, it is clear that working with a non-self-
adjoint Hamiltonian $H$ is a possible way to extend the standard dynamical approach
to quantum mechanics, see Sections 2.4 and 2.7, where $H$ is always taken to be self-
adjoint. A different possibility consists in introducing what we will call a rule to
the time evolution, i.e., some effect which cannot be easily modeled adding some
further term in the Hamiltonian $H$ of the system, or admitting that its eigenvalues
could be complex. This gives rise to a series of interesting results and considera-
tions, which will be discussed, at a rather general level, in this section, together
with some preliminary applications. More, and more interesting, applications will
be considered in the second part of this book. As it will appear clear from our treat-
ment, the idea of considering the dynamics as driven by some $H = H^\dagger$ and some
further rule not included in $H$ links the general framework of quantum dynamics
with the possibility that the dynamics may be periodically disturbed because of
some external (or internal) action on the system, whose effect cannot be (easily)
described by any extra term in the self-adjoint Hamiltonian. This is what we call
$(H, \rho)$-induced dynamics.

Let $\mathcal{S}$ be our physical system, and $Q_j$ ($j = 1, \ldots, M$) a set of $M$ commuting self-
adjoint operators with eigenvectors $\varphi_{\alpha_n}^{(j)}$ and eigenvalues $\alpha_{n_j}^{(j)}$ needed for a full de-
scription of $\mathcal{S}$. Then

$$[Q_j, Q_k] = Q_jQ_k - Q_kQ_j = 0, \quad Q_j = Q_j^\dagger, \quad Q_j \varphi_{n_j}^{(j)} = \alpha_{n_j}^{(j)} \varphi_{n_j}^{(j)}, \quad (2.60)$$

$j, k = 1, 2, \ldots, M, n_j = 1, 2, 3, \ldots, N_j$. Setting $\mathbf{n} = (n_1, n_2, \ldots, n_M)$, the vector

$$\varphi_{\mathbf{n}} = \varphi_{n_1}^{(1)} \otimes \varphi_{n_2}^{(2)} \otimes \cdots \otimes \varphi_{n_M}^{(M)} \quad (2.61)$$
2.8 The \((\mathbf{H}, \rho)\)-Induced Dynamics

represents an eigenstate of all the operators \(Q_j\), say

\[
Q_j \phi_n = \alpha^{(j)}_{n_j} \phi_n.
\]

(2.62)

The existence of a common eigenstate for all the operators \(Q_j\) is guaranteed by the fact that they all mutually commute. It is convenient to assume that these vectors are mutually orthogonal and normalized:

\[
\langle \phi_n, \phi_m \rangle = \delta_{n,m} = \prod_{j=1}^{M} \delta_{n_j,m_j}.
\]

(2.63)

This is the case, of course, if the eigenvalues are all nondegenerate, which we will assume here to simplify the treatment. However, it is not hard to extend our results to the case in which this is not true. The Hilbert space \(\mathcal{H}\) where \(S\) is defined as the closure of the linear span of all the vectors \(\phi_n\), which in this way turn out to be, by construction, an o.n. basis for \(\mathcal{H}\). Now, let \(H\) be a (quadratic, time-independent) self-adjoint Hamiltonian, describing the kinetic term and the interactions occurring in \(S\). Notice that \(H\), in general, does not commute with the \(Q_j\)'s. Therefore, these operators are not, in principle, integrals of motion. In absence of any other information, the wave function \(\Psi(t)\), describing \(S\) at time \(t\), evolves according to the Schrödinger equation \(i\dot{\Psi}(t) = H\Psi(t)\), where \(\Psi(0) = \Psi_0\) describes the initial status of \(S\). As we know, the formal solution of the Schrödinger equation is, since \(H\) does not depend explicitly on \(t\), \(\Psi(t) = e^{-iHt}\Psi(0) = e^{-iHt}\Psi_0\). We can now compute the mean value of each operator \(Q_j\) in the state \(\Psi(t)\),

\[
q_j(t) = \langle \Psi(t), Q_j \Psi(t) \rangle,
\]

(2.64)

and we use the various functions \(q_j(t)\) to define an \(M\)-dimensional time-dependent vector \(\mathbf{q}(t) = (q_1(t), q_2(t), \ldots, q_M(t))\). Of course, there is a dual possibility in which \(\Psi_0\) stays constant and \(Q_j\) evolves in time according to the Heisenberg recipe\(^{15}\): \(Q_j(t) = e^{iHt}Q_j e^{-iHt}\). And in a similar way we can construct again the same vector \(\mathbf{q}(t)\), with \(q_j(t) = \langle \Psi_0, Q_j(\Psi) \Psi_0 \rangle\).

We are now ready to introduce two different ways of extending the dynamics through the introduction of a specific rule.

In the first approach, the rule \(\rho\) is a map from \(\mathcal{H}\) to \(\mathcal{H}\). Its explicit action depends on the expression of \(\mathbf{q}(t)\) at particular instants \(k\tau\), where \(k \in \mathbb{N}\) and \(\tau \in \mathbb{R}^+\) is some fixed time which is relevant (for some reason) for \(S\). For instance, some external action could take place on \(S\) for \(t = \tau, 2\tau, 3\tau, \ldots\). In other words, according to what \(\mathbf{q}(\tau)\) looks like, \(\rho\) maps the input vector \(\Psi(\tau)\) (here \(k = 1\)) into a (possibly different) output vector \(\Psi_{\text{new}}\), and we write \(\rho(\Psi(\tau)) = \Psi_{\text{new}}\). This is not very different from what happens in scattering theory, where an incoming state, after the occurrence

\(^{15}\) We recall that \(H = H^\dagger\), here.
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of the scattering, is transformed into a different outgoing state [31]. However, a
big difference exists, which has to do with the fact that \( \Psi_{\text{new}} \) is connected, in some
way still to be specified, to the vector \( q(\tau) \), while in a scattering process the only
essential fact is that the scattering must take place. Now, the new vector \( \Psi_{\text{new}} \) can
be considered as the initial state of the system \( S \) and its time evolution is driven
again, for another time interval of length \( \tau \), by the Hamiltonian \( H \). At \( t = 2\tau \) the
rule is applied again, and a (second) new vector is introduced, which also evolves
according to \( H \). This procedure can be iterated several times. As an example, in
[5], the rule \( \rho \) is used to map an incoming state into one of the elements of a
fixed o.n. basis of \( \mathcal{H} \). We have used this rule to extend the game of life, see also
Chapter 8. Our approach has some aspects in common with the repeated quantum
measurements, according to which the state of a system is perturbed by a nontrivial
quantum measurement, and subsequent measurements of the same system reveal
the presence of disturbances if compared to the situation in which no previous
measurements were performed. In our case, the disturbance manifests with the
creation of the new state \( \Psi_{\text{new}} \), which is prepared as the result of the choice of the
rules. We also like to mention that the repeated action of the rule \( \rho \) in our approach
reminds of what is done in the context of the quantum Zeno effect (see [49] and
references therein), where some measures on a given quantum system are repeated
again and again, projecting the system on some particular state connected to the
result of the measure. We will say more on this connection later.

Another aspect of the \((H, \rho)\)-induced dynamics, which makes this idea interesting
and useful for applications, has to do with the fact that the rule does not
necessarily act to modify the state of the system, as discussed so far. A second
possibility exists, in which the rule \( \rho \) works on the space of the parameters of the
Hamiltonian \( H \), rather than on the wave function. In this case, if our self-adjoint
Hamiltonian involves \( p \) real parameters, the rule \( \rho \) is a map from \( \mathbb{R}^p \) to \( \mathbb{R}^p \) that,
again at particular instants \( k\tau \), depending on some check performed on the system,
modifies some of the values of these parameters. In this way the model adjusts
itself, by slightly modifying some details of \( H \), and is able to describe more com-
plex (and possibly more realistic) behaviors. As one can easily understand, this
approach introduces a sort of time dependence in \( H \), but this time dependence is, in
a sense, rather specific: \( H \) is, in fact, piecewise time-independent. This is, of course,
technically much simpler than when \( H \) presents a more complicated time depen-
dence, since in each interval \([k\tau, (k + 1)\tau]\) one can solve the Schrödinger (or the
Heisenberg) equation of motion as if \( H \) were time-independent, and then glue the
solutions at the various instants \( k\tau \), requiring some suitable regularity conditions.
We will discuss more on the two different kind of rules in Sections 2.8.1 and 2.8.2,
while some concrete applications of \((H, \rho)\)-induced dynamics will be considered
in Chapters 3 and 8.
2.8 The (H, ρ)-Induced Dynamics

2.8.1 The Rule ρ as a Map from H to H

We begin our analysis by discussing the case in which the rule ρ is a map from H to H. First, we observe that there exists a one-to-one correspondence between the vector \( \varphi_n \) in Equation (2.61) and its label \( n \): once we know \( n \), \( \varphi_n \) is clearly identified, and vice versa. Suppose now that, at time \( t = 0 \), the system \( S \) is in a state \( n^0 \) or, which is the same, \( S \) is described by the vector \( \varphi_{n^0} \). Then, once fixed a positive value of \( \tau \), this vector evolves in the time interval \( [0, \tau] \) according to the Schrödinger recipe: \( e^{-iHt} \varphi_{n^0} \). Here \( H = H^\dagger \) is the Hamiltonian of \( S \). Let us set

\[
\Psi(\tau^-) = \lim_{t \to \tau^-} e^{-iHt} \varphi_{n^0},
\]

where \( t \) converges to \( \tau \) from below.\(^{16}\) Now, at time \( t = \tau \), \( \rho \) is applied to \( \Psi(\tau^-) \), and the output of this action is a new vector which we assume here to be again an eigenstate of each operator \( Q_j \), but with different eigenvalues, \( \varphi_{n^1} \). In other words, \( \rho \) looks at the explicit expression of \( \Psi(\tau^-) \) and, according to its form, returns a new vector \( n^1 = (n^1_1, n^1_2, \ldots, n^1_M) \) and, as a consequence, a new vector \( \varphi_{n^1} \) of \( H: \varphi_{n^1} = \rho(\Psi(\tau^-)) \). Some preliminary examples of how \( \rho \) explicitly acts will be briefly presented in Section 2.9.

Remark: As we have already said, this is not the only possibility to set up a rule. In fact, other possibilities can also be considered. In particular, it is not necessary to stay inside the set of eigenstates of the \( Q_j \)'s: the rule could produce a vector \( \Phi \) which has nothing to do with any vector \( \varphi_k \) in Equation (2.61). The common aspect for all possible choices of \( \rho \) is that it behaves as a control over the system \( S \) and modifies some of its ingredients according to the result of this check. This aspect will be discussed further in Section 2.8.2.

Now, the procedure is iterated, taking \( \varphi_{n^1} \) as the initial vector and letting it evolve with \( H \) for another time interval of length \( \tau \). Hence we compute

\[
\Psi(2\tau^-) = \lim_{t \to \tau^-} e^{-iHt} \varphi_{n^1},
\]

and the new vector \( \varphi_{n^2} \) is deduced as the result of the action of rule \( \rho \) on \( \Psi(2\tau^-) \): \( \varphi_{n^2} = \rho(\Psi(2\tau^-)) \). In general, for all \( k \geq 1 \), we have

\[
\Psi(k\tau^-) = \lim_{t \to \tau^-} e^{-iHt} \varphi_{n^k}, \quad \text{and then} \quad \varphi_{n^k} = \rho(\Psi(k\tau^-)).
\]

Now, let \( X \) be a generic operator on \( H \), bounded or unbounded. In this latter case, we need to check that the various \( \varphi_{n^k} \) belong to the domain of \( X(t) = e^{iHt}Xe^{-iHt} \) for all \( t \in [0, \tau] \). In the applications we have considered so far, this condition is always satisfied.

\(^{16}\) We use here \( \tau^- \), \( 2\tau^- \), ..., as argument of \( \Psi \) to emphasize that, for instance, before \( \tau \) the time evolution is due only to \( H \). In fact, as we will see now, \( \rho \) really acts at \( t = \tau \).
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Definition 2.8.1 The sequence of functions

\[ x_{k+1}(t) := \langle \phi_n^k, X(t) \phi_n^k \rangle , \]  

for \( t \in [0, \tau] \) and \( k \in \mathbb{N}_0 \), is called the \((H, \rho)\)-induced dynamics of \( X \).

Some consequences of Definition 2.8.1 and some properties of the sequence \( X(\tau) = (x_1(\tau), x_2(\tau), ...) \) have been discussed in [5]. Moreover, from \( X(t) = (x_1(t), x_2(t), ...) \) it is possible to define a new function of time, which can be understood as the evolution of the observable \( X \) of \( S \) under the action of \( H \) and \( \rho \), in the following way:

\[
\tilde{X}(t) = \begin{cases} 
  x_1(t) & t \in [0, \tau], \\
  x_2(t-\tau) & t \in [\tau, 2\tau], \\
  x_3(t-2\tau) & t \in [2\tau, 3\tau], \\
  \ldots
\end{cases}
\]  

(2.69)

It is clear that \( \tilde{X}(t) \) may have discontinuities in \( k\tau \), for positive integers \( k \). The meaning of \( \tilde{X}(t) \) is the following: For \( t \in [0, \tau] \) \( \tilde{X}(t) \) is just the mean value of the operator \( X(t) \) in the state defined by \( \phi_n^0 \). For larger values of \( t \), and in particular if \( t \in [\tau, 2\tau] \), then \( \tilde{X}(t) = \langle \phi_n^1, X(t-\tau) \phi_n^1 \rangle \), which is again the mean value of \( X(t) \) shifted back in time, but in a state labeled by \( \phi_n^1 \), and so on. In other words, \( \tilde{X}(t) \) is constructed by considering a set of mean values of \( X(s) \) with \( s \in [0, \tau] \), but on different states, \( \phi_n^0, \phi_n^1, \phi_n^2 \) and so on, which are the vectors identified by \( \rho \) by its successive actions. This is because, in our interpretation of the rule, we assume that the time evolution of any observable starts again and again any time the rule acts on the system.

2.8.2 The Rule \( \rho \) as a Map in the Space of the Parameters of \( H \)

Formula (2.67) shows that the action of the rule considered in the previous section produces a change in the state of the system, from an input to an output vector – and, we would add, a very special change: from an eigenstate of the \( Q_j \)’s to another eigenstate of the same compatible operators. The other ingredients of \( S \), and in particular its Hamiltonian \( H \), are not modified by the rule and stay unchanged. As we have already mentioned, this is not the only possibility. We show now how a rule can also act on \( S \), changing some details of the Hamiltonian of \( S \), and in particular adjusting the values of some of the parameters of \( H \).

To be concrete, we assume \( S \) is a system described in terms of \( M \) fermionic (or bosonic) modes, and we suppose that its evolution is ruled by the following quadratic time-independent self-adjoint Hamiltonian:
2.8 \textit{The} \((\text{H}, \rho)\)-\textit{Induced Dynamics}

\begin{equation}
H = \sum_{j=1}^{M} \omega_j a_j^{\dagger} a_j + \sum_{j=1}^{M-1} \sum_{k=j+1}^{M} \lambda_{j,k} (a_j^{\dagger} a_k^{\dagger} + a_k a_j),
\end{equation}

(2.70)

involving the \(p = M(M + 1)/2\) real parameters (not necessarily all different from zero) \(\omega_j\) and \(\lambda_{j,k}\), and where \(a_j\) and \(a_j^{\dagger}\), \(j = 1, \ldots, M\), are annihilation and creation operators, respectively.

The time evolution of the lowering operators \(a_j\)'s can be deduced from the following linear system of ordinary differential equations, resulting from Equation (2.43) applied to the present situation:

\begin{equation}
\dot{a}_j(t) = i \left( -\omega_j a_j(t) + \sum_{k=1, k \neq j}^{M} \lambda_{j,k} a_k(t) \right), \quad j = 1, \ldots, M.
\end{equation}

(2.71)

Restricting ourselves to the fermionic case, these are \(M \times 2^M\) linear differential equations which should be solved, considering the initial conditions for the matrices representing the operators \(a_j\): \(a_j(0) = a_j\).

\textit{Remark:} It is worth observing that, if we replace the CAR with the CCR, then the set Equation (2.71) produces an infinite number of linear differential equations, for which finding an analytic solution is, in principle, much harder (if possible). However, cases when this can be done do exist, see [9] as an example, but in the analysis presented here we prefer to avoid difficulties related to working in infinite-dimensional Hilbert spaces. In fact, Equation (2.71) is already a set of linear differential equations. Hence, an analytic solution can be found. But this solution, easy to be formally deduced, could be very difficult to manage to get \textit{numbers}.

Going back to (2.71), and noticing that the system is linear, we may write it in the following compact form:

\begin{equation}
\dot{A}(t) = UA(t),
\end{equation}

(2.72)

where \(A(t) = (a_1(t), a_2(t), \ldots, a_M(t))^T\), and \(U\) is an \(M \times M\) constant matrix such that \(U_{j,i} = -i \omega_j\), \(U_{j,k} = i \lambda_{j,k}\), and each component of \(A\) is a \(2^M \times 2^M\) matrix. The formal solution is immediately deduced, namely

\begin{equation}
A(t) = e^{Ut} A(0) = V(t) A(0).
\end{equation}

(2.73)

Thus, if \(V_{\ell,m}(t)\) is the generic entry of matrix \(V(t)\), we have

\begin{equation}
a_\ell(t) = \sum_{k=1}^{M} V_{\ell,k}(t) a_k(0), \quad \ell = 1, \ldots, M.
\end{equation}

(2.74)
Now, we need to compute the mean value of the number operator for the $\ell$-th mode (which is intended to represent a physical quantity which is relevant for the description of $S$)

$$\hat{n}_\ell(t) = a_\ell^\dagger(t)a_\ell(t)$$ (2.75)

on an eigenvector $\varphi_{n_1,n_2,\ldots,n_M}$ of all the $\hat{n}_\ell(0)$,

$$\hat{n}_\ell(0)\varphi_{n_1,n_2,\ldots,n_M} = n_\ell\varphi_{n_1,n_2,\ldots,n_M}, \quad \ell = 1, 2, \ldots, M.$$ (2.76)

Simple computations show that

$$n_\ell(t) = \langle \varphi_{n_1,n_2,\ldots,n_M}, \hat{n}_\ell(t)\varphi_{n_1,n_2,\ldots,n_M} \rangle = \sum_{k=1}^M |V_{\ell,k}(t)|^2 n_k, \quad \ell = 1, \ldots, M.$$ (2.77)

This is what we get from a quadratic Hamiltonian such as the one in (2.70). Of course, even without entering into the detailed meaning of the model, which is not particularly relevant here, we see that the dynamics is related to the details of the matrix $V(t)$. But, since $V(t) = e^{Ut}$, these are related to the matrix elements of $U$ which, in turn, are fixed by the parameters of the Hamiltonian $H$. Hence, we may enrich the dynamics of $S$ by introducing a rule, acting several times at specific instants, and accounting for a sort of dependence of the parameters $\omega_j$ and $\lambda_{j,k}$ in Equation (2.70) on the state of the system at the instants in which the rule acts. In some sense, as we have already observed, the model adjusts itself as a consequence of its evolution. However, the way in which $S$ is adjusted is quite special: the rule modifies the value of the parameters of $H$, while keeping unchanged its analytical expression. In other words, once $\rho$ is applied, the values of $\omega_j$ and $\lambda_{j,k}$ in $H$ can be modified, but the expression of $H$ in Equation (2.70) cannot. This is not the only choice we can adopt. In fact, also the interacting part of $H$ could be modified after the action of the rule. However, we will not consider this possibility here, since going from a quadratic to a, say, cubic Hamiltonian would produce equations of motion for which we cannot possibly find any exact analytical solution.

Let us now go more in the details of how our approach works concretely. We start considering a self-adjoint, time-independent, quadratic Hamiltonian operator $H^{(0)}$, obtained by choosing a first set of values of the parameters in (2.70). Then, the time evolution of a certain observable $X$ driven by $H^{(0)}$ is

$$X(t) = e^{iH^{(0)}t}Xe^{-iH^{(0)}t},$$ (2.78)

and its mean value is

$$x(t) = \langle \varphi_{n_1,n_2,\ldots,n_M}, X(t)\varphi_{n_1,n_2,\ldots,n_M} \rangle$$ (2.79)
in a time interval of length $\tau > 0$ on a vector $\varphi_{n_1,n_2,...,n_M}$, which can again be
assumed to be a common eigenstate of the operators $Q_j$’s. Then, according to the
particular value of $x(\tau)$, we modify some of the parameters appearing in the definition
of $H^{(0)}$. In this way, we get a new Hamiltonian operator, $H^{(1)}$, having the same
functional form as $H^{(0)}$, but (in general) different values of (some of) its parame-
ters, and the system now evolves under the action of this new Hamiltonian for the
next time interval of length $\tau$. We are not restarting the time evolution of the sys-
tem from a new initial condition, but simply continuing to follow the evolution with
the only difference that for $t \in ]\tau, 2\tau]$ the Hamiltonian $H^{(1)}$ rules the process, rather
than $H^{(0)}$ and so on. Therefore, the rule now has to be thought of as a map from $\mathbb{R}^p$
into $\mathbb{R}^p$ acting on the space of the parameters involved in the Hamiltonian, and the
global evolution is governed by a sequence of analogous Hamiltonian operators,
with the same analytic expression but with different value of the parameters. In a
few words, these parameters of $H$ become nothing but piecewise (in time) constant
functions, whose values are not fixed a priori, but decided by the evolution itself,
and by the rule.

In general, in the $k$-th subinterval $[k\tau, (k+1)\tau]$, the dynamics will be driven
by the Hamiltonian $H^{(k)}$. Hence, the global dynamics arises from the sequence of Hamiltonians

\[H^{(0)} \xrightarrow{\tau} H^{(1)} \xrightarrow{\tau} H^{(2)} \xrightarrow{\tau} \ldots\]  \hspace{1cm} (2.80)

In every subinterval we therefore have a system like

\[\dot{A}(t) = U^{(k)}A(t), \quad t \in [k\tau, (k+1)\tau],\]  \hspace{1cm} (2.81)

$k \geq 0$, where $U^{(k)}$ is constructed out of the parameters of $H^{(k)}$. The analytical solu-
tion of this system is not difficult, due to the fact that each equation is linear. The
complete evolution is obtained by gluing the local evolutions.

Some numerical aspects of our procedure are discussed in [6], which we refer to
for more details on this particular aspect of the strategy proposed here.

2.9 A Two-Mode System

In Section 2.7 we have discussed how some observable of a system $S$ can reach
an equilibrium. In particular, we have shown that this is not possible, within the
approach discussed in Section 2.4, if $S$ lives in a finite-dimensional Hilbert space
and if its dynamics is driven by a self-adjoint Hamiltonian. On the other hand, in
presence of an (infinitely extended) environment, decay processes can be modeled
and stable asymptotic values can be found. Also, as discussed in Section 2.7, it is
possible to use very specific non-self-adjoint Hamiltonians with the same purpose,
but this approach should be pursued *cum grano salis*, since several complications
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arise. To avoid these difficulties, we are going to show how the \((H, \rho)\)-induced dynamics can be useful to produce, after some time, a nontrivial equilibrium for the time evolution of the mean value of some observables of the system. This will now be shown by considering a very simple toy model considered first in [7].

Let us consider a system \(S\), having two (fermionic) degrees of freedom, and described by the Hamiltonian

\[
H = H_0 + \lambda H_1, \quad H_0 = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2, \quad H_1 = a_1^\dagger a_2 + a_2^\dagger a_1, \tag{2.82}
\]

where \(\omega_j\) and \(\lambda\) are real (and positive) quantities in order to ensure that \(H\) is self-adjoint. The operators \(a_j\) and \(a_j^\dagger\) are assumed to satisfy the following CAR:

\[
\{a_i, a_j^\dagger\} = \delta_{ij} \mathbb{I}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0, \tag{2.83}
\]

\(i, j = 1, 2\), where, as usual, \(\mathbb{I}\) is the identity operator. Of course, when \(\lambda = 0\), the two agents of \(S\) are not interacting. Notice that \(H\) in (2.82) is a fermionic version of the Hamiltonian for \(S \cup \tilde{S}\) introduced in Section 2.7, for which we proved that no equilibrium exists at all.

The eigenstates of the number operators \(\hat{n}_j := a_j^\dagger a_j\) are easily obtained: if \(\varphi_{0,0}\) is the ground vector of \(S\), \(a_1 \varphi_{0,0} = a_2 \varphi_{0,0} = 0\), an o.n. basis of the four-dimensional Hilbert space \(\mathcal{H}\) of \(S\) is given by the following vectors:

\[
\varphi_{0,0}, \quad \varphi_{0,1} := a_1^\dagger \varphi_{0,0}, \quad \varphi_{0,2} := a_2^\dagger \varphi_{0,0}, \quad \varphi_{1,1} := a_1^\dagger a_2^\dagger \varphi_{0,0}. \tag{2.84}
\]

We have

\[
\hat{n}_1 \varphi_{n_1, n_2} = n_1 \varphi_{n_1, n_2}, \quad \hat{n}_2 \varphi_{n_1, n_2} = n_2 \varphi_{n_1, n_2}. \tag{2.85}
\]

The equations of motion for the annihilation operators \(a_j(t)\) are

\[
\dot{a}_1(t) = -i\omega_1 a_1(t) - i\lambda a_2(t), \quad \dot{a}_2(t) = -i\omega_2 a_2(t) - i\lambda a_1(t), \tag{2.86}
\]

which are formally identical to those in Equation (2.47). They can be solved imposing the initial conditions \(a_1(0) = a_1\) and \(a_2(0) = a_2\), and the solution is

\[
a_1(t) = \frac{1}{2\delta} (a_1 ((\omega_1 - \omega_2)\Phi_-(t) + \delta \Phi_+(t)) + 2\lambda a_2 \Phi_-(t)), \tag{2.87}
\]

\[
a_2(t) = \frac{1}{2\delta} (a_2 (-(\omega_1 - \omega_2)\Phi_-(t) + \delta \Phi_+(t)) + 2\lambda a_1 \Phi_-(t)),
\]

where

\[
\delta = \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2},
\]

\[
\Phi_+(t) = 2 \exp \left( -\frac{it(\omega_1 + \omega_2)}{2} \right) \cos \left( \frac{\delta t}{2} \right), \tag{2.88}
\]

\[
\Phi_-(t) = -2i \exp \left( -\frac{it(\omega_1 + \omega_2)}{2} \right) \sin \left( \frac{\delta t}{2} \right).
\]
Then, the functions \( n_j(t) := \langle \varphi_{n_1, n_2}, \hat{n}_j(t) \varphi_{n_1, n_2} \rangle \) are

\[
\begin{align*}
n_1(t) &= \frac{n_1(\omega_1 - \omega_2)^2}{\delta^2} + \frac{4\lambda^2}{\delta^2} \left( n_1 \cos^2 \left( \frac{\delta t}{2} \right) + n_2 \sin^2 \left( \frac{\delta t}{2} \right) \right), \\
n_2(t) &= \frac{n_2(\omega_1 - \omega_2)^2}{\delta^2} + \frac{4\lambda^2}{\delta^2} \left( n_2 \cos^2 \left( \frac{\delta t}{2} \right) + n_1 \sin^2 \left( \frac{\delta t}{2} \right) \right),
\end{align*}
\]

which oscillate in time as deduced for their bosonic counterparts in (2.47).

These functions could be interpreted, in agreement with other similar applications, as the densities of two species, \( S_1 \) and \( S_2 \), interacting as in Equation (2.82) in a given (small) region.\(^\text{17}\) The interaction Hamiltonian \( H_I \) in Equation (2.82) describes a sort of predator-prey mechanism, and this is reflected by the solutions in Equation (2.89), which show how the two densities, because of the interaction between \( S_1 \) and \( S_2 \), oscillate in the interval \([0, 1]\). Otherwise, if \( \lambda = 0 \), \( n_j(t) = n_j \); the densities stay constant, and nothing interesting happens in \( S \). We observe that the formulas in (2.89) automatically imply that \( n_1(t) + n_2(t) = n_1 + n_2 \), independently of \( t \) and \( \lambda \): the oscillations are such that they sum up to zero. We refer to [7] for the role of this Hamiltonian in modeling migration, which is achieved considering a 2D version of the \( H \) in Equation (2.82), with an additional term responsible for the diffusion of the two species along a rectangular lattice. Here, we exploit the possibility of getting some limiting values for \( n_1(t) \) and \( n_2(t) \), or for some functions naturally related to these, for large values of \( t \), when \( \lambda \neq 0 \).

The first trivial remark is that the functions \( n_1(t) \) and \( n_2(t) \) in Equation (2.89) do not admit any asymptotic limit, except when \( n_1 = n_2 \) (or when \( \lambda = 0 \), which is excluded here). In this case, clearly, \( n_1(t) = n_2(t) = n_1 = n_2 \). On the other hand, if \( n_1 \neq n_2 \), then both \( n_1(t) \) and \( n_2(t) \) always oscillate in time. This is not surprising since we have already proved that, if \( S \) is a system living in a finite-dimensional Hilbert space, and if its dynamics is driven by a time-independent, self-adjoint, Hamiltonian \( \tilde{H} \), then its evolution is necessarily periodic or quasi-periodic. Then the conclusion is that, if we are interested in getting (nontrivial) asymptotic values, we need to modify the way in which the time evolution is taken, at least for large \( t \). This is exactly what we will do next, by introducing a rule \( \rho \) in the analysis: this is much simpler, in principle, than considering the interaction of the system with a reservoir, as proposed in Section 2.7, and this simplicity makes the concept of the \((H, \rho)\)-induced dynamics interesting, for practical purposes. Also, we will discuss later that \( \rho \) has, quite often, a specific physical interpretation, which makes the model more realistic.

\(^{17}\) Other interpretations are also possible. For instance, they can play the role of the decision functions of two interacting agents, trying to decide on some binary question.
2.9.1 The Rule and the Existence of an Asymptotic Value

In order to see how a rule can be useful for us, we first rewrite Equation (2.89) as

\[ N(t) = T_t N(0), \quad (2.90) \]

where

\[ N(t) = \begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix}, \quad T_t = \frac{1}{\delta^2} \begin{pmatrix} \delta^2 - 4\lambda^2 \sin^2 \left( \frac{\delta t}{2} \right) & 4\lambda^2 \sin^2 \left( \frac{\delta t}{2} \right) \\ 4\lambda^2 \sin^2 \left( \frac{\delta t}{2} \right) & \delta^2 - 4\lambda^2 \sin^2 \left( \frac{\delta t}{2} \right) \end{pmatrix}. \quad (2.91) \]

Of course, the components of \( N(t) \) return the expressions of \( n_1(t) \) and \( n_2(t) \) for all times. Let us now see what happens if we insert a certain rule \( \rho \) in the time evolution of the system.

Here, we can think of \( \rho \) as a measure of \( n_1(t) \) and \( n_2(t) \) repeated at time \( \tau \), \( 2\tau \), \( 3\tau \), ... We know that performing a measure on a quantum system is a delicate operation, which usually modifies the system itself [49]. Therefore, there is no reason a priori to imagine that the result of a measure at time \( k\tau \) (after having measured the system at time \( \tau \), \( 2\tau \), ...) would be exactly the same as the result of a single measure performed on the system at time \( k\tau \). This is, in fact, what we are going to show next.

In the first case, the first measure at \( t = \tau \) gives \( N_1(\tau) := N(\tau) = T_\tau N(0) \), which is just the effect of a single application of our rule. Then, according to what discussed in Section 2.8.1, we let the system evolve out of this new initial condition \( N_1(\tau) \) for another time interval: \( N_2(\tau) := T_\tau N_1(\tau) = T_{2\tau} N(0) \), and so on. It is quite natural to call the rule \( \rho \) considered here a \textit{stop and go} rule: apparently, in fact, \( \rho \) just stops the time evolution at \( \tau \), \( 2\tau \), \( 3\tau \) and so on, and then lets the time evolution start again. Repeating this procedure again and again we produce a sequence of results

\[ N_k(\tau) = T_{k\tau} N(0), \quad (2.92) \]

for \( k \geq 1 \). On the other hand, if we perform a single measure at time \( t = k\tau \), the result is \( N(k\tau) = T_{k\tau} N(0) \), see Equations (2.90) and (2.91).

We want to show that \( N_k(\tau) \), when \( k \) diverges, can converge to some nontrivial limit, at least under suitable conditions, while this is not possible for \( N(k\tau) \): this vector never converges when \( k \) diverges, except for some trivial cases which are not interesting for us. In order to compute \( N_k(\tau) \), and its limit for \( k \) diverging, we observe that \( T_t \) in (2.91) is a self-adjoint matrix, so it can be easily diagonalized. In particular, we get

\[ U^{-1}T_t U = \begin{pmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{pmatrix} =: \Lambda_t, \quad (2.93) \]
where
\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \lambda_1(t) = 1, \quad \lambda_2(t) = \frac{1}{\delta^2} \left( \delta^2 - 8\lambda^2 \sin^2 \left( \frac{\delta t}{2} \right) \right). \] (2.94)

Then
\[ T_k^\tau = U A_k^\tau U^{-1} = U \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2^k(\tau) \end{pmatrix} U^{-1}, \] (2.95)

so that \( N_k(\tau) = T_k^\tau N(0) \) can converge if \( \lambda_2^k(\tau) \) does converge when \( k \) diverges. This is what happens whenever the parameters \( \delta, \tau \) and \( \lambda \) satisfy the following inequalities:
\[ 0 < 8\lambda^2 \sin^2 \left( \frac{\delta \tau}{2} \right) < \delta^2. \] (2.96)

In fact, when this is true, \( \lambda_2(\tau) \in ]0, 1[ \), and, therefore, \( \lim_{k \to \infty} \lambda_2^k(\tau) = 0 \). Hence,
\[ \lim_{k \to \infty} N_k(\tau) = \begin{pmatrix} n_1(0) \\ 0 \end{pmatrix}, \] (2.97)

which clearly shows that a nontrivial equilibrium can be reached in this case. However, if the parameters do not satisfy (2.96), the asymptotic behavior of \( N_k(\tau) \) can be something completely different. In fact, taking, for instance, \( \tau = \frac{\pi}{2} \) and \( \lambda = \sqrt{\frac{3}{8}} \delta \), and then fixing \( \delta = 1 \) for simplicity, we deduce that \( \lambda_2(\tau) = -2 \), so that \( \lim_{\tau \to \infty} |\lambda_2(\tau)|^k = \infty \); thus, it is evident that the role of the parameters of \( H \) is in fact essential for the existence of some finite asymptotic value.

The conclusion of the analysis of this simple example is the following: even in presence of a self-adjoint Hamiltonian, a simple two-modes fermionic system admits a nontrivial asymptotic limit for a large range of values of the parameters of the model, at least if a stop and go rule is assumed. However, the same rule can also produce a non-converging dynamics for other choices of the parameters. Of course, if we adopt different rules we may likely obtain different results. Other examples of rules, in different contexts, will be discussed in the second part of the book.

Remarks: (1) As already observed, the rule could be seen as an alternative, and somehow simplified, way to deal with Hamiltonians with some explicit time dependence. The fact that the \( (H, \rho) \)-induced dynamics is technically simpler than the use of a Hamiltonian \( H(t) \) follows from the fact that, in each interval \( [k\tau, (k + 1)\tau] \), the standard formulas for time-independent Hamiltonians can be used, and the only difficulty is to match the solutions for \( t = (k + 1)\tau, k \geq 0 \).
(2) We stress once more that what it is discussed here is not very different from what is observed in the quantum Zeno effect, in which repeated measures are performed on a quantum system. We refer to [49] for a detailed review on this effect. However, it is not even so close! Differences are many, especially when we look at the rule as a map in the space of parameters of the Hamiltonian. In this case, in fact, we are not really modifying the state of the system, as it happens when we perform a measure (or a repeated measure) on a quantum system, but we are modifying the dynamics itself, changing the values of the parameters needed to fix the Hamiltonian. In principle, there is no reason not to modify also the analytic expression of the Hamiltonian according to the rule. The main reason why we do not consider this possibility is because, in this way, we get in general different differential equations in each time interval, and this will make the solution of the dynamics quite complicated.