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A Compositional Shuffle Conjecture Specifying Touch Points of the Dyck Path

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Abstract. We introduce a q,t-enumeration of Dyck paths that are forced to touch the main diagonal at specific points and forbidden to touch elsewhere and conjecture that it describes the action of the Macdonald theory ∇ operator applied to a Hall–Littlewood polynomial. Our conjecture refines several earlier conjectures concerning the space of diagonal harmonics including the "shuffle conjecture" (Duke J. Math. **126** (2005), pp. 195–232) for $\nabla e_n[X]$. We bring to light that certain generalized Hall–Littlewood polynomials indexed by compositions are the building blocks for the algebraic combinatorial theory of q,t-Catalan sequences, and we prove a number of identities involving these functions.

1 Introduction

Our study concerns the combinatorics behind the character of the space of diagonal harmonics DH_n and identities involving Macdonald polynomials that can be used to form expressions for this character. At the root of this theory is a linear operator ∇, introduced in [2], under which the modified Macdonald polynomials $\widetilde{H}_\mu[X;q,t]$ are eigenfunctions. In [11], Haiman proved that the Frobenius image of the character of DH_n equals ∇e_n. This gives an explicit expression involving rational functions in q,t for the multiplicity of an irreducible indexed by a partition λ in the character of DH_n.

An important open problem in this area is the "shuffle conjecture" of [9], which asserts that the coefficient of m_λ in ∇e_n simplifies to a q,t statistic on lattice paths. A major breakthrough in this direction was made with the conjectured combinatorial formula of [6] for the coefficient of $m_{1^n} = s_{1^n}$ in ∇e_n. In this case, the coefficient is a bi-graded version of the sign character, and it is called the q,t-Catalan $C_n(q,t)$, since it reduces to the n-th Catalan number when $q = t = 1$. The combinatorial formula for $C_n(q,t)$ was proved in [3,4], and pursuant work [7] also settled the shuffle conjecture for partitions of hook-shape. However, the general case remains a mystery.

An unrelated study of Macdonald polynomials [14] led to the discovery of a new family of symmetric functions called k-Schur functions $s_\lambda^{(k)}[X;t]$, which were conjectured to refine the special combinatorial properties held by Schur functions. The k-Schur functions have a number of conjecturally equivalent characterizations, and it has now been established in [13,17] that those introduced in [16] refine combinatorial, geometric, and representation theoretic aspects of Schur theory. This prompted Bergeron, Descouens, and Zabrocki to explore the role of k-Schur functions in the

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q,t-Catalan theory. To this end, they conjectured in [1] that the coefficient of s_{1^n} in $\nabla s_{1^n}^{(k)}[X;t]$ is a positive polynomial in q,t and proved their conjecture for the case $t=1$.

Our work here was initially motivated by a desire to find a combinatorial description for this coefficient in general, ideally in terms of a q,t-statistic on lattice paths as with the q,t-Catalans. We found such a description, but, more remarkably, this led us to discover that a natural setting for the combinatorial theory of DH_n is created by applying ∇ to the general set of Hall–Littlewood polynomials indexed by compositions. To be precise, it was proved in [15] that the k-Schur function $s_{1^n}^{(k)}[X;t]$ is merely a certain Hall–Littlewood polynomial. This led us to study ∇ on a Hall–Littlewood polynomial indexed by any partition λ. But in fact, our work carries through to the family of polynomials $C_\alpha[X;q]$, for any composition α, defined in terms of operators similar to Jing operators.

A key component in the proof of the q,t-Catalan conjecture [4] is the use of symmetric functions $E_{n,k}[X;q]$ that decompose e_n into pieces that remain positive under the action of ∇. We have discovered that the $C_\alpha[X;q]$ can be used as building blocks in the q,t-Catalan theory that decompose the $E_{n,k}[X;q]$ into finer pieces, still positive under the action of ∇. Our conjectures on these building blocks thus refine earlier conjectures involving $E_{n,k}[X;q]$, the conjectures in [1], the shuffle conjecture, and the conjectures in [2] asserting that ∇ applied to Hall–Littlewood functions have q,t-positive Schur coefficients. Loehr and Warrington [18] introduced an intricate conjecture for the combinatorics of ∇ applied to a Schur function s_λ. Our conjecture is extremely simple, describes the action of ∇ on a larger set of symmetric functions than just a basis, and refines the conjecture of Loehr and Warrington when ∇ acts on the Schur function $s_{(n-k,1^k)}$ [18, Conjecture 3] as explained at the end of Section 4.

Garsia, Xin, and Zabrocki [5], using a combinatorial argument of A. Hicks, have now proven our generalized q,t-Catalan conjecture and expanded the result giving a "compositional q,t-Schröder" theorem.

2 Definitions and Notation

2.1 Combinatorics

A Dyck path is a lattice path in the first quadrant of the xy-plane from the point $(0,0)$ to the point (n,n) with steps $+(0,1)$ and $+(1,0)$ that stays above the line $x=y$. For a Dyck path D, the cells in the i-th row are those unit squares in the xy-plane that are below the path and fully above the line $x=y$ and whose NE corner has a y coordinate of i. The set of Dyck paths from $(0,0)$ to (n,n) will be denoted DP^n, and the number of paths in this set is well known to be the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

For a Dyck path D, let $a_i = a_i(D)$ equal the number of cells in the i^{th} row of D. It is always true that $a_1 = 0$ and $0 \le a_{i+1} \le a_i + 1$. We define the arm sequence $\operatorname{arm}(D) = (a_1, a_2, \ldots, a_n)$ and note that this completely determines D. We consider

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two statistics (non-negative integers) on Dyck paths. The *area* statistic is the number of whole cells that are below the path and above the diagonal, or

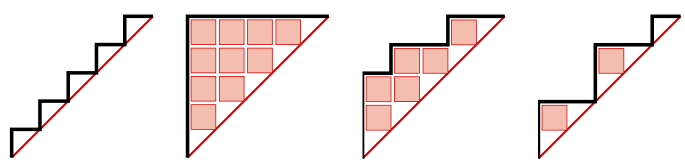
$$\operatorname{area}(D) = \sum_{i=1}^{n} a_i .$$

The *dinv* statistic is defined as

$$\operatorname{dinv}(D) = \sum_{1 \le i < j \le n} \chi\big(a_i - a_j \in \{0, 1\}\big) ,$$

where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

Remark 2.1 The Dyck path D with arm sequence (0^n) has $\operatorname{area}(D) = 0$ and $\operatorname{dinv}(D) = \binom{n}{2}$. The Dyck path D' with arm sequence $(0, 1, 2, 3, \dots, n-1)$ has $\operatorname{area}(D') = \binom{n}{2}$ and $\operatorname{dinv}(D') = 0$.



arm sequence	(0,0,0,0,0)	(0,1,2,3,4)	(0,1,2,2,1)	(0,1,0,1,0)
area	0	10	6	2
dinv	10	0	4	7

Remark 2.2 The original proof of the combinatorial interpretation of the q, t-Catalan polynomial was stated in terms of a third statistic *bounce*(D). Since we are able to formulate our results more cleanly in terms of the $\operatorname{dinv}(D)$ statistic, we choose to state all the results in this paper in terms of the $\operatorname{dinv}(D)$ statistic; however, the reference [8, p. 50] describes an automorphism ϕ on DP^n such that $\operatorname{area}(\phi(D)) = \operatorname{bounce}(D)$ and $\operatorname{dinv}(\phi(D)) = \operatorname{area}(D)$.

We make use of a partial order on Dyck paths; namely $D_1 \le D_2$ if $\operatorname{arm}(D_1) \le \operatorname{arm}(D_2)$, component-wise. In this case we say that D_1 is "below" D_2 because D_1 will not cross D_2 and is hence weakly "between" D_2 and the diagonal.

A composition α of n, denoted $\alpha \models n$, is an integer sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$ with $\alpha_i \ge 1$ and where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_r = n$. The length of α is $\ell(\alpha) = r$. We shall also use $\overleftarrow{\alpha} = (\alpha_{\ell(\alpha)}, \alpha_{\ell(\alpha)-1}, \dots, \alpha_2, \alpha_1)$. For any composition α, we define

$$n(\alpha) = \sum_{i=1}^{\ell(\alpha)} (i-1)\alpha_i .$$

The descent set of a composition α is defined to be

$$\mathrm{Des}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell(\alpha)-1}\}.$$

There is a common partial order defined on compositions $\alpha, \beta \models n$ by letting $\alpha \leq \beta$ when α is "finer" than β, *i.e.,* $\mathrm{Des}(\beta) \subseteq \mathrm{Des}(\alpha)$. If α is a composition of n, $DP(\alpha)$ represents the Dyck path consisting of α_1 steps in the North $(0,1)$ direction followed by α_1 steps in the East $(1,0)$ direction, α_2 $(0,1)$ steps followed by α_2 $(1,0)$ steps, etc.

A partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a non-increasing sequence of positive integers. When λ is a partition of n, denoted $\lambda \vdash n$, $|\lambda| = \sum \lambda_i = n$. The length of λ is $\ell(\lambda) = r$. Given a partition λ, we set

$$m(\lambda) = \left(m_1(\lambda), m_2(\lambda), m_3(\lambda), \ldots, m_{|\lambda|}(\lambda)\right),$$

where the numbers $m_i(\lambda)$ represent the number of parts of size i in λ. The conjugate of a partition λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$, where λ'_i is the number of parts of λ that are at least i. Partitions are generally considered to be compositions with parts arranged in non-increasing order. Hence, notions defined on compositions apply to partitions as well. Generally, we will use the symbols α, β, γ to represent compositions and λ, μ, ν to represent partitions.

For a given Dyck path D, $\mathrm{touch}(D)$ denotes the composition

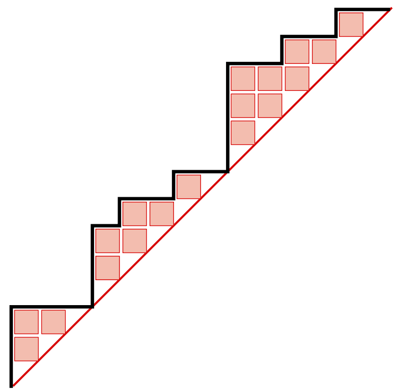
$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{\ell(\gamma)}) \models n$$

that specifies in which rows the Dyck path "touches" the diagonal. That is, for $\mathrm{arm}(D) = (a_1, a_2, \ldots, a_n)$, $a_k = 0$ if and only if $k = 1$ or $k - 1 \in \mathrm{Des}(\gamma)$. The title of this paper comes from the notion of the touch composition. By requiring that $\mathrm{touch}(D) = \alpha$ for a fixed composition α, we have specified that the Dyck path will touch the diagonal in rows $1, 1+\alpha_1, 1+\alpha_1+\alpha_2, \ldots, 1+\alpha_1+\alpha_2+\cdots+\alpha_{\ell(\alpha)-1}$ and is forbidden to touch the diagonal in the other rows. Note that under this definition, we view all paths as touching the diagonal in row 1, but none touching in row $n+1$, and we say the path touches the diagonal $\ell(\alpha)$ times. The partial order on compositions is consistent with the partial order on Dyck paths in the sense that if D_1 and D_2 are Dyck paths such that $D_1 \leq D_2$, then $\mathrm{touch}(D_1) \leq \mathrm{touch}(D_2)$.

Using these notions, we introduce a new statistic $\mathrm{doff}_\alpha(D)$ for a given Dyck path D with $\mathrm{touch}(D) \leq \alpha$. If $\mathrm{arm}(D) = (a_1, a_2, \ldots, a_n)$, let r_1 be the number of rows such that $a_i = 0$ for $1 \leq i \leq \alpha_1$, r_2 be the number of rows such that $a_i = 0$ for $\alpha_1 < i \leq \alpha_1 + \alpha_2$, and more generally $r_k = \#\{i : a_i = 0 \text{ and } \sum_{j=1}^{k-1} \alpha_j < i \leq \sum_{j=1}^{k} \alpha_j\}$. We then set

$$\mathrm{doff}_\alpha(D) = \sum_{k=1}^{\ell(\alpha)} (\ell(\alpha) - k) r_k.$$

Remark 2.3 If $\mathrm{arm}(D) = (0,1,2,0,1,2,2,1,0,1,2,3,2,1)$, then $\mathrm{touch}(D) = (3,5,6)$. Taking $\alpha = (8,6)$, we have $\mathrm{doff}_{(8,6)}(D) = 2$.



The only Dyck path with $\mathrm{touch}(D) = (1^n)$ has $\mathrm{arm}(D) = (0^n)$. There are C_{n-1} Dyck paths with $\mathrm{touch}(D) = (n)$, and, more generally, there are $\prod_{i=1}^{\ell(\alpha)} C_{\alpha_i - 1}$ Dyck paths such that $\mathrm{touch}(D) = \alpha$. Note that if D and E are two Dyck paths with $\mathrm{touch}(D) = \mathrm{touch}(E) \leq \alpha$, then $\mathrm{doff}_\alpha(D) = \mathrm{doff}_\alpha(E)$.

The results we have mentioned so far are stated in terms of Dyck paths, but we will require the notion of parking functions to state the generalization of the shuffle conjecture. For a Dyck path D in DP^n with $\mathrm{arm}(D) = (a_1, \dots, a_n)$, let $\mathcal{WP}_D$ be the set of words of length n in the alphabet $\{1, 2, \dots, n\}$ such that $w_i < w_{i+1}$ if $a_i < a_{i+1}$. We use the notation x^w to denote the monomial $x_{w_1} x_{w_2} \cdots x_{w_n}$. We also define an extension of the dinv statistic for words in $\mathcal{WP}_D$ by setting

$$\begin{aligned}\mathrm{dinv}(w) = |\{(i,j) : 1 \leq i < j \leq n, a_i = a_j \text{ and } w_i < w_j\}| \\ + |\{(i,j) : 1 \leq i < j \leq n, a_i = a_j + 1 \text{ and } w_i > w_j\}|.\end{aligned}$$

2.2 Symmetric Functions

Let X represent a sum of an infinite set of variables $X = x_1 + x_2 + x_3 + \cdots$ considered as elements of the ring of polynomial series in an infinite number of variables of bounded degree. For $r > 0$, let p_r represent a linear and algebraic morphism that acts on polynomial series by $p_r[x] = x^r$. That is for two polynomial series of bounded degree A and B,

$$\begin{aligned} p_r[A+B] &= p_r[A] + p_r[B], \\ p_r[A-B] &= p_r[A] - p_r[B], \\ p_r[AB] &= p_r[A] p_r[B] \end{aligned}$$

and in particular, $p_r[X] = x_1^r + x_2^r + x_3^r + \cdots$ represents the r-th power sum in the variables $\{x_1, x_2, x_3, \dots\}$. The ring of symmetric functions over the field F is defined to be the polynomial ring

$$\Lambda = F\big[\, p_1[X], p_2[X], p_3[X], \dots \big].$$

For our purposes, we choose the field F to be the ring of rational power series in the variables q, t, u, z over $\mathbb{Q}$ where each of the parameters q, t, u, z has the property that $p_r[a] = a^r$ for each $a = q, t, z, u$.

Generally our symmetric functions f will be considered as polynomials in the elements p_r so the notation $f[A]$ represents f with each p_r replaced by $p_r[A]$. The degree of p_r is r and the degree of a symmetric function f is determined by the degree of the monomials in the power sums that appear in f. Following the notation of Macdonald [19], we have the power sum basis $p_\lambda[X]$, Schur basis $s_\lambda[X]$, homogeneous basis $h_\lambda[X]$, and elementary basis $e_\lambda[X]$.

In the expressions of variables it is useful to have a special symbol ϵ that will represent a value of negative one but behaves differently than a negative symbol. If f is of homogeneous degree r,

$$f[\epsilon X] = (-1)^r f[X], \quad f[-\epsilon X] = \omega(f[X]),$$

where ω is an involution on symmetric functions such that

$$\omega(p_\lambda[X]) = (-1)^{|\lambda|+\ell(\lambda)} p_\lambda[X], \ \omega(e_n[X]) = h_n[X], \ \text{ and } \ \omega(s_\lambda[X]) = s_{\lambda'}[X].$$

We will also make use of the standard Hall scalar product, which is defined by

$$\big\langle p_\lambda[X], p_\mu[X]/z_\mu \big\rangle = \big\langle s_\lambda[X], s_\mu[X] \big\rangle = \chi(\lambda = \mu),$$

where $z_\mu = \prod_{i\geq 1} m_i(\mu)! i^{m_i(\mu)}$.

For any symmetric function f, multiplication by f is an operation on symmetric functions that raises the degree of the symmetric function by $\deg(f)$. If we define $f^\perp$ to be the operation that is dual to multiplication in the sense that

$$\big\langle f^\perp g, h \big\rangle = \langle g, fh \rangle,$$

then $f^\perp$ is an operator that lowers the degree of the symmetric function by $\deg(f)$. It is not difficult to show that

$$f[X+z] = \sum_{k\geq 0} z^k (h_k^\perp f)[X],$$

$$f[X-z] = \sum_{k\geq 0} (-z)^k (e_k^\perp f)[X].$$

In addition we will refer to the form of the Macdonald basis $\widetilde{H}_\lambda[X; q, t]$ that is relevant to the study of the $n!$ Theorem [10] and the q, t-Catalan numbers. The relation of this basis to the integral form $J_\mu[X; q, t]$ of [19] is

$$\widetilde{H}_\mu[X; q, t] = t^{n(\mu)} J_\mu\Big[\frac{X}{1-1/t}; q, 1/t\Big].$$

It is also characterized as the unique basis such that

$$\big\langle \widetilde{H}_\mu\big[X(1-1/t); q, t\big], \widetilde{H}_\lambda\big[X(1-q); q, t\big] \big\rangle = 0$$

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if $\lambda \neq \mu$ and $\langle \widetilde{H}_\mu[X;q,t], h_n[X] \rangle = 1$.

We are particularly interested in the Hall–Littlewood symmetric functions. Following the notation of Macdonald we define the functions $Q'_\lambda[X;q]$ to be the basis of the symmetric functions that satisfy

$$\Big\langle Q'_\lambda\big[X(1-q);q\big]\, , Q'_\mu[X;q] \Big\rangle = 0$$

if $\lambda \neq \mu$ and $\langle Q'_\lambda[X;q], h_n[X] \rangle = q^{n(\lambda)}$. Relating the definitions of the Hall–Littlewood and Macdonald symmetric functions, we note that

$$Q'_\lambda[X;q] = \widetilde{H}_\lambda[X;0,1/q]q^{n(\lambda)} = \sum_\lambda K_{\lambda\mu}(q) s_\lambda[X].$$

The operator ∇ was introduced in [2] and is defined by

$$\nabla \widetilde{H}_\lambda[X;q,t] = t^{n(\lambda)} q^{n(\lambda')} \widetilde{H}_\lambda[X;q,t].$$

This operator has been fundamental to the study of the q,t-combinatorial identities associated with DH_n and Macdonald polynomials. Its definition is chosen so that

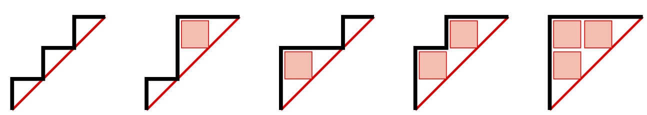
$$\Big\langle \nabla\big(e_n[X]\big)\, , e_n[X] \Big\rangle = C_n(q,t),$$

where $C_n(q,t)$ is the q,t-Catalan polynomial. References [3, 6, 8] showed that

$$C_n(q,t) = \sum_{D \in DP^n} t^{\mathrm{area}(D)} q^{\mathrm{dinv}(D)} \tag{2.1}$$

with the sum over all Dyck paths of length n.

Remark 2.4 A small example is $C_3(q,t) = q^3 + qt + q^2t + qt^2 + t^3$, whose terms can be computed (in order) from the 5 Dyck paths of length 3 with respective arm sequences $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$ and $(0,1,2)$.



We will make use of the Newton element

$$\Omega[X] = \sum_\lambda p_\lambda[X]/z_\lambda = \sum_{m \geq 0} h_m[X],$$

where we have the identities

$$\Omega[X+Y] = \Omega[X]\Omega[Y],$$
$$\Omega[X-Y] = \Omega[X]/\Omega[Y],$$
$$\Omega[x_1 + \epsilon x_2 - x_3 - \epsilon x_4] = \frac{(1-x_3)(1+x_4)}{(1-x_1)(1+x_2)},$$
$$\Omega[XY] = \sum_\lambda s_\lambda[X]s_\lambda[Y] = \sum_\lambda p_\lambda[X]p_\lambda[Y]/z_\lambda.$$

Jing [12] introduced a family of operators $\mathbb{H}_m$ indexed by $m \in \mathbb{Z}$ using the following formal power series in the parameter z:

$$\mathbb{H}(z)P[X] = \sum_{m\in\mathbb{Z}} z^m \mathbb{H}_m P[X] := P\left[X - \frac{1-q}{z}\right]\Omega[zX] \tag{2.2}$$
$$= \sum_{m\in\mathbb{Z}} z^m \sum_{r\geq 0}(-1)^r h_{m+r}[X] e_r[(1-q)X]^{\perp} P[X].$$

He proved that these operators create the Hall–Littlewood polynomials by adding rows.

Proposition 2.5 ([12]) *For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$,*

$$Q'_\lambda[X;q] = \mathbb{H}_{\lambda_1}\mathbb{H}_{\lambda_2}\cdots\mathbb{H}_{\lambda_\ell}(1).$$

3 Two Families of Hall–Littlewood Symmetric Functions

Our primary focus is the study of two families of symmetric functions and the combinatorics surrounding them. These functions arise from the following operators $\mathbb{B}_m$ and $\mathbb{C}_m$, closely related to Jing's $\mathbb{H}_m$ operators from equation (2.2):

$$\mathbb{B}(z)P[X] = \sum_{m\in\mathbb{Z}} z^m \mathbb{B}_m P[X] := P\Big[X + \epsilon\frac{(1-q)}{z}\Big]\,\Omega[-\epsilon zX] \tag{3.1}$$
$$= \sum_{m\in\mathbb{Z}} z^m \sum_{r\geq 0}(-1)^r e_{m+r}[X] h_r[X(1-q)]^{\perp} P[X]$$
$$\mathbb{C}(z)P[X] = \sum_{m\in\mathbb{Z}} z^m \mathbb{C}_m P[X] := -q\,P\Big[X + \epsilon\frac{(1-q)}{z}\Big]\,\Omega[\epsilon(z/q)X]$$
$$= \sum_{m\in\mathbb{Z}} (-1/q)^{m-1} z^m \sum_{r\geq 0} q^{-r} h_{m+r}[X] h_r[X(1-q)]^{\perp} P[X].$$

The symmetric functions of particular interest here are those defined, for any composition α, by setting

$$B_\alpha[X;q] = \mathbb{B}_{\alpha_{\ell(\alpha)}}\mathbb{B}_{\alpha_{\ell(\alpha)-1}}\cdots\mathbb{B}_{\alpha_1}(1),$$
$$C_\alpha[X;q] = \mathbb{C}_{\alpha_1}\mathbb{C}_{\alpha_2}\cdots\mathbb{C}_{\alpha_{\ell(\alpha)}}(1).$$

Note that the operators generating B_α and C_α are both indexed by the parts of α, but are applied in reverse order with respect to one another. This is done so that the associated combinatorial and algebraic identities are more uniform.

These operators are related by way of the equation:

$$\mathbb{B}(z) = \omega\mathbb{H}(z)\omega \quad \text{and} \quad \mathbb{C}(z) = (-q)\mathbb{H}^{q\to 1/q}(-z/q),$$

or equivalently

$$\mathbb{C}_m = (-1/q)^{m-1}\mathbb{H}_m^{q\to 1/q} = (-1/q)^{m-1}\omega\mathbb{B}_m^{q\to 1/q}\omega. \tag{3.2}$$

Thus the functions themselves are related as:

$$Q'_\lambda[X;q] = \omega B_{\overleftarrow{\lambda}}[X;q] = (-q)^{\ell(\lambda)-|\lambda|}C_\lambda[X;1/q].$$

The Jing operators create Hall–Littlewood polynomials indexed by partitions that form a basis for the symmetric function ring. The C_α and B_α symmetric functions are indexed by compositions and are not linearly independent. The equations above detail how Hall–Littlewood symmetric functions are included in these families, and therefore, Schur positive expansions of the C_α and B_α hold in certain cases. However, they are not Schur positive in complete generality. The smallest examples that are not uniformly Schur positive or Schur negative are $B_{(3,1)}[X;q]$ and $C_{(1,3)}[X;q]$.

Remark 3.1 The following is a table of the symmetric functions $B_\alpha[X;q]$ and $C_\alpha[X;q]$ for $\alpha \models 4$. Notice that both $B_{(3,1)}[X;q]$ and $C_{(1,3)}[X;q]$ have mixed signs in their coefficients.

$$\begin{bmatrix} B_{(1,1,1,1)}[X;q] \\ B_{(1,1,2)}[X;q] \\ B_{(1,2,1)}[X;q] \\ B_{(2,1,1)}[X;q] \\ B_{(1,3)}[X;q] \\ B_{(2,2)}[X;q] \\ B_{(3,1)}[X;q] \\ B_{(4)}[X;q] \end{bmatrix} = \begin{bmatrix} q^6 & q^3+q^4+q^5 & q^2+q^4 & q+q^2+q^3 & 1 \\ q^3 & q+q^2 & q & 1 & 0 \\ q^4 & q^2+q^3 & q^2 & q & 0 \\ q^5 & q^3+q^4 & q^3 & q^2 & 0 \\ q & 1 & 0 & 0 & 0 \\ q^2 & q & 1 & 0 & 0 \\ q^3 & q^2 & q-1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_{(1,1,1,1)}[X;q] \\ s_{(2,1,1)}[X;q] \\ s_{(2,2)}[X;q] \\ s_{(3,1)}[X;q] \\ s_{(4)}[X;q] \end{bmatrix}$$

$$\begin{bmatrix} C_{(1,1,1,1)}[X;q] \\ C_{(1,1,2)}[X;q] \\ C_{(1,2,1)}[X;q] \\ C_{(2,1,1)}[X;q] \\ C_{(1,3)}[X;q] \\ C_{(2,2)}[X;q] \\ C_{(3,1)}[X;q] \\ C_{(4)}[X;q] \end{bmatrix} =$$

$$\begin{bmatrix} 1 & q^{-3}+q^{-2}+q^{-1} & q^{-4}+q^{-2} & q^{-5}+q^{-4}+q^{-3} & q^{-6} \\ 0 & -q^{-3} & -q^{-4} & -q^{-5}-q^{-4} & -q^{-6} \\ 0 & -q^{-2} & -q^{-3} & -q^{-4}-q^{-3} & -q^{-5} \\ 0 & -q^{-1} & -q^{-2} & -q^{-3}-q^{-2} & -q^{-1} \\ 0 & 0 & q^{-3}-q^{-2} & q^{-4} & q^{-5} \\ 0 & 0 & q^{-2} & q^{-3} & q^{-4} \\ 0 & 0 & 0 & q^{-2} & q^{-3} \\ 0 & 0 & 0 & 0 & q^{-3} \end{bmatrix} \begin{bmatrix} s_{(1,1,1,1)}[X;q] \\ s_{(2,1,1)}[X;q] \\ s_{(2,2)}[X;q] \\ s_{(3,1)}[X;q] \\ s_{(4)}[X;q] \end{bmatrix}$$

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To manipulate these symmetric functions, we derive commutation relations between the symmetric function operators. Our first result enables us to expand an element $C_\alpha[X;q]$, for any composition α, in terms of the $C_\lambda[X;q]$ indexed by partitions λ.

Proposition 3.2 *For $m, n \in \mathbb{Z}$, we have*

$$q\mathbb{C}_m\mathbb{C}_n - \mathbb{C}_{m+1}\mathbb{C}_{n-1} = \mathbb{C}_n\mathbb{C}_m - q\mathbb{C}_{n-1}\mathbb{C}_{m+1}. \tag{3.3}$$

In particular for $m \in \mathbb{Z}$, $\mathbb{C}_m\mathbb{C}_{m+1} = \frac{1}{q}\mathbb{C}_{m+1}\mathbb{C}_m$.

Proof We begin with the expressions for the $\mathbb{C}$-operators from equation (3.1). For ease of notation we shall use h_m in place of $h_m[X]$ and $h_r^{q\perp}$ in place of the expression $h_r[X(1-q)]^\perp$. We compute that

$$\begin{aligned} h_r^{q\perp}(h_m P[X]) &= h_m[X+(1-q)z]P\big[X+(1-q)z\big]\Big|_{z^r} \\ &= \sum_{i\geq 0} h_{m-i}h_i[1-q]P\big[X+(1-q)z\big]\Big|_{z^{r-i}} \\ &= \sum_{i\geq 0} h_{m-i}h_i[1-q]h_{r-i}^{q\perp}P[X]. \end{aligned} \tag{3.4}$$

We also know that

$$h_r[1-q] = \begin{cases} 0 & \text{if } r<0, \\ 1 & \text{if } r=0, \\ 1-q & \text{if } r>0. \end{cases} \tag{3.5}$$

These two identities imply that

$$\begin{aligned} q\mathbb{C}_m\mathbb{C}_n &= (-1/q)^{m+n-2}\sum_{i\geq 0}\sum_{r\geq 0}\sum_{s\geq 0} q^{-r-s-i+1}h_{m+r+i}h_{n+s-i}h_i[1-q]h_r^{q\perp}h_s^{q\perp} \\ &= (-1/q)^{m+n-2}\sum_{i\geq 0}\sum_{r\geq 0}\sum_{s\geq 0} q^{-r-s-i+1}h_{m+r+i}h_{n+s-i}h_r^{q\perp}h_s^{q\perp} \\ &\quad - (-1/q)^{m+n-2}\sum_{i\geq 1}\sum_{r\geq 0}\sum_{s\geq 0} q^{-r-s-i+2}h_{m+r+i}h_{n+s-i}h_r^{q\perp}h_s^{q\perp} \\ &= (-1/q)^{m+n-2}\sum_{i\geq 0}\sum_{r\geq 0}\sum_{s\geq 0} q^{-r-s-i+1}h_{m+r+i}h_{n+s-i}h_r^{q\perp}h_s^{q\perp} \\ &\quad - (-1/q)^{m+n-2}\sum_{i\geq 0}\sum_{r\geq 0}\sum_{s\geq 0} q^{-r-s-i+1}h_{m+r+i+1}h_{n+s-i-1}h_r^{q\perp}h_s^{q\perp} \\ &= (-1/q)^{m+n-2}\sum_{r\geq 0}\sum_{s\geq 0}\sum_{i\geq 0} q^{-r-s-i+1}s_{(m+r+i,n+s-i)}h_r^{q\perp}h_s^{q\perp}. \end{aligned} \tag{3.6}$$

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where to arrive at (3.6) we have introduced the Schur function $s_{(a,b)} = h_a h_b - h_{a+1}h_{b-1}$. Similarly,

$$\mathbb{C}_{m+1}\mathbb{C}_{n-1} = (-1/q)^{m+n-2}\sum_{r\geq 0}\sum_{s\geq 0}\sum_{i\geq 0} q^{-r-s-i} s_{(m+r+i+1,n+s-i-1)} h_r^{q\perp} h_s^{q\perp}.$$

From these identities, we find that all terms in the difference $q\mathbb{C}_m\mathbb{C}_n - \mathbb{C}_{m+1}\mathbb{C}_{n-1}$ cancel except the $i = 0$ term in (3.6):

$$\begin{aligned} q\mathbb{C}_m\mathbb{C}_n - \mathbb{C}_{m+1}\mathbb{C}_{n-1} &= (-1/q)^{m+n-2}\sum_{r\geq 0}\sum_{s\geq 0} q^{-r-s+1} s_{(m+r,n+s)} h_r^{q\perp} h_s^{q\perp} \\ &\quad + (-1/q)^{m+n-2}\sum_{r\geq 0}\sum_{s\geq 0}\sum_{i\geq 1} q^{-r-s-i+1} s_{(m+r+i,n+s-i)} h_r^{q\perp} h_s^{q\perp} \\ &\quad - (-1/q)^{m+n-2}\sum_{r\geq 0}\sum_{s\geq 0}\sum_{i\geq 0} q^{-r-s-i} s_{(m+r+i+1,n+s-i-1)} h_r^{q\perp} h_s^{q\perp} \\ &= (-1/q)^{m+n-2}\sum_{r\geq 0}\sum_{s\geq 0} q^{-r-s+1} s_{(m+r,n+s)} h_r^{q\perp} h_s^{q\perp}. \end{aligned}$$

We can then compute $\mathbb{C}_n\mathbb{C}_m - q\mathbb{C}_{n-1}\mathbb{C}_{m+1}$ from this by replacing $m \to n-1$ and $n \to m+1$. In particular,

$$q\mathbb{C}_{n-1}\mathbb{C}_{m+1} - \mathbb{C}_n\mathbb{C}_m = (-1/q)^{m+n-2}\sum_{r\geq 0}\sum_{s\geq 0} q^{-r-s+1} s_{(n-1+r,m+1+s)} h_r^{q\perp} h_s^{q\perp}.$$

The identity $-s_{(b-1,a+1)} = s_{(a,b)}$ and the commutation of $h^\perp$ then imply our claim. ■

An important consequence of this result is that if α is a composition of length ℓ, then $C_\alpha[X;q]$ can be written as a linear combination of the $C_\lambda[X;q]$ where λ are partitions that also have length ℓ.

Remark 3.3 The symmetric function $C_{(1,3)}[X;q]$ can be expressed in terms of $C_{(3,1)}[X;q]$ and $C_{(2,2)}[X;q]$ using this commutation relation,
since $q\mathbb{C}_1\mathbb{C}_3 = \mathbb{C}_2\mathbb{C}_2 + \mathbb{C}_3\mathbb{C}_1 - q\mathbb{C}_2\mathbb{C}_2$. Consequently,

$$C_{(1,3)}[X;q] = (1/q-1)C_{(2,2)}[X;q] + 1/qC_{(3,1)}[X;q].$$

The relation of $\mathbb{C}$ to $\mathbb{B}$ given in (3.2) enables us to derive an identity on $\mathbb{B}$ from Theorem 3.2. In particular, we simply apply ω to (3.3) and replace q by $1/q$.

Corollary 3.4 *For $m \in \mathbb{Z}$,*

$$\mathbb{B}_m\mathbb{B}_n - q\mathbb{B}_{m+1}\mathbb{B}_{n-1} = q\mathbb{B}_n\mathbb{B}_m - \mathbb{B}_{n-1}\mathbb{B}_{m+1}.$$

In particular, letting $n = m+1$ gives $\mathbb{B}_m\mathbb{B}_{m+1} = q\mathbb{B}_{m+1}\mathbb{B}_m$.

In fact, we can also pin down commutation relations between the $\mathbb{B}$ and $\mathbb{C}$ operators if $m+n>0$ (note, the relation does not hold when $m+n\leq 0$).

Proposition 3.5 *If $m+n>0$, then $\mathbb{B}_n\mathbb{C}_m = q\mathbb{C}_m\mathbb{B}_n$.*

Proof We use identities (3.4) and (3.5) to compute an expression for $\mathbb{B}_m\mathbb{C}_n$:

$$\begin{aligned}\mathbb{B}_m\mathbb{C}_n =&(-1/q)^{n-1}\sum_{r\geq 0}\sum_{s\geq 0}\sum_{i\geq 0}(-1)^r q^{-s} e_{m+r}h_{n+s-i}h_i[1-q]h_{r-i}^{q\perp}h_s^{q\perp}\\ =&(-1/q)^{n-1}\sum_{r\geq 0}\sum_{s\geq 0}(-1)^r q^{-s}e_{m+r}h_{n+s}h_r^{q\perp}h_s^{q\perp}\\ &+(-1/q)^{n-1}\sum_{i\geq 1}\sum_{r\geq 0}\sum_{s\geq 0}(-1)^{r+i}q^{-s}e_{m+r+i}h_{n+s-i}(1-q)h_r^{q\perp}h_s^{q\perp}.\end{aligned}$$

Analogously, we also have the equations

$$h_r^{q\perp}e_m = \sum_{i\geq 0}e_{m-i}e_i[1-q]h_{r-i}^{q\perp} \quad\text{and}\quad e_r[1-q] = (-q)^{r-1}h_r[1-q]$$

if $r>0$. From this we derive a similar expression for $q\mathbb{C}_n\mathbb{B}_m$:

$$\begin{aligned}q\mathbb{C}_n\mathbb{B}_m =&(-1/q)^{n-1}\sum_{s\geq 0}\sum_{r\geq 0}\sum_{i\geq 0}(-1)^r q^{-s+1}h_{n+s}e_{m+r-i}e_i[1-q]h_{s-i}^{q\perp}h_r^{q\perp}\\ =&(-1/q)^{n-1}\sum_{s\geq 0}\sum_{r\geq 0}(-1)^r q^{-s+1}e_{m+r}h_{n+s}h_s^{q\perp}h_r^{q\perp}\\ &+(-1/q)^{n-1}\sum_{i\geq 1}\sum_{s\geq 0}\sum_{r\geq 0}(-1)^{r+i+1}q^{-s}e_{m+r-i}h_{n+s+i}(1-q)h_s^{q\perp}h_r^{q\perp}.\end{aligned}$$

Their difference is

$$\begin{aligned}&\mathbb{B}_m\mathbb{C}_n - q\mathbb{C}_n\mathbb{B}_m\\ &\quad= (1-q)(-1/q)^{n-1}\sum_{s\geq 0}\sum_{r\geq 0}q^{-s}(-1)^r e_{m+r}h_{n+s}h_r^{q\perp}h_s^{q\perp}\\ &\qquad+(1-q)(-1/q)^{n-1}\sum_{r\geq 0}\sum_{s\geq 0}q^{-s}\Big(\sum_{i\geq 1}(-1)^{r+i}e_{m+r-i}h_{n+s+i}\Big)h_r^{q\perp}h_s^{q\perp}\\ &\qquad+(1-q)(-1/q)^{n-1}\sum_{r\geq 0}\sum_{s\geq 0}q^{-s}\Big(\sum_{i\geq 1}(-1)^{r+i}e_{m+r+i}h_{n+s-i}\Big)h_r^{q\perp}h_s^{q\perp}.\end{aligned}$$

In fact, the right-hand side reduces to zero, since $m+n>0$ implies that for each $r,s\geq 0$, $m+n+r+s>0$ and

$$\begin{aligned}(-1)^r e_{m+r}h_{n+s}+\sum_{i\geq 1}(-1)^{r+i}e_{m+r-i}h_{n+s+i}+\sum_{i\geq 1}(-1)^{r+i}e_{m+r+i}h_{n+s-i}&\\ =\sum_{i=-n-s}^{m+r}(-1)^{r+i}e_{m+r-i}h_{n+s+i}&=0\end{aligned}$$

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by the identity $\sum_{i=0}^{d}(-1)^i e_{d-i}h_i = 0$ for all $d > 0$. ■

Jing's operators generalize operators of Bernstein (see [19]), defined by

$$\begin{aligned}\mathbb{S}(z)P[X] &= \sum_{m\in\mathbb{Z}} z^m \mathbb{S}_m P[X] = P\left[X - \frac{1}{z}\right]\Omega[zX]\\ &= \sum_{m\in\mathbb{Z}} z^m \sum_{r\geq 0}(-1)^r h_{m+r}[X]e_r[X]^{\perp}P[X].\end{aligned}$$

These are creation operators for the Schur functions, since $\mathbb{S}_{\lambda_1}\mathbb{S}_{\lambda_2}\cdots\mathbb{S}_{\lambda_\ell}(1) = s_\lambda[X]$, and they satisfy the commutation relation $\mathbb{S}_m\mathbb{S}_n = -\mathbb{S}_{n-1}\mathbb{S}_{m+1}$. We can write the Schur creation operators in terms of the $\mathbb{C}_a$ operators, which will help us to write Schur functions in terms of the C_α in Section 5.

Proposition 3.6 *For $m \in \mathbb{Z}$,*

$$\mathbb{S}_m = (-q)^{m-1}\sum_{i\geq 0}\mathbb{C}_{m+i}e_i^{\perp}. \tag{3.7}$$

Proof We will use the identity

$$h_r[(1-q)X] = \sum_{j\geq 0} h_j[X]h_{r-j}[-qX] = \sum_{j\geq 0}(-q)^{r-j}h_j[X]e_{r-j}[X]$$

and calculate directly

$$\begin{aligned}&(-q)^{m-1}\sum_{i\geq 0}\mathbb{C}_{m+i}e_i^{\perp}\\ &\quad= (-q)^{m-1}\sum_{i\geq 0}(-1/q)^{m+i-1}\sum_{r\geq 0}q^{-r}h_{m+i+r}[X]h_r[X(1-q)]^{\perp}e_i^{\perp}\\ &\quad= \sum_{r\geq 0}\sum_{i\geq 0}\sum_{j\geq 0}(-1/q)^{i-r+j}q^{-r}h_{m+i+r}[X]h_j^{\perp}e_{r-j}^{\perp}e_i^{\perp}\\ &\quad= \sum_{d\geq 0}\sum_{j=0}^{d}\sum_{r\geq 0}(-1)^{d-r}q^{-d}h_{m+d+r-j}[X]h_j^{\perp}e_{r-j}^{\perp}e_{d-j}^{\perp}\\ &\quad= \sum_{d\geq 0}\sum_{j=0}^{d}\sum_{r\geq 0}(-1)^{d-r-j}q^{-d}h_{m+d+r}[X]h_j^{\perp}e_r^{\perp}e_{d-j}^{\perp}\\ &\quad= \sum_{r\geq 0}(-1)^{-r}h_{m+r}[X]e_r^{\perp} = \mathbb{S}_m,\end{aligned}$$

where the last equality follows because $\sum_{j=0}^{d}(-1)^j h_j e_{d-j} = 0$ for all $d > 0$, so the remaining sum is only the part where $d = 0$. ■

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Remark 3.7 In reference [5], the operator $\mathbb{C}_a$ is presented in a slightly different but equivalent expression. We note that a series $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$ has the property that $f(\epsilon z/q)|_{z^a} = (-1/q)^a f(z)|_{z^a}$. For this reason,

$$\begin{aligned}\mathbb{C}_a P[X] &= -qP\Big[X + \epsilon\frac{(1-q)}{z}\Big]\,\Omega[\epsilon(z/q)X]\Big|_{z^a}\\ &= -q(-1/q)^a P\Big[X + \frac{(1-q)}{qz}\Big]\,\Omega[zX]\Big|_{z^a}\\ &= (-1/q)^{a-1} P\Big[X - \frac{1-1/q}{z}\Big]\,\Omega[zX]\Big|_{z^a}.\end{aligned}$$

4 The Combinatorics of ∇ Applied to Hall–Littlewood Polynomials

Recall that in the special case where α is a partition, $B_\alpha[X;q]$ and $C_\alpha[X;q]$ are closely related to the Hall–Littlewood symmetric functions. It was conjectured in [2, Conjecture II and III] (partially attributed to A. Lascoux) that applying ∇ to a Hall–Littlewood polynomial produces a Schur positive function. Our main discovery is that including all compositions α in the study of $\nabla(B_\alpha[X;q])$ and $\nabla(C_\alpha[X;q])$ leads to a natural refinement for the combinatorics of Dyck paths. Moreover, our combinatorial exploration led us to discover new symmetric function identities.

One useful tool in the exploration of the operator ∇ is the fact that $\nabla^{q=1}$ is a multiplicative operator. Since we can deduce from the operator definitions of our symmetric functions that $B_\alpha[X;1] = e_\alpha[X]$ and $C_\alpha[X;1] = h_\alpha[X]$, we have

$$\nabla^{q=1}\big(B_\alpha[X;1]\big) = \nabla^{q=1}\big(e_{\alpha_\ell}[X]\big)\,\nabla^{q=1}\big(e_{\alpha_{\ell-1}[X]}\big)\cdots\nabla^{q=1}\big(e_{\alpha_1}[X]\big)$$

and

$$\nabla^{q=1}\big(C_\alpha[X;1]\big) = \nabla^{q=1}\big(h_{\alpha_1}[X]\big)\,\nabla^{q=1}\big(h_{\alpha_2}[X]\big)\cdots\nabla^{q=1}\big(h_{\alpha_\ell}[X]\big).$$

From this we can deduce the coefficient of $e_n[X]$. In particular, the coefficients of $e_n[X]$ in $\nabla(e_n[X])$ and in $\nabla(h_n[X])$ are the q,t-Catalan numbers $C_n(q,t)$ and $C_{n-1}(q,t)$, respectively. Thus, the coefficient of $e_n[X]$ in $\nabla^{q=1}(B_\alpha[X;1])$ and in $\nabla^{q=1}(C_\alpha[X;1])$ is $\prod_i C_{\alpha_i}(1,t)$ and $\prod_i C_{\alpha_i-1}(1,t)$, respectively. The combinatorial interpretation for $C_n(1,t)$ then gives combinatorial meaning to these coefficients. Namely, $\langle e_n[X], \nabla^{q=1}(B_\alpha[X;1])\rangle$ is the t-enumeration of Dyck paths (with weight t raised to the area) that lie below the staircase consisting of α_1 steps up and over, α_2 steps up and over, etc., and $\langle e_n[X], \nabla^{q=1}(C_\alpha[X;1])\rangle$ is a t-enumeration of Dyck paths that touch the diagonal only in rows $1, 1+\alpha_1, 1+\alpha_1+\alpha_2$ steps, etc.

Proposition 4.1 *For α a composition of n,*

$$\Big\langle \nabla^{q=1}\big(B_\alpha[X;1]\big), e_n[X]\Big\rangle = \sum_{D\le DP(\alpha)} t^{\mathrm{area}(D)}$$

and

$$\Big\langle \nabla^{q=1}\big(C_\alpha[X;1]\big), e_n[X]\Big\rangle = \sum_{\mathrm{touch}(D)=\alpha} t^{\mathrm{area}(D)}.$$

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Remarkably, we have empirical evidence to suggest that, in general, there is a combinatorial interpretation for the coefficients of $e_n[X]$ in $\nabla(B_\alpha[X;q])$ and in $\nabla(C_\alpha[X;q])$ that naturally generalizes the beautiful combinatorics of the q,t-Catalan.

Conjecture 4.2 *For $\alpha \models n$,*

$$\Big\langle \nabla\big(B_\alpha[X;q]\big), e_n[X] \Big\rangle = \sum_{D \leq DP(\alpha)} t^{\mathrm{area}(D)} q^{\mathrm{dinv}(D)+\mathrm{doff}_\alpha(D)}.$$

Conjecture 4.3 *For $\alpha \models n$,*

$$\Big\langle \nabla\big(C_\alpha[X;q]\big), e_n[X] \Big\rangle = \sum_{\mathrm{touch}(D)=\alpha} t^{\mathrm{area}(D)} q^{\mathrm{dinv}(D)}.$$

Our work was inspired by the work of [1] where they considered coefficients $\nabla(B_{\overleftarrow{\lambda}}[X;q])$ for λ a hook partition (since for that case $\mathrm{doff}_{\overleftarrow{\lambda}}(D) = 0$). The innovation in these identities is to consider symmetric functions indexed by compositions which allowed us to conjecture the action of ∇ on a spanning set of the symmetric functions.

More generally, we have conjectures for the expansion of $\nabla(B_\alpha[X;q])$ and $\nabla(C_\alpha[X;q])$ into monomials.

Conjecture 4.4

$$\nabla(B_\alpha[X;q]) = \sum_{D \leq DP(\alpha)} \sum_{w \in \mathcal{WP}_D} t^{\mathrm{area}(w)} q^{\mathrm{dinv}(w)+\mathrm{doff}_\alpha(D)} x^w. \tag{4.1}$$

Conjecture 4.5

$$\nabla(C_\alpha[X;q]) = \sum_{\mathrm{touch}(D)=\alpha} \sum_{w \in \mathcal{WP}_D} t^{\mathrm{area}(D)} q^{\mathrm{dinv}(w)} x^w. \tag{4.2}$$

By the arguments in [9] (see also [8, p. 99]) Conjectures 4.4 and 4.5 imply Conjectures 4.2 and 4.3. The case $\alpha = (n)$ of (4.1) reduces to the shuffle conjecture, since $B_{(n)}[X;q] = e_n[X]$. Also, because of the expansion of $s_{(n-k,1^k)}$ in Proposition 5.3, (4.2) implies the special case of the Loehr–Warrington conjecture [18, Conjecture 3] involving the action of ∇ on the Schur function $s_{(n-k,1^k)}$.

We will prove in the next section that Conjectures 4.4 and 4.5 are equivalent to each other (and by consequence Conjectures 4.2 and 4.3 are equivalent as well). In work building on our results here, [5] with contributions from A. Hicks, proved Conjecture 4.3.

5 Symmetric Function Identities

The exploration of q,t-Catalans led [3] to the special symmetric function elements $E_{n,k}[X;q]$, defined by the algebraic identity

$$e_n\Big[X\frac{1-z}{1-q}\Big] = \sum_{k=1}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k}[X;q],$$

where $(z;q)_k = (1-z)(1-qz)\cdots(1-q^{k-1}z)$. These elements play a fundamental role in the proof that the q,t-Catalan polynomial is the q,t-enumeration of Dyck paths as given in (2.1). Namely, the proof follows by showing that

(5.1)
$$\left\langle \nabla E_{n,k}[X;q], e_n[X] \right\rangle = q^{\binom{k}{2}} t^{n-k} \sum_{r=0}^{n-k} \begin{bmatrix} r+k-1 \\ r \end{bmatrix}_q \left\langle \nabla\bigl(E_{n-k,r}[X;q]\bigr), e_{n-r}[X] \right\rangle,$$

where
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

and the combinatorial interpretation for $\langle \nabla E_{n,k}[X;q], e_n[X] \rangle$ in terms of Dyck paths satisfies the same recurrence.

In particular, the coefficient of $e_n[X]$ in $\nabla(E_{n,k}[X;q])$ q,t-enumerates the Dyck paths that touch the diagonal k times. From this, Conjecture 4.3 leads us to expect that
$$\left\langle e_n[X], \nabla(E_{n,k}[X;q]) \right\rangle = \sum_{\alpha\models n, \ell(\alpha)=k} \left\langle e_n[X], \nabla(C_\alpha[X;q]) \right\rangle.$$

In fact, we have discovered much more generally that
$$E_{n,k}[X;q] = \sum_{\substack{\alpha\models n \\ \ell(\alpha)=k}} C_\alpha[X;q].$$

This section is devoted to proving this surprising result, which suggests that the $C_\alpha[X;q]$ are the building blocks in q,t-Catalan theory.

Remark 5.1 A key step in the proof of our Conjectures 4.2 and 4.3 relies on extending the recurrence (5.1) to involve Dyck paths that touch the diagonal at certain points. This is carried out in [5].

Our point of departure is a simple expression for $e_n[X]$ in terms of the Hall–Littlewood symmetric functions $C_\alpha[X;q]$.

Proposition 5.2

$$e_n[X] = \sum_{\alpha\models n} C_\alpha[X;q]. \tag{5.2}$$

Proof Assume by induction on n that equation (5.2) holds (the base cases of $n=0$ and 1 are easily verified). Since the operator $\mathbb{S}_m$ is a creation operator for the Schur functions, by (3.7) we have

$$e_n[X] = s_{(1^n)}[X] = \mathbb{S}_1(s_{(1^{n-1})}[X]) = \sum_{i=0}^{n-1} \mathbb{C}_{1+i} s_{(1^{n-i-1})}[X],$$

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which, by induction, gives

$$e_n[X] = \sum_{i=0}^{n-1} \sum_{\alpha \models n-i-1} \mathbb{C}_{i+1} C_\alpha[X;q] = \sum_{\alpha \models n} C_\alpha[X;q]. \qquad \blacksquare$$

Proposition 5.2 can be stated in a more general form, suggesting that any Schur function may expand nicely in terms of our Hall–Littlewood spanning set.

Proposition 5.3 *For $0 \le k < n$,*

$$s_{(n-k,1^k)}[X] = (-q)^{n-k-1} \sum_{\substack{\alpha \models n \\ \alpha_1 \ge n-k}} C_\alpha[X;q].$$

Proof Again using that $\mathbb{S}$ is a Schur function creation operator, by (3.7) we have

$$s_{(n-k,1^k)}[X] = \mathbb{S}_{n-k}\big(s_{1^k}[X]\big) = (-q)^{n-k-1} \sum_{i=0}^{k} \mathbb{C}_{n-k+i}\big(s_{(1^{k-i})}[X]\big).$$

The previous proposition then implies that

$$\begin{aligned} s_{(n-k,1^k)}[X] &= (-q)^{n-k-1} \sum_{i=0}^{k} \mathbb{C}_{n-k+i}\Big(\sum_{\alpha \models k-i} C_\alpha[X;q]\Big) \\ &= (-q)^{n-k-1} \sum_{i=0}^{k} \sum_{\alpha \models k-i} C_{(n-k+i,\alpha)}[X;q] \\ &= (-q)^{n-k-1} \sum_{\substack{\alpha \models n \\ \alpha_1 \ge n-k}} C_\alpha[X;q]. \qquad \blacksquare \end{aligned}$$

We are now in the position to prove that $E_{n,k}[X;q]$ can be decomposed canonically in terms of the $C_\alpha[X;q]$.

Proposition 5.4 *For $0 \le k < n$,*

$$E_{n,k}[X;q] = \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{-n(\mu)-k+M(\mu)} \begin{bmatrix} k \\ m(\mu) \end{bmatrix}_q C_\mu[X;q], \tag{5.3}$$

where $M(\mu) = \sum_{i=1}^{n} \binom{m_i(\mu)+1}{2}$ and $\left[{k \atop m(\mu)} \right]_q = (q;q)_k / \prod_{i=1}^{n} (q;q)_{m_i(\mu)}$.

Proof Recall that the expansion of the elementary symmetric functions in the Macdonald basis is given by (see [3])

$$e_n\left[X\frac{1-z}{1-q}\right] = \sum_{\mu \vdash n} \frac{\widetilde{H}_\mu[X;q,t]\widetilde{H}_\mu[(1-z)(1-t);q,t]}{\widetilde{h}_\mu(q,t)\widetilde{h}'_\mu(q,t)},$$

where $\widetilde{h}_\mu(q,t) = \prod_{c\in\mu}(q^{a(c)} - t^{l(c)+1})$ and $\widetilde{h}'_\mu(q,t) = \prod_{c\in\mu}(t^{l(c)} - q^{a(c)+1})$. When $t = 0$, these expressions reduce to

$$\widetilde{h}_\mu(q,0) = \prod_{c\in\mu}(q^{a(c)} - 0^{l(c)+1}) = q^{n(\mu')},$$

$$\begin{aligned}\widetilde{h}'_\mu(q,0) &= \prod_{\substack{c\in\mu\\ l(c)=0}}(1-q^{a(c)+1}) \prod_{\substack{c\in\mu\\ l(c)\neq 0}}(-q^{a(c)+1})\\ &= (-1)^{n-\mu_1} q^{n+n(\mu')-M(\mu')} \prod_{i=1}^{n}(q;q)_{m_i(\mu')}.\end{aligned}$$

Therefore, since $\widetilde{H}_\mu[X;q,t] = \widetilde{H}_{\mu'}[X;t,q]$, when we set $t = 0$ everywhere, we have

$$e_n\Big[X\frac{1-z}{1-q}\Big] = \sum_{\mu\vdash n}(-1)^{n-\mu_1} q^{-n-2n(\mu')+M(\mu')} \frac{\widetilde{H}_{\mu'}[X;0,q]\widetilde{H}_\mu[(1-z);q,0]}{\prod_{i=1}^{n}(q;q)_{m_i(\mu')}}. \tag{5.4}$$

Now the evaluation

$$\widetilde{H}_\mu[(1-z);q,t] = \prod_{c\in\mu}(1 - zt^{l'(c)}q^{a'(c)})$$

also yields $\widetilde{H}_\mu[(1-z);q,0] = (z;q)_{\mu_1}$. Thus, replacing μ by μ' in (5.4), and thereby exchanging μ_1 and $\ell(\mu)$, gives

$$e_n\Big[X\frac{1-z}{1-q}\Big] = \sum_{\mu\vdash n}(-1)^{n-\ell(\mu)} q^{-n-2n(\mu)+M(\mu)} \frac{\widetilde{H}_\mu[X;0,q](z;q)_{\ell(\mu)}}{\prod_{i=1}^{n}(q;q)_{m_i(\mu)}}.$$

Since $\widetilde{H}_\mu[X;0,q] = (-1)^{n-\ell(\mu)} q^{n(\mu)+n-\ell(\mu)} C_\mu[X;q]$, we have

$$e_n\Big[X\frac{1-z}{1-q}\Big] = \sum_{k=1}^{n}\frac{(z;q)_k}{(q;q)_k}\sum_{\substack{\mu\vdash n\\ \ell(\mu)=k}} q^{-k-n(\mu)+M(\mu)}\frac{C_\mu[X;q](q;q)_k}{\prod_{i=1}^{n}(q;q)_{m_i(\mu)}},$$

which implies our claim. ∎

The q-binomial coefficients that appear in equation (5.3) suggest that there is a relation between the terms of the $C_\lambda[X;q]$ basis and subsets of a k element set. It turns out that Proposition 5.4 can be more cleanly written over compositions using a different expansion.

Corollary 5.5 *For $0 \le k < n$,*

$$E_{n,k}[X;q] = \sum_{\substack{\alpha\models n\\ \ell(\alpha)=k}} C_\alpha[X;q].$$

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Proof Using the straightening relations of the $\mathbb{C}_m$ operators, if α is a composition of n and λ is a partition of n such that $\ell(\lambda) \neq \ell(\alpha)$, then

$$C_\alpha[X;q]\Big|_{C_\lambda[X;q]} = 0.$$

Now for $\ell(\lambda) = k$, by Proposition 5.4 and the fact that $e_n[X] = \sum_{k=1}^n E_{n,k}[X;q]$,

$$\begin{aligned} E_{n,k}[X;q]\Big|_{C_\lambda[X;q]} = e_n[X]\Big|_{C_\lambda[X;q]} &= \sum_{\alpha \models n} C_\alpha[X;q]\Big|_{C_\lambda[X;q]} \\ &= \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} C_\alpha[X;q]\Big|_{C_\lambda[X;q]}. \end{aligned}$$

Furthermore, if $\ell(\lambda) \neq k$, then

$$E_{n,k}[X;q]\Big|_{C_\lambda[X;q]} = 0 = \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} C_\alpha[X;q]\Big|_{C_\lambda[X;q]}.$$

Since the functions $C_\lambda[X;q]$ are a basis, this implies that

$$E_{n,k}[X;q] = \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} C_\alpha[X;q]. \qquad \blacksquare$$

We have seen in (3.2) that $\mathbb{C}$ is naturally related to $\mathbb{B}$. Here we pin down the relationship between the symmetric functions $B_\alpha[X;q]$ and $C_\alpha[X;q]$. A by-product of this identity is that Conjecture 4.5 implies Conjecture 4.4.

Theorem 5.6 *For $n \geq 0$ and any composition $\alpha \models n$,*

$$B_\alpha[X;q] = \sum_{\beta \leq \alpha} q^{\mathrm{doff}_\alpha(DP(\beta))} C_\beta[X;q]. \tag{5.5}$$

Proof We show this result by induction on the number of parts of α. The base case follows, since $\mathbb{B}_m(1) = e_m[X]$ which is equal to $\sum_{\gamma \models m} C_\gamma[X;q]$ by Proposition 5.2. Assume by induction that (5.5) holds for a composition α of length ℓ and consider a composition (m, α). We then have

$$B_{(\alpha,m)}[X;q] = \mathbb{B}_m\big(B_\alpha[X;q]\big) = \sum_{\beta \leq \alpha} q^{\mathrm{doff}_\alpha(DP(\beta))} \mathbb{B}_m\big(C_\beta[X;q]\big). \tag{5.6}$$

Now consider $\mathbb{B}_m(C_\beta[X;q]) = \mathbb{B}_m \circ \mathbb{C}_{\beta_1} \circ \mathbb{C}_{\beta_2} \circ \cdots \circ \mathbb{C}_{\beta_{\ell(\beta)}}(1)$. The commutation relation between the $\mathbb{C}_n$ and $\mathbb{B}_m$ from Theorem 3.5 implies

$$\mathbb{B}_m(C_\beta[X;q]) = q^{\ell(\beta)} \mathbb{C}_{\beta_1} \circ \mathbb{C}_{\beta_2} \circ \cdots \circ \mathbb{C}_{\beta_{\ell(\beta)}} \circ \mathbb{B}_m(1).$$

By Proposition 5.2, we then have

$$\begin{aligned}(5.7)\qquad \mathbb{B}_m(C_\beta[X;q]) &= q^{\ell(\beta)}\mathbb{C}_{\beta_1}\circ\mathbb{C}_{\beta_2}\circ\cdots\circ\mathbb{C}_{\beta_{\ell(\beta)}}\Big(\sum_{\gamma\models m} C_\gamma[X;q]\Big)\\ &= q^{\ell(\beta)}\sum_{\gamma\models m} C_{(\beta,\gamma)}[X;q].\end{aligned}$$

Putting (5.7) into (5.6), we thus find that

$$B_{(\alpha,m)}[X;q] = \sum_{\beta\leq\alpha} q^{\mathrm{doff}_\alpha(DP(\beta))+\ell(\beta)}\sum_{\gamma\models m} C_{(\beta,\gamma)}[X;q].$$

For each term in the sum, the composition (β,γ) is finer than the composition (α,m). Moreover, if we let r_i be the number of times that $DP(\beta)$ touches the diagonal below the i-th bump of the Dyck path $DP(\alpha)$, then

$$\begin{aligned}\mathrm{doff}_{(\alpha,m)}(DP(\beta,\gamma)) &= \sum_{i=1}^{\ell(\alpha)} r_i(\ell(\alpha)+1-i) = \sum_{i=1}^{\ell(\alpha)} r_i(\ell(\alpha)-i)+\sum_{i=1}^{\ell(\alpha)} r_i\\ &= \mathrm{doff}_\alpha(DP)+\ell(\beta).\end{aligned}$$ ∎

To prove that Conjectures 4.5 and 4.4 are in fact equivalent, we need to express $C_\alpha[X;q]$ in terms of $B_\beta[X;q]$.

Lemma 5.7 *Let γ,α be compositions with $\gamma\leq\alpha$. Then*

$$\sum_{\substack{\beta\\ \gamma\leq\beta\leq\alpha}} (-1)^{\ell(\alpha)-\ell(\beta)} q^{\ell(\alpha)-\ell(\beta)+\mathrm{doff}_\beta(DP(\gamma))-\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta}))} = \begin{cases} 0 \text{ if } \gamma<\alpha,\\ 1 \text{ if } \gamma=\alpha.\end{cases}$$

Proof First assume that $\gamma<\alpha$ and consider the difference of the descent sets

$$\mathrm{Des}(\gamma)\backslash\mathrm{Des}(\alpha) = \{i_1,i_2,\dots,i_d\}.$$

There are 2^d compositions β such that $\gamma\leq\beta\leq\alpha$, and we will pair them up with a sign reversing involution.

If $i_1\in\mathrm{Des}(\beta)$, let $\widetilde{\beta}$ be the composition with $\mathrm{Des}(\widetilde{\beta})=\mathrm{Des}(\beta)\backslash\{i_1\}$ (the terms with $i_1\in\mathrm{Des}(\beta)$ will match with the terms $i_1\notin\mathrm{Des}(\widetilde{\beta})$). There is some $r>1$ such that $\alpha_1=\beta_1,\alpha_2=\beta_2,\dots,\alpha_r>\beta_r$, because $\mathrm{Des}(\beta)$ contains the descent i_1 that is not in $\mathrm{Des}(\alpha)$. Calculating directly we have that

$$\begin{aligned}\ell(\alpha)-\ell(\beta) &= \ell(\alpha)-\ell(\widetilde{\beta})-1,\\ \mathrm{doff}_\beta(DP(\gamma)) &= \mathrm{doff}_{\widetilde{\beta}}(DP(\gamma))+r,\\ -\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta})) &= -\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\widetilde{\beta}}))-r+1.\end{aligned}$$

Therefore,

$$\ell(\alpha) - \ell(\beta) + \mathrm{doff}_\beta\big(DP(\gamma)\big) - \mathrm{doff}_{\overleftarrow{\alpha}}\big(DP(\overleftarrow{\beta})\big) =$$
$$\ell(\alpha) - \ell(\widetilde{\beta}) + \mathrm{doff}_{\widetilde{\beta}}\big(DP(\gamma)\big) - \mathrm{doff}_{\overleftarrow{\alpha}}\big(DP(\overleftarrow{\widetilde{\beta}})\big)$$

and the signs of these terms are different in the sum. This provides a sign reversing involution and hence those terms with $\mathrm{touch}(D) < \alpha$ sum to 0 matching those terms with $\mathrm{Des}(\beta)$ that include the smallest descent.

Now for those terms with $\mathrm{touch}(D) = \alpha$, we have $\gamma = \beta = \alpha$ and

$$\ell(\alpha) - \ell(\beta) + \mathrm{doff}_\beta(DP(\gamma)) - \mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta})) = 0,$$

since $\mathrm{doff}_\alpha(DP(\alpha)) = \binom{\ell(\alpha)}{2}$. ■

Theorem 5.8 *For $n \geq 0$ and for any composition $\alpha \models n$,*

$$C_\alpha[X;q] = \sum_{\beta \leq \alpha} (-q)^{\ell(\alpha)-\ell(\beta)} q^{-\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta}))} B_\beta[X;q].$$

Proof By Theorem 5.6 we have

$$\begin{aligned}
&\sum_{\beta \leq \alpha} (-q)^{\ell(\alpha)-\ell(\beta)} q^{-\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta}))} B_\beta[X;q] \\
&\quad = \sum_{\beta \leq \alpha} (-q)^{\ell(\alpha)-\ell(\beta)} q^{-\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta}))} \sum_{\gamma \leq \beta} q^{\mathrm{doff}_\beta(DP(\gamma))} C_\gamma[X;q] \\
&\quad = \sum_{\gamma} C_\gamma[X;q] \sum_{\substack{\beta \\ \gamma \leq \beta \leq \alpha}} (-q)^{\ell(\alpha)-\ell(\beta)} q^{-\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta}))} q^{\mathrm{doff}_\beta(DP(\gamma))},
\end{aligned}$$

and our claim now follows by Lemma 5.7. ■

Theorem 5.9 *Conjecture 4.5 is true if and only if Conjecture 4.4 is true.*

Proof Theorem 5.6 gives $B_\alpha[X;q]$ in terms of $C_\beta[X;q]$, to which we apply ∇:

$$\nabla(B_\alpha[X;q]) = \sum_{\beta \leq \alpha} q^{\mathrm{doff}_\alpha(DP(\beta))} \nabla(C_\beta[X;q]).$$

Given that Conjecture 4.5 holds, we then have that

$$\begin{aligned}
\nabla(B_\alpha[X;q]) &= \sum_{\beta \leq \alpha} \sum_{\substack{D \\ \mathrm{touch}(D)=\beta}} \sum_{w \in \mathcal{WP}_D} t^{\mathrm{area}(D)} q^{\mathrm{dinv}(w)+\mathrm{doff}_\alpha(DP(\beta))} x^w \\
&= \sum_{D \leq DP(\beta)} \sum_{w \in \mathcal{WP}_D} t^{\mathrm{area}(D)} q^{\mathrm{dinv}(w)+\mathrm{doff}_\alpha(DP(\beta))} x^w.
\end{aligned}$$

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On the other hand, assuming Conjecture 4.4 holds, Theorem 5.8 gives

$$\begin{aligned}(5.8)\quad \nabla(C_\alpha[X;q]) &= \sum_{\beta\leq\alpha}(-q)^{\ell(\alpha)-\ell(\beta)}q^{-\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta}))}\nabla(B_\beta[X;q])\\ &= \sum_{\beta\leq\alpha}\sum_{D\leq DP(\beta)}\sum_{w\in\mathcal{WP}_D} t^{\mathrm{area}(D)}(-q)^{\ell(\alpha)-\ell(\beta)}q^{\mathrm{dinv}(w)+\mathrm{doff}_\beta(D)-\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta}))}x^w.\end{aligned}$$

For a Dyck path D in the sum, let $\gamma = \mathrm{touch}(D)$. Note that $D \leq DP(\beta)$ implies that $\gamma \leq \beta \leq \alpha$, and thus we may rearrange sums as follows:

$$\sum_{\beta\leq\alpha}\sum_{D\leq DP(\beta)}(-q)^{\ell(\alpha)-\ell(\beta)}q^{\mathrm{doff}_\beta(D)-\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta}))} = \sum_{\gamma\leq\alpha}\sum_{\substack{D\\ \mathrm{touch}(D)=\gamma}}\sum_{\substack{\beta\\ \gamma\leq\beta\leq\alpha}}(-1)^{\ell(\alpha)-\ell(\beta)}q^{\ell(\alpha)-\ell(\beta)+\mathrm{doff}_\beta(D)-\mathrm{doff}_{\overleftarrow{\alpha}}(DP(\overleftarrow{\beta}))}.$$

Lemma 5.7 allows us to conclude that (5.8) reduces to Conjecture 4.5. ■

Corollary 5.10 *Conjecture* 4.2 *is true if and only if Conjecture* 4.3 *is true.*

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