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UNIFORMLY BOUNDED COMPOSITION OPERATORS

DOROTA GŁAZOWSKA✉ and JANUSZ MATKOWSKI

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Abstract

We prove that if a uniformly bounded (or equidistantly uniformly bounded) Nemytskij operator maps the space of functions of bounded φ-variation with weight function in the sense of Riesz into another space of that type (with the same weight function) and its generator is continuous with respect to the second variable, then this generator is affine in the function variable (traditionally, in the second variable).

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1. Introduction

Given a real interval I, normed spaces X, Y, a closed set $C \subset X$ and a function $h : I \times C \to Y$, the mapping $H : X^I \to Y^I$ defined by

$$H(f)(x) := h(x, f(x)), \quad f \in X^I, x \in I$$

(where X^I denotes the set of all functions $f : I \to X$), is called the composition Nemytskij operator of the generator h.

The theory of this operator and its role in applications can be found in [1]. Recall, for instance, that for every locally defined self-mapping the set of continuous functions must be a composition Nemytskij operator (see [7, 10–14]). According to a well-known result of Krasnosielskij, H is a self-map of the set of real continuous functions into X if and only if its generator h is continuous. In this situation it is rather unexpected that there are discontinuous functions $h : I \times \mathbb{R} \to \mathbb{R}$ generating composition operators H which map the space of continuously differentiable functions $C^1(I, \mathbb{R})$ into itself (see [1, page 209] and [7]). Another astonishing property of this nonlinear operator H was discovered in [8]: if H is a Lipschitzian self-map of the Banach space $\mathrm{Lip}(I, \mathbb{R})$, then

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in \mathbb{R}, \tag{$*$}$$

for some Lipschitz functions a and b, that is, the generator h of H is affine in the the second (or function) variable. This result has been extended to some other function

Banach spaces (see [1] for other references). Results of this type show that the presence of locally defined operators (operators with memory) in some operator equations may essentially restrict the possibility of application of the Banach contraction principle.

In [3] it has been proved that if H maps the space $GRV_{\varphi,\lambda}(I, C)$ of functions of bounded φ-variation with weight λ in the sense of Riesz into the space $GRV_{\psi,\lambda}(I, Y)$ and is uniformly continuous, then h, the generator function of the operator H, is affine in the second variable.

In [9] it is proved that any uniformly bounded composition operator (see Definition 2.3) acting between general Lipschitz function normed spaces must be of form (∗). Similar results hold for the Banach space of absolutely continuous functions (see [5]), and the space of functions of bounded (p, k)-variation in the sense of Riesz and Popoviciu, for $p \geq 1$ and integer k (see [2]).

The main result of this paper says that, under a weak regularity condition, the generator of every uniformly bounded (or equidistantly uniformly bounded) composition operator mapping the space $GRV_{\varphi,\lambda}(I, C)$ of functions of bounded φ-variation with weight λ in the sense of Riesz into the space $GRV_{\psi,\lambda}(I, Y)$ is an affine function with respect to the second variable.

2. Preliminaries

In this section we introduce useful notation and definitions and recall some results concerning the Riesz φ-variation with weight.

Let $\mathcal{F}$ be the set of all convex continuous functions $\varphi : [0, \infty) \to [0, \infty)$ such that

$$\varphi(0) = 0, \quad \varphi(t) > 0 \quad \text{for all } t > 0, \quad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

Obviously, every $\varphi \in \mathcal{F}$ is strictly increasing.

Given $\varphi \in \mathcal{F}$, we define the function $\delta_\varphi : (0, \infty) \to (0, \infty)$ by

$$\delta_\varphi(r) := r\varphi^{-1}(1/r), \quad r > 0,$$

where φ^{-1} stands for the inverse of φ. Note that the function δ_φ is continuous, concave,

$$\lim_{r\to 0^+} \delta_\varphi(r) = \lim_{t\to\infty} \frac{t}{\varphi(t)} = 0,$$

and, consequently, it is subadditive (cf. [4]).

Let $(X, |\cdot|)$ be a real normed space and $I \subset \mathbb{R}$ be an arbitrary interval. For $\varphi \in \mathcal{F}$ and a continuous strictly increasing function $\lambda : I \to \mathbb{R}$ we define the Riesz φ-variation $RV_{\varphi,\lambda}(f)$ of the function $f : I \to X$ with respect to the *weight function* λ by (cf. [4])

$$RV_{\varphi,\lambda}(f) := \sup\{RV_{\varphi,\lambda}(f, \tau) : \tau \in \mathcal{P}(I)\},$$

where $\mathcal{P}(I)$ denotes the set of all finite strictly increasing sequences (called partitions of I) $\tau = (t_0, \dots, t_m)$ such that $t_i \in I$ for $i = 0, 1, \dots, m$, and

$$RV_{\varphi,\lambda}(f, \tau) := \sum_{i=1}^{m} \varphi\Big(\frac{|f(t_i) - f(t_{i-1})|}{\lambda(t_i) - \lambda(t_{i-1})}\Big)(\lambda(t_i) - \lambda(t_{i-1})).$$

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We say that f has a bounded Riesz φ-variation with weight λ on I, if $RV_{\varphi,\lambda}(f) < \infty$. By $BRV_{\varphi,\lambda}(I)$ we denote the set of all functions of bounded Riesz φ-variation with weight λ on I.

Remark 2.1 (Cf. [4, Lemma 3]). The set $BRV_{\varphi,\lambda}(I)$ is a linear space if and only if the function φ satisfies the Δ_2 condition, that is, there exist $t_0 \geq 0$ and $C > 0$ such that $\varphi(2t) \leq C\varphi(t)$ for all $t \geq t_0$.

We denote by $GRV_{\varphi,\lambda}(I, X)$ the set of all functions $f \in X^I$ such that $RV_{\varphi,\lambda}(\gamma f) < \infty$ for some $\gamma > 0$. Clearly,

$$BRV_{\varphi,\lambda}(I, X) \subset GRV_{\varphi,\lambda}(I, X).$$

Moreover, the set $GRV_{\varphi,\lambda}(I, X)$ is a linear space (cf. [4]).

In the space $GRV_{\varphi,\lambda}(I, X)$ we define the norm $\|\cdot\|_\varphi$ by

$$\|f\|_\varphi := |f(x_0)| + \mathbf{p}_\varphi(f), \quad f \in GRV_{\varphi,\lambda}(I, X),$$

where $x_0 \in I$ is arbitrarily fixed and

$$\mathbf{p}_\varphi(f) := \inf\{\varepsilon > 0 : RV_{\varphi,\lambda}(f/\varepsilon) \leq 1\}$$

is the Luxemburg–Nakano seminorm. Moreover, if X is a Banach space, then $GRV_{\varphi,\lambda}(I, X)$ equipped with the norm $\|\cdot\|_\varphi$ becomes a Banach space.

In what follows, for a subset C of X, the symbol $GRV_{\varphi,\lambda}(I, C)$ stands for the set of all functions $f \in GRV_{\varphi,\lambda}(I, X)$ such that $f : I \to C$.

Lemma 2.2 (Cf. [4, Lemma 4]). *Let $\varphi \in \mathcal{F}$ and $f \in GRV_{\varphi,\lambda}(I, X)$.*

(a) *If $t, s \in I$, then $|f(t) - f(s)| \leq \delta_\varphi(|\lambda(t) - \lambda(s)|)\mathbf{p}_\varphi(f)$.*

(b) *If $\mathbf{p}_\varphi(f) > 0$, then $RV_{\varphi,\lambda}(f/\mathbf{p}_\varphi(f)) \leq 1$.*

(c) *If $r > 0$, then:*

  (c_1) *$\mathbf{p}_\varphi(f) \leq r$ if and only if $RV_{\varphi,\lambda}(f/r) \leq 1$;*

  (c_2) *if $RV_{\varphi,\lambda}(f/r) = 1$, then $\mathbf{p}_\varphi(f) = r$ (but not vice versa in general).*

By $\mathcal{A}(X, Y)$ we denote the space of all additive mappings $a : X \to Y$.

We now recall definitions of uniformly bounded and equidistantly uniformly bounded mappings introduced in [9].

Definition 2.3. Let $\mathcal{Y}$ and $\mathcal{Z}$ be two metric (or normed) spaces. We say that a mapping $H : \mathcal{Y} \to \mathcal{Z}$ is:

(i) *uniformly bounded* if, for any $t > 0$, there is a nonnegative real number $\gamma(t)$ such that, for any nonempty set $B \subset \mathcal{Y}$,

$$\operatorname{diam} B \leq t \Longrightarrow \operatorname{diam} H(B) \leq \gamma(t);$$

(ii) *equidistantly uniformly bounded* if, for every $t > 0$, there is a nonnegative real number $\gamma(t)$ such that, for all $u, v \in B \subset \mathcal{Y}$,

$$\operatorname{diam}\{u, v\} = t \Longrightarrow \operatorname{diam}\{H(u), H(v)\} \leq \gamma(t).$$

Notice that every operator of bounded range is uniformly bounded.

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3. Main results

We begin this section with the following theorem.

THEOREM 3.1. *Let $I \subset \mathbb{R}$ be an interval, $\lambda : I \to \mathbb{R}$ a fixed continuous strictly increasing function and $\varphi, \psi \in \mathcal{F}$. Suppose that $(X, |\cdot|)$ is a real normed space, $(Y, |\cdot|)$ is a real Banach space, $C \subset X$ is a closed and convex set, and the function $h : I \times C \to Y$ is continuous with respect to the second variable. If there exists a function $\gamma : [0, \infty) \to [0, \infty)$ such that the composition operator H of the generator h maps the set $GRV_{\varphi,\lambda}(I, C)$ into $GRV_{\psi,\lambda}(I, Y)$ and satisfies the inequality*

$$\|H(f_1) - H(f_2)\|_\psi \le \gamma(\|f_1 - f_2\|_\varphi), \quad f_1, f_2 \in GRV_{\varphi,\lambda}(I, C), \tag{3.1}$$

then there exist an additive function $a : I \to Y$ and a function $b : I \to Y$ such that

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in C.$$

PROOF. Obviously, for arbitrary $y \in C$ the constant function $f : I \to C$ defined by

$$f(x) = y, \quad x \in I,$$

belongs to $GRV_{\varphi,\lambda}(I, C)$. Since the composition operator H maps $GRV_{\varphi,\lambda}(I, C)$ into $GRV_{\psi,\lambda}(I, Y)$, for every $y \in C$, the function $h(\cdot, y)$ belongs to $GRV_{\psi,\lambda}(I, Y)$, and the function h is continuous with respect to the first variable.

From the definition of the norm $\|\cdot\|_\psi$ we get

$$\mathbf{p}_\psi(H(f_1) - H(f_2)) \le \|H(f_1) - H(f_2)\|_\psi, \quad f_1, f_2 \in GRV_{\varphi,\lambda}(I, C).$$

Hence, in view of Lemma 2.2 and inequality (3.1), we obtain

$$RV_{\psi,\lambda}\Big(\frac{H(f_1) - H(f_2)}{\gamma(\|f_1 - f_2\|_\varphi)}\Big) \le 1 \quad \text{if } \gamma(\|f_1 - f_2\|_\varphi) > 0.$$

Now, by the definitions of $RV_{\psi,\lambda}(\cdot)$ and H, for any $f_1, f_2 \in GRV_{\varphi,\lambda}(I, C)$ and $\alpha, \beta \in I$, $\alpha < \beta$, we get

$$\psi\Big(\frac{|h(\beta, f_1(\beta)) - h(\beta, f_2(\beta)) - h(\alpha, f_1(\alpha)) + h(\alpha, f_1(\alpha))|}{\gamma(|f_1 - f_2|)(\lambda(\beta) - \lambda(\alpha))}\Big) \le \frac{1}{\lambda(\beta) - \lambda(\alpha)}.$$

Hence,

$$\begin{aligned} &|h(\beta, f_1(\beta)) - h(\beta, f_2(\beta)) - h(\alpha, f_1(\alpha)) + h(\alpha, f_2(\alpha))| \\ &\quad \le \gamma(|f_1 - f_2|)(\lambda(\beta) - \lambda(\alpha))\psi^{-1}\Big(\frac{1}{\lambda(\beta) - \lambda(\alpha)}\Big), \end{aligned}$$

and, consequently,

$$|h(\beta, f_1(\beta)) - h(\beta, f_2(\beta)) - h(\alpha, f_1(\alpha)) + h(\alpha, f_2(\alpha))| \le \gamma(|f_1 - f_2|)\delta_\psi(\lambda(\beta) - \lambda(\alpha)), \tag{3.2}$$

for any $f_1, f_2 \in GRV_{\varphi,\lambda}(I, C)$ and all $\alpha, \beta \in I$, $\alpha < \beta$.

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For $\alpha, \beta \in I$, $\alpha < \beta$, define the function $\eta_{\alpha,\beta} : \mathbb{R} \to [0, 1]$ by the formula

$$\eta_{\alpha,\beta}(x) := \begin{cases} 0 & x \le \alpha, \\ \dfrac{\lambda(x) - \lambda(\alpha)}{\lambda(\beta) - \lambda(\alpha)} & \alpha \le x \le \beta, \\ 1 & \beta \le x, \end{cases}$$

and for every $y_1, y_2 \in C$, $y_1 \neq y_2$, consider the functions $f_1, f_2 : I \to X$ given by

$$f_j(x) := \tfrac{1}{2}(\eta_{\alpha,\beta}(x)(y_1 - y_2) + y_j + y_2), \quad x \in I, j = 1, 2.$$

Note that the functions $f_1, f_2 \in GRV_{\varphi,\lambda}(I, C)$ have the following properties:

$$f_1(\beta) = y_1, \quad f_2(\beta) = \frac{y_1 + y_2}{2}, \quad f_1(\alpha) = \frac{y_1 + y_2}{2}, \quad f_2(\alpha) = y_2,$$
$$f_1(x) - f_2(x) = \frac{y_1 - y_2}{2}, \quad x \in I,$$
$$\|f_1 - f_2\|_\varphi = \frac{|y_1 - y_2|}{2}.$$

Hence, by inequality (3.2),

$$\left| h(\beta, y_1) - h\Big(\beta, \frac{y_1 + y_2}{2}\Big) - h\Big(\alpha, \frac{y_1 + y_2}{2}\Big) + h(\alpha, y_2) \right| \le \gamma\Big(\frac{|y_1 - y_2|}{2}\Big)\delta_\psi(\lambda(\beta) - \lambda(\alpha)),$$

for all $y_1, y_2 \in C$, $y_1 \neq y_2$. Letting α and β tend to $x \in [\alpha, \beta]$ and making use of the continuity of the function h with respect to the first variable,

$$h\Big(x, \frac{y_1 + y_2}{2}\Big) = \frac{h(x, y_1) + h(x, y_2)}{2}, \quad x \in I, y_1, y_2 \in C,$$

that is, for every $x \in I$, the function $h(x, \cdot)$ satisfies the Jensen functional equation. The assumed continuity of h with respect to the second variable implies that (cf. [6, Theorem 1, page 315]) there exist an additive function $a : I \to Y$ and $b : I \to Y$ such that

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in C,$$

which finishes the proof. □

Applying this theorem we get our main results.

Theorem 3.2. *Let $I \subset \mathbb{R}$ be an interval, $\lambda : I \to \mathbb{R}$ a fixed continuous strictly increasing function and $\varphi, \psi \in \mathcal{F}$. Suppose that $(X, |\cdot|)$ is real normed space, $(Y, |\cdot|)$ is a real Banach space, $C \subset X$ is a closed and convex set, and $h : I \times C \to Y$ is continuous with respect to the second variable. If the composition operator H of the generator h maps the set $GRV_{\varphi,\lambda}(I, C)$ into the Banach space $GRV_{\psi,\lambda}(I, Y)$ and is uniformly bounded, then there exist functions $a \in \mathcal{A}(I, Y)$ and $b \in Y^I$ such that*

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in C.$$

Proof. Take any $t \geq 0$ and arbitrary $f, g \in GRV_{\varphi,\lambda}(I, C)$ such that $\|f - g\|_\varphi = t$. The uniform boundedness of H implies that diam $H(\{f, g\}) \leq \gamma(t)$, that is,

$$\|H(f) - H(g)\|_\psi = \operatorname{diam} H(\{f, g\}) \leq \gamma(\|f - g\|_\varphi),$$

so it is enough to apply Theorem 3.1. □

Remark 3.3. Clearly, the continuity of γ at 0 and $\gamma(0) = 0$, imply that the uniformly bounded operator H is uniformly continuous. It follows that Theorem 3.2 improves the result of [3] where H is assumed to be uniformly continuous.

Similarly, by Theorem 3.1, we obtain the following result.

Theorem 3.4. *Let $I \subset \mathbb{R}$ be an interval, $\lambda : I \to \mathbb{R}$ a fixed continuous strictly increasing function and $\varphi, \psi \in \mathcal{F}$. Suppose that $(X, |\cdot|)$ is real normed space, $(Y, |\cdot|)$ is a real Banach space, $C \subset X$ is a closed and convex set, and $h : I \times C \to Y$ is continuous with respect to the second variable. If the composition operator H of the generator h maps the set $GRV_{\varphi,\lambda}(I, C)$ into the Banach space $GRV_{\psi,\lambda}(I, Y)$ and is equidistantly uniformly bounded, then*

$$h(x, y) = a(x)y + b(x), \quad x \in I, y \in C,$$

for some functions $a : I \to \mathcal{A}(I, Y)$ and $b \in Y^I$.

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DOROTA GŁAZOWSKA, Faculty of Mathematics,
Computer Science and Econometrics,
University of Zielona Góra, Szafrana 4a,
65-516 Zielona Góra, Poland
e-mail: D.Glazowska@wmie.uz.zgora.pl

JANUSZ MATKOWSKI, Faculty of Mathematics,
Computer Science and Econometrics,
University of Zielona Góra, Szafrana 4a,
65-516 Zielona Góra, Poland
e-mail: J.Matkowski@wmie.uz.zgora.pl