ON HOMEOMORPHISMS BETWEEN EXTENSION SPACES

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Introduction. In this note, conditions are obtained which will ensure that two topological spaces are homeomorphic when they have homeomorphic extension spaces of a certain kind. To discuss this topic in suitably general terms, an unspecified extension procedure, assumed to be applicable to some class of topological spaces, is considered first, and it is shown that simple conditions imposed on the extension procedure and its domain of operation easily lead to a condition of the desired kind. After the general result has been established it is shown to be applicable to a number of particular extensions, such as the Stone-Čech compactification and the Hewitt Q-extension of a completely regular Hausdorff space, Katětov's maximal Hausdorff-closed extension of a Hausdorff space, the maximal zero-dimensional compactification of a zero-dimensional space, the maximal Hausdorff-minimal extension of a semi-regular space, and Freudenthal's compactification of a rim-compact space. The case of the Hewitt Q-extension was first discussed by Heider **(6)**.

1. Preliminaries. A pair (Γ, γ) consisting of a class Γ of Hausdorff spaces and a mapping $\gamma : \Gamma \to \Gamma$ such that γX contains X as a dense subspace for each $X \in \Gamma$ will be called an *extension structure.* An extension structure (Γ, γ) will be called *normal* if for any $X \in \Gamma$ the subspaces $X - \{a\}$, $a \in X$, of X also belong to Γ and γ has the properties:

$C1$. Any sequence of dense imbeddings

$$X \xrightarrow{f} Y \xrightarrow{g} \gamma X \qquad (X, Y \in \Gamma)$$

such that $g \circ f$ is the identity mapping on X can be extended to a sequence of homeomorphisms

$$\gamma X \xrightarrow{f^\gamma} \gamma Y \xrightarrow{g^\gamma} \gamma X.$$

$C2$. For each $a \in X$ there exists a dense imbedding

$$h_a : \gamma X - \{a\} \to \gamma(X - \{a\})$$

which induces the identity mapping on $X - \{a\}$.

It follows easily from $C1$ that the extension space γX is a topological invariant of X in the sense that any homeomorphism $X \to Y$ can be extended to a homeomorphism $\gamma X \to \gamma Y$. Also, one has

LEMMA 1. $\gamma(\gamma X) = \gamma X$ *for any* $X \in \Gamma$.

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Proof. Consider the sequence of dense imbeddings

$$X \xrightarrow{j} \gamma X \xrightarrow{i} \gamma X$$

where j is the natural injection of X into γX and i the identity mapping. Then, $C1$ gives, as an extension of this, the sequence of homeomorphisms

$$\gamma X \xrightarrow{j^\gamma} \gamma(\gamma X) \xrightarrow{i^\gamma} \gamma X.$$

Now, if $u \in \gamma(\gamma X) - \gamma X$ and $v = (j^\gamma)^{-1}u \in \gamma X$ then $v = (i^\gamma \circ j^\gamma)v = i^\gamma u$, but also $v = iv = i^\gamma v$ and hence $i^\gamma u = i^\gamma v$ which leads to $u = v \in \gamma X$, a contradiction.

In view of Lemma 1 one can say that $C1$ implies a certain *minimality* of the extension space γX of X: If $Y \in \Gamma$ is such that $\gamma Y = Y$ and $X \subseteq Y \subseteq \gamma X$ then $Y = \gamma X$, and thus γX is minimal in the class of all spaces $Y \supseteq X$, $Y \in \Gamma$, with $\gamma Y = Y$.

For a number of known normal extension structures (Γ, γ) the following further condition is found to hold:

$C3$. Any dense imbedding $f : X \to Y(X, Y \in \Gamma)$ can be extended to a continuous mapping f^γ of γX onto γY.

Such extension structures will be called *strongly normal*. Strong normality can usually be checked by means of

LEMMA 2. *$C3$ and Lemma 1 together imply $C1$.*

Proof. If $f : X \to Y$ and g: $Y \to \gamma X$ are dense imbeddings such that $g \circ f$ is the identity mapping on X then the extensions

$$f^\gamma : \gamma X \to \gamma Y \quad \text{and} \quad g^\gamma : \gamma Y \to \gamma(\gamma X) = \gamma X$$

given by $C3$ are necessarily homeomorphisms since $g^\gamma \circ f^\gamma$ is the identity mapping on γX.

The conditions of strong normality imply a certain *maximality* of the extension space γX of X, which complements the above mentioned minimality: If X is a dense subspace of Y with $\gamma Y = Y(X, Y \in \Gamma)$, then the natural injection $X \to Y$ can be extended to a continuous mapping of γX onto Y in the case of strongly normal (Γ, γ).

2. The principal result.

Let (Γ, γ) be a normal extension structure throughout this section and denote by $\gamma_0 X (X \in \Gamma)$ the subspace (not necessarily belonging to Γ) of X consisting of all those $a \in X$ for which the natural injection $i : X - \{a\} \to X$ cannot be extended to a homeomorphism of $\gamma(X - \{a\})$ onto γX. These subspaces $\gamma_0 X$ have the following basic property:

LEMMA 3. *For any sequence of dense imbeddings*

$$X \xrightarrow{f} Y \xrightarrow{g} \gamma X \qquad (X, Y \in \Gamma)$$

such that $g \circ f$ is the identity mapping on X one has $f(\gamma_0 X) = \gamma_0 Y$.

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Proof. First, it will be shown that $\gamma_0 Y \subseteq fX$. For any $c \in Y - fX$, the given sequence of dense imbeddings leads to a new such sequence

$$X \xrightarrow{h} Y - \{c\} \xrightarrow{k} \gamma X \qquad (h, k \text{ induced by } f, g)$$

which has the extension

$$\gamma X \xrightarrow{h^\gamma} \gamma(Y - \{c\}) \xrightarrow{k^\gamma} \gamma X;$$

also, the given sequence itself can be extended to

$$\gamma X \xrightarrow{f^\gamma} \gamma Y \xrightarrow{g^\gamma} \gamma X.$$

It follows that $f^\gamma \circ k^\gamma$ is a homeomorphism of $\gamma(Y - \{c\})$ onto γY, and since it maps the dense subset fX of $Y - \{c\}$ identically it extends the identity mapping $Y - \{c\} \to Y$. Hence, $c \notin \gamma_0 Y$ and thus $\gamma_0 Y \subseteq fX$.

Next, take $a \in X$ and $b = fa$. Here, one has the sequence of dense imbeddings

$$X - \{a\} \xrightarrow{h} Y - \{b\} \xrightarrow{k} \gamma X - \{a\} \xrightarrow{h_a} \gamma(X - \{a\})$$

where h and k are again restrictions of f and g respectively and h_a is given by $C2$. From $C2$ and the assumption that $g \circ f$ is the identity mapping on X one concludes that $(h_a \circ k) \circ h$ is the identity mapping on $X - \{a\}$; hence, $C1$ is applicable and one has a sequence of homeomorphisms

$$(1) \qquad \gamma(X - \{a\}) \xrightarrow{h^\gamma} \gamma(Y - \{b\}) \xrightarrow{(h_a \circ k)^\gamma} \gamma(X - \{a\}).$$

Now, suppose $a \notin \gamma_0 X$ and let i^* be the homeomorphism $\gamma(X - \{a\}) \to \gamma X$ extending the natural injection $i : X - \{a\} \to X$. Then, one obtains from (1) the sequence of homeomorphisms

$$(2) \qquad \gamma(Y - \{b\}) \xrightarrow{(h_a \circ k)^\gamma} \gamma(X - \{a\}) \xrightarrow{i^*} \gamma X \xrightarrow{f^\gamma} \gamma Y,$$

which has the following effect on any point fz, $z \in X - \{a\}$:

$$fz \to z \to z \to fz.$$

Hence, the homeomorphism $\gamma(Y - \{b\}) \to \gamma Y$ given by (2) maps the dense subset $f(X - \{a\})$ of $Y - \{b\}$ identically, and therefore the same holds for the whole of $Y - \{b\}$. Thus, one has $b \notin \gamma_0 Y$ or $f(X - \gamma_0 X) \subseteq Y - \gamma_0 Y$, and from $\gamma_0 Y \subseteq fX$ it now follows that $\gamma_0 Y \subseteq f(\gamma_0 X)$.

Conversely, assume $b \notin \gamma_0 Y$ and let j^* be the homeomorphism $\gamma(Y - \{b\}) \to \gamma Y$ extending the natural injection j: $Y - \{b\} \to Y$. Again, one obtains from (1) a sequence of homeomorphisms

$$(3) \qquad \gamma(X - \{a\}) \xrightarrow{h^\gamma} \gamma(Y - \{b\}) \xrightarrow{j^*} \gamma Y \xrightarrow{g^\gamma} \gamma X,$$

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giving a homeomorphism $\gamma(X - \{a\}) \to \gamma X$ which clearly extends the natural injection $X - \{a\} \to X$. This shows $a \notin \gamma_0 X$ which implies $f(\gamma_0 X) \subseteq \gamma_0 Y$; in all, $f(\gamma_0 X) = \gamma_0 Y$ is hereby established.

The particular case where $f : X \to Y$ is a homeomorphism and $g : Y \to \gamma X$ taken as $y \to f^{-1}y$, $y \in Y$, makes it obvious that any homeomorphism $f : X \to Y$ maps $\gamma_0 X$ homeomorphically onto $\gamma_0 Y$, that is, the subspaces $\gamma_0 X$ are topological invariants of the spaces X.

Lemma 3 now leads immediately to

PROPOSITION 1. *If $\Gamma_0 \subseteq \Gamma$ is the class of all $X \in \Gamma$ such that $\gamma_0 X = X$ then any homeomorphism $f : X' \to Y'$ between spaces X', $Y' \in \Gamma$ such that $X \subseteq X' \subseteq \gamma X$, $Y \subseteq Y' \subseteq \gamma Y$ and $X, Y \in \Gamma_0$ is the extension of a homeomorphism $X \to Y$.*

Proof. One has $\gamma_0 X = \gamma_0 X'$, $\gamma_0 Y = \gamma_0 Y'$ and $f(\gamma_0 X') = \gamma_0 Y'$ from Lemma 3 and hence from the given equations $X = \gamma_0 X$ and $Y = \gamma_0 Y$ also $fX = Y$.

3. Characterization of Γ_0 for strongly normal (Γ, γ). Proposition 1 naturally leads to the question whether there exist other subclasses of Γ, (Γ, γ) being any normal extension structure, for which the analogous proposition holds. Of course, this is trivially so for any subclass of Γ_0. Similarly, there may exist trivial enlargements of Γ_0 with this property: If, for instance, Γ contains an X such that $\gamma X = X$ but $\gamma_0 X' \neq X'$ for any dense subspace of X, then Γ_0 does not contain any space homeomorphic to X, and $\Gamma_0 \cup \{X\}$ would be of the said type. Consequently, one has to look for further properties of Γ_0 which together with Proposition 1 will give rise to some characterization of Γ_0. This will be done in the present section at least for the case of strongly normal extension structures (Γ, γ).

A preliminary result is:

LEMMA 4. *If (Γ, γ) is a strongly normal extension structure and $f : X \to Y$ $(X, Y \in \Gamma)$ a dense imbedding then $(f^\gamma)^{-1} fX = X$.*

Proof. Let $a \in X$ and $b \in \gamma X - X$ be such that $fa = f^\gamma b$. This has to be shown to lead to a contradiction. Since γX is Hausdorff there exist disjoint neighbourhoods U and V of a and b respectively in γX. Also, since f is an imbedding there exists a neighbourhood W of fa in γY such that $f(U \cap X) = W \cap fX$. Now, by the continuity of f^γ there exists a neighbourhood $V_0 \subseteq V$ of b in γX such that $f^\gamma V_0 \subseteq W$. This implies $f(V_0 \cap X) \subseteq W \cap fX = f(U \cap X)$ and therefore $V_0 \cap X \subseteq U \cap X$ since f is one-to-one; this, however, leads to the contradiction $\phi \neq V_0 \cap X \subseteq U \cap V = \phi$.

The desired property of Γ_0 will be obtained from the following

LEMMA 5. *If (Γ, γ) is a strongly normal extension structure then $\gamma_0 X - \{a\} \subseteq \gamma_0(X - \{a\})$ for any $a \in X$, $X \in \Gamma$.*

Proof. Let $b \in X - \{a\}$ be such that there exists a homeomorphism h:

$\gamma(X - \{a, b\}) \to \gamma(X - \{a\})$ inducing the identity mapping on $X - \{a, b\}$. Then b is not isolated in X and one has the following diagram of continuous mappings

$$\begin{array}{ccc} \gamma(X - \{b\}) - \{a\} & \xrightarrow{g} & \gamma(X - \{a, b\}) \\ k \downarrow & & \downarrow h \\ \gamma X - \{a\} & \xrightarrow[j]{} & \gamma(X - \{a\}) \end{array}$$

where g and j are the imbeddings given by $C2$ and k is the restriction of the continuous extension i^γ of the natural injection $i\colon X - \{b\} \to X$ by $C3$ which is known to map $\gamma(X - \{b\}) - \{a\}$ onto $\gamma X - \{a\}$ by Lemma 4. Since all mappings induce the identity on $X - \{a, b\}$ the diagram is commutative. It follows that k must be one-to-one since g and h are, and consequently i^γ: $\gamma(X - \{b\}) \to \gamma X$ is also one-to-one. This means that i^γ has an inverse f, and this coincides with the h_b (of $C2$) on $\gamma X - \{b\}$ as one sees immediately from the sequence

$$\gamma X - \{b\} \xrightarrow{h_b} \gamma(X - \{b\}) \xrightarrow{i^\gamma} \gamma X.$$

This shows that the restriction of f to $\gamma X - \{b\}$ is continuous. On the other hand, the restriction of f to $\gamma X - \{a\}$ is merely k^{-1}. Now, $h^{-1} \circ j$ and g map $\gamma X - \{a\}$ and $\gamma(X - \{b\}) - \{a\}$ respectively onto the same subspace of $\gamma(X - \{a, b\})$; thus g^{-1} is defined at each $h^{-1}(jx)$, $x \in \gamma X - \{a\}$, and clearly $k^{-1}x = g^{-1}(h^{-1}(jx))$. Since $h^{-1} \circ j$ and g are dense imbeddings it follows that k^{-1} is continuous. Therefore, the restriction of f to $\gamma X - \{a\}$ is also continuous, and this shows f to be continuous, hence i^γ to be a homeomorphism and finally $b \notin \gamma_0 X$.

With this it is proved that $b \notin \gamma_0(X - \{a\})$ implies $b \notin \gamma_0 X$ which immediately gives the desired result $\gamma_0 X - \{a\} \subseteq \gamma_0(X - \{a\})$.

After these preparations, the following characterization of Γ_0 can easily be established.

PROPOSITION 2. *If* (Γ, γ) *is a strongly normal extension structure then* $X \in \Gamma_0$ *implies* $X - \{a\} \in \Gamma_0$ *for any* $a \in X$ *and* Γ_0 *is the largest class of spaces* $X \in \Gamma$ *for which this condition and Proposition* 1 *hold.*

Proof. By Lemma 5, $\gamma_0 X = X$ implies $X - \{a\} = \gamma_0 X - \{a\} \subseteq \gamma_0(X - \{a\})$ and thus $\gamma_0(X - \{a\}) = X - \{a\}$, as stated. Now, let Γ_1 be any subclass of Γ such that $X - \{a\} \in \Gamma_1$ for any $X \in \Gamma_1$, $a \in X$, and Proposition 1 holds for Γ_1 (in place of Γ_0). Then, if $X \in \Gamma_1$ does not belong to Γ_0 there must be an $a \in X$ such that the natural injection $X - \{a\} \to X$ can be extended to a homeomorphism $\gamma(X - \{a\}) \to \gamma X$. However, this homeomorphism clearly does not induce a homeomorphism $X - \{a\} \to X$, and this contradicts the assumptions for Γ_1. It follows that $\Gamma_1 \subseteq \Gamma_0$.

Remark. We do not know whether Proposition 2 might not be true for *any* normal extension structure (Γ, γ). It is clear that Γ_0 is characterized as above

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whenever $X - \{a\} \in \Gamma_0$ for all $a \in X$, $X \in \Gamma_0$, and even for those (Γ, γ) considered below which are not (or not known to be) strongly normal this can actually be verified explicitly.

With this, the general considerations are concluded and the following sections deal with their application to particular extension structures, that is, with the proofs of the normality or strong normality and the explicit descriptions of Γ_0 for several instances of (Γ, γ). For the latter, it is useful to observe that for any normal (Γ, γ) and $X \in \Gamma$ the points $a \notin \gamma_0 X$ of X cannot be isolated simply because the inverse of the supposed homeomorphism $\gamma(X - \{a\}) \to \gamma X$ maps $a \in X$ into a point of $\gamma(X - \{a\}) - (X - \{a\})$ and any neighbourhood of such a point must meet $X - \{a\}$.

4. Stone-Čech compactifications and Hewitt Q-extensions. Let (B, β) be the extension structure for which B is the class of all completely regular Hausdorff spaces and βX the Stone-Čech compactification of $X \in \mathrm{B}$. Obviously, $X \in \mathrm{B}$ implies $X - \{a\} \in \mathrm{B}$ for any $a \in X$, $\beta(\beta X) = \beta X$ holds for all $X \in \mathrm{B}$ and any dense imbedding $X \to Y$ has a continuous extension mapping βX onto βY by the well-known maximality property of βX.

To obtain, for each $a \in X$, an imbedding of $\beta X - \{a\}$ into $\beta(X - \{a\})$ as described in $C3$ it is sufficient to prove that every bounded continuous real function f on $X - \{a\}$ can be continuously extended to $\beta X - \{a\}$ **(7)**. To show this, let $u \in \beta X - X$ be any point, V and W disjoint closed neighbourhoods in βX of u and a respectively and g a continuous function on βX such that $gV = \{1\}$ and $gW = \{0\}$. Now, with the restriction h of g to $X - \{a\}$ the product fh is continuous on $X - \{a\}$ and vanishes on $W \cap (X - \{a\})$; it can therefore be extended continuously to X with value 0 at a, and the resulting function has a continuous extension f^* to βX. f^* satisfies $f^*x = fx \cdot hx = fx$ for all $x \in V \cap X$, hence

$$\lim_{\substack{x \to u \\ x \in X - \{a\}}} fx = \lim_{\substack{x \to u \\ x \in X.}} f^*x = f^*u.$$

The existence of this limit for any $u \in \beta X - \{a\}$ implies **(4**, ch. I, § 6**)** that f can be continuously extended to $\beta X - \{a\}$.

In all, it is then established that β satisfies the conditions $C1$—$C3$. Further, it is obvious that for any non-isolated $a \in X$ the extension of the natural injection $X - \{a\} \to X$ to $\beta(X - \{a\})$ is a homeomorphism if and only if any bounded continuous real function on $X - \{a\}$ has a continuous extension to X. Hence one has:

PROPOSITION 3. *The extension structure* (B, β) *is strongly normal and* B_0 *is the class of all* $X \in \mathrm{B}$ *such that for any non-isolated* $a \in X$ *there exist bounded continuous real functions on* $X - \{a\}$ *without a continuous extension to* X.

Remark. The class B_0 can also be described in a variety of other ways such as: the filter on $X - \{a\}$ given by the sets $V - \{a\}$, V the neighbourhoods of a in X, is not maximal completely regular (**4**, ch. IX, p. 15) on $X - \{a\}$ for any $a \in X$; or, as in (**6**), the finite open normal coverings of $X - \{a\}$ are not all induced by such coverings of X. In the case of particular types of spaces X, simpler topological conditions can be given. Thus, for fully normal X, $X \in \mathrm{B}_0$ is equivalent to the condition that $\{a\} = A \cap B$ with closed sets A, B both different from $\{a\}$, for each $a \in X$. Similarly, for locally compact $X, X \in B_0$ amounts to the existence of compact $A, B \subseteq X$ with $A \cap B = \{a\}$, A and B different from $\{a\}$, for each $a \in X$.

If one considers the Hewitt Q-extension υX for completely regular Hausdorff spaces X, one can easily see, by essentially the same argument as above, that any continuous real function on $X - \{a\}$ can be extended continuously to $\upsilon X - \{a\}$. Using the standard properties of υX, one then obtains for the class Υ of completely regular Hausdorff spaces and the operation υ of Hewitt Q-extension:

PROPOSITION 4. *The extension structure* (Υ, υ) *is normal and* Υ_0 *is the class of all* $X \in Y$ *such that for any* $a \in X$ *there exist continuous real functions on* $X - \{a\}$ *without a continuous extension to* X.

Remark 1. Proposition 1 for $(\Gamma, \gamma) = (\Upsilon, \upsilon)$ which is hereby established is due to Heider (**6**) in the case where $X' = \upsilon X$ and $Y' = \upsilon Y$.

Remark 2. Although (Υ, υ) is normal it is not strongly normal since a space E may have extension spaces X and Y such that $E \subseteq X \subset Y \subseteq \beta E$, $\upsilon X = X$ and $\upsilon Y = Y$, in which case $C3$ breaks down for the natural injection $X \to Y$. Nevertheless, it is still true that $X \in \Upsilon_0$ implies $X - \{a\} \in \Upsilon_0$ for any $a \in X$ and hence the characterization of Υ_0 as in Proposition 2 still applies.

5. Katětov extensions. For any Hausdorff space X which is not absolutely closed, Katětov (**7**) introduced an absolutely closed extension which can be described as follows: Corresponding to each non-convergent maximal open (that is, with a basis consisting of open sets) filter $\mathfrak{M}$ on X a new point $x_{\mathfrak{M}}$ is adjoined to X and on this enlarged set the collection of all sets $V \cup \{x_{\mathfrak{M}}\}$, V open in X and $V \in \mathfrak{M}$, is taken as a basis for the open sets. If one considers on the class K of all Hausdorff spaces the operator κ which assigns to each $X \in \mathrm{K}$ its Katětov extension κX ($\kappa X = X$ if X absolutely closed), one has:

PROPOSITION 5. (K, κ) *is a strongly normal extension structure and* K_0 *is the class of all* $X \in \mathrm{K}$ *such that for any non-isolated* $a \in X$ *there exists an open* $U \subseteq X$ *such that* $a \in \bar{U}$ *but* $U \cup \{a\}$ *is not open.*

Proof. $\kappa(\kappa X) = \kappa X$ and the basic properties of κX proved in (**7**) implies $C1$ and $C3$ for (K, κ). $C2$ follows from the fact that there is a natural one-to-one correspondence between the maximal non-convergent open filters on X and those filters of this kind on $X - \{a\}$ which do not converge to a in X given by $\mathfrak{M} \to \{A | a \notin A \in \mathfrak{M}\}$.

To obtain the stated description of K_0, one first observes that $a \notin \kappa_0 X$ holds if and only if the filter consisting of the sets $V - \{a\}$, V the neighbourhoods of a in X, is a maximal open filter in $X - \{a\}$. Next, this is obviously the case if and only if any open U in $X - \{a\}$ which meets all these $V - \{a\}$ is itself one of them. Finally, in terms of the topology of X itself, this condition means that for any open $U \subseteq X$ such that $a \in \bar{U} - U$, $U \cup \{a\}$ is open in X. Therefore, $a \in \kappa_0 X$ is equivalent, for non-isolated $a \in X$, to the existence of an open $U \subseteq X$ with $a \in \bar{U}$ for which $U \cup \{a\}$ is not open.

Remark. Proposition 1 for $(\Gamma, \gamma) = (\mathrm{K}, \kappa)$ which is thus obtained was proved by Katětov **(7)**.

6. Maximal zero-dimensional compactifications. For any zero-dimensional Hausdorff space X there is defined a maximal zero-dimensional compact extension ζX which can be considered as the completion of X with respect to the uniform structure of X given by the finite partitions of X into open-closed sets **(1)**. Alternatively, ζX is the maximal ideal space of the Boolean algebra of all open-closed sets in X, X imbedded in this as usual by identifying each point with the corresponding fixed ideal. Yet another description of ζX is as follows: Let $\Phi_\zeta(X)$ be the set of all maximal open-closed filters $\mathfrak{M}$ (that is, with a basis consisting of open-closed sets) in X with void adherence. Then, corresponding to each $\mathfrak{M} \in \Phi_\zeta(X)$ a new point $x_{\mathfrak{M}}$ is adjoined to X, and on this enlarged set the collection of all sets $V \cup \{x_{\mathfrak{M}} | V \in \mathfrak{M} \in \Phi_\zeta(X)\}$, V open-closed in X, is taken as a basis for the open sets.

If Z denotes the class of all zero-dimensional Hausdorff spaces and ζ the operator which assigns to each $X \in Z$ its extension ζX one has:

PROPOSITION 6. *(Z, ζ) is a strongly normal extension structure and Z_0 is the class of all $X \in \mathrm{Z}$ such that for any non-isolated $a \in X$ there exists an open $U \subseteq X$ for which $\bar{U} = U \cup \{a\}$ and $U \cup \{a\}$ is not open.*

Proof. One has $\zeta(\zeta X) = \zeta X$ for any $X \in \mathrm{Z}$ and $C3$ follows immediately from the maximality property of ζX which states **(1)** that any zero-dimensional compact extension of X is the continuous image of ζX under a mapping which extends the identity mapping on X. $C2$ is again obtained from the fact that (i) $\mathfrak{M} \to \mathfrak{M}_* = \{A | a \notin A \in \mathfrak{M}\}$ is a one-to-one correspondence between the $\mathfrak{M} \in \Phi_\zeta(X)$ and those $\mathfrak{N} \in \Phi_\zeta(X - \{a\})$ which do not converge to a and (ii) $V \in \mathfrak{M}$ is equivalent to $V - \{a\} \in \mathfrak{M}_*$.

As to the characterization of Z_0, it follows immediately from the definition of $\zeta(X - \{a\})$ in terms of maximal open-closed filters that $a \notin \zeta_0 X$ holds if and only if the open-closed filter on $X - \{a\}$ consisting of the sets $V - \{a\}$, V the neighbourhoods of a in X, is a maximal open-closed filter in $X - \{a\}$. This will be the case if and only if any open-closed W in $X - \{a\}$ which meets all these $V - \{a\}$ is itself one of them. Expressed in terms of the topology of X this means that for any open $W \subseteq X$ with $\bar{W} = W \cup \{a\}$ the set $W \cup \{a\}$ is open. Therefore, $a \in \zeta_0 X$ holds for non-isolated $a \in X$ exactly if there exists an open $U \subseteq X$ with $\bar{U} = U \cup \{a\}$ for which $U \cup \{a\}$ is not open.

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A consequence of Proposition 1 for $(\Gamma, \gamma) = (Z, \zeta)$, obtained from the description of ζX in terms of the Boolean algebra $B(X)$ of the open-closed sets of X, is the following:

COROLLARY. *If X, $Y \in Z_0$, then any isomorphism $B(X) \to B(Y)$ is induced by a homeomorphism $X \to Y$.*

7. Maximal Hausdorff-minimal extensions. A space X is called Hausdorff-minimal if it is Hausdorff and its topology is minimal in the partially ordered set of all Hausdorff topologies on X (order by inclusion between the collections of open sets); in other words, if any continuous one-to-one mapping of X is a homeomorphism. It is known **(3)** that any semi-regular space X (that is, X is Hausdorff and the interiors of closed sets in X form a basis for the open sets) possesses a Hausdorff-minimal extension σX such that for any Hausdorff-minimal extension $W \supseteq X$ (X dense in W), the natural injection $X \to W$ can be extended to a continuous mapping of σX onto W. A description of σX can be given as follows: Let a filter in X be called semi-regular if it has a basis consisting of regular open sets (= the interiors of closed sets) and denote by $\Phi_\sigma(X)$ the set of all maximal semi-regular filters in X whose adherence is void. Then, adjoin to X a new point $x_{\mathfrak{M}}$ for each $\mathfrak{M} \in \Phi_\sigma(X)$ and take as basis for the open sets on this enlarged set the collection of all sets $V \cup \{x_{\mathfrak{M}} | V \in \mathfrak{M} \in \Phi_\sigma(X)\}$ where V is a regular open set in X.

Now, if Σ denotes the class of all semi-regular spaces and σ the operator which assigns to each $X \in \Sigma$ its extension σX one has:

PROPOSITION 7. *(Σ, σ) is a strongly normal extension structure and Σ_0 is the class of all $X \in \Sigma$ such that any non-isolated point of X belongs to the closures of two disjoint open sets.*

Proof. The strong normality follows from the mentioned properties of σX in the same way as it was obtained for (Z, ζ). To determine Σ_0 one again observes that $a \in \sigma_0 X$ ($a \in X$ non-isolated) means that the semi-regular filter on $X - \{a\}$ consisting of all $V - \{a\}$, V the neighbourhoods of a in X, is not a maximal such filter on $X - \{a\}$. Since finite intersections of regular open sets are regular open, this means that there exists a regular open set W of $X - \{a\}$ which meets all $V - \{a\}$ but is not itself one of them, that is, for which $W \cup \{a\}$ is not open. Such a W is the interior, in $X - \{a\}$, of its closure $\bar{W} - \{a\}$ in $X - \{a\}$, and since $X - \{a\}$ is open in X this is the same as saying that W is the interior of $\bar{W} - \{a\}$ in X. Also, since $W \cup \{a\}$ is not open W is merely the interior of $\bar{W}$ in X, that is, a regular open set of X, and a belongs to $\bar{W}$ as well as to the closure of its complement. Therefore, $a \in \bar{U} \cap \bar{W}$ with open disjoint U and W. Conversely, if this is the case then a cannot belong to the interior W_0 of $\bar{W}$ since $W_0 \cap U = \phi$, and hence $W_0 \subseteq X - \{a\}$ is the interior, in $X - \{a\}$, of the closed set $\bar{W} - \{a\}$ in $X - \{a\}$, that is, a regular open set of $X - \{a\}$. Also, $a \in \bar{W} = \bar{W}_0$ shows that W_0 meets all $V - \{a\}$. Finally, $W_0 \cup \{a\}$ is not open, for if it were $a \in \bar{W}$ would imply $a \in W_0$.

8. Freudenthal extensions. A Hausdorff space X is called rim-compact (also: semi-compact) if the open sets $V \subseteq X$ whose boundary $B(V) = \bar{V} \cap CV$ (CV the complement of V) is compact form a basis for the open sets in X. For any such space X there exists, according to **(5)**, a compact extension φX with the property that (i) every point in φX has arbitrarily small neighbourhoods whose boundaries lie in X and (ii) for any other compact extension $W \supseteq X$ satisfying (i) the natural injection $X \to W$ has a continuous extension $\varphi X \to W$. The extension φX has been described in the following way: If $\Phi_\varphi(X)$ denotes the set of all filters in X which have a basis consisting of open sets with compact boundary (such filters will be called rim-compact here), are maximal with respect to this property and have void adherence, then φX is obtained by adjoining to X a new point $x_{\mathfrak{M}}$ for each $\mathfrak{M} \in \Phi_\varphi(X)$ and taking the sets $V \cup \{x_{\mathfrak{M}} | V \in \mathfrak{M} \in \Phi_\varphi(X)\}$, $V \subseteq X$ open with compact $B(V)$, as basis for the open sets in this enlarged set **(5)**. Alternatively, φX is the completion of X with respect to the uniform structure of X which is defined by the finite coverings of X by open sets with compact boundary **(8)**.

For the class Φ of all rim-compact Hausdorff spaces and the operator which assigns to each $X \in \Phi$ the extension φX the following holds:

PROPOSITION 8. *(Φ, φ) is a normal extension structure and Φ_0 is the class of all $X \in \Phi$ such that any non-isolated $a \in X$ lies on the boundary $B(U)$ of some open $U \subseteq X$ for which $B(U) - \{a\}$ is compact and $U \cup \{a\}$ not open.*

Proof. To verify $C1$ for (Φ, φ) it is sufficient to consider $X, Y \in \Phi$ such that $X \subseteq Y \subseteq \varphi X$ and prove that the sequence of natural injections

$$X \xrightarrow{i} Y \xrightarrow{j} \varphi X$$

can be extended to a sequence of homeomorphisms

$$\varphi X \xrightarrow{i^\varphi} \varphi Y \xrightarrow{j^\varphi} \varphi X.$$

This amounts to the same as the existence of a homeomorphism $\varphi Y \to \varphi X$ which extends the natural injection j. Now, the existence of a continuous extension k of j mapping φY onto φX follows from the maximality property of φY and since φY and φX are compact it is enough to show k to be one-to-one. Therefore, consider $\mathfrak{L}, \mathfrak{M} \in \Phi_\varphi(Y)$ such that $kx_{\mathfrak{L}} = kx_{\mathfrak{M}} = x_{\mathfrak{N}}$, $\mathfrak{N} \in \Phi_\varphi(X)$. If $\mathfrak{L}_X$ and $\mathfrak{M}_X$ are the filters on X obtained from $\mathfrak{L}$ and $\mathfrak{M}$ respectively by intersecting all their sets with X one has $\mathfrak{N} \subseteq \mathfrak{L}_X, \mathfrak{M}_X$ by the continuity of k and the limit relations $\lim \mathfrak{L} = x_{\mathfrak{L}}$, $\lim \mathfrak{M} = x_{\mathfrak{M}}$ in φY. Now, $\mathfrak{L}_X$ and $\mathfrak{M}_X$ are rim-compact filters: For any open $U \in \mathfrak{L}$ there exists a $V \in \mathfrak{L}$ such that $\bar{V} \subseteq U$ and $B(V)$ is compact (topological operations all in Y). This

$B(V)$ can be covered by finitely many open sets W_i in φX such that $V_i = W_i \cap Y \subseteq U$ and the boundaries $B(V_i)$ are compact and lie in X. For the open set $V^* = V \cup \bigcup V_i$ one has $B(V^*) \subseteq B(V) \cap \bigcup B(V_i)$ which implies $B(V^*) \subseteq \bigcup B(V_i)$ since $B(V) \subseteq \bigcup V_i$ shows that no $x \in B(V)$ belongs to $B(V^*)$. It follows that $B(V^*)$ is compact and lies in X; hence $X \cap V^*$ is an open set with compact boundary of X. As $V^* \in \mathfrak{L}$ (by $V^* \supseteq V$) and $V^* \subseteq U$ where U was an arbitrary open set in $\mathfrak{L}$, this means that $\mathfrak{L}_X$ is a rim-compact filter. This result now leads to the equations $\mathfrak{L}_X = \mathfrak{N} = \mathfrak{M}_X$ which in turn give $\mathfrak{L} = \mathfrak{M}$; finally, this proves that k is one-to-one.

The proof of $C2$ for (Φ, φ) is of the same nature as that for (Z, ζ), and the remaining thing is to characterize Φ_0. For this one uses the fact that a rim-compact filter $\mathfrak{S}$ on a space is maximal if and only if any open U with compact boundary which meets all sets of $\mathfrak{S}$ itself belongs to $\mathfrak{S}$; this can easily be deduced from the relation $B(U \cap V) \subseteq B(U) \cap B(V)$. It follows that a non-isolated $a \in X$ belongs to $\varphi_0 X$ if and only if there exists an open $U \subseteq X - \{a\}$ which meets all neighbourhoods of a, that is , $a \in \bar{U}$, and whose boundary in $X - \{a\}$ is compact, but for which $U \cup \{a\}$ is not open in X. Since $a \notin U$ and $a \in \bar{U}$ means $a \in B(U)$, the boundary of U in X (U is open in X), the boundary of U in $X - \{a\}$ is $B(U) - \{a\}$. This completes the proof of Proposition 8.

Remark. (Φ, φ) is not strongly normal: If X is the open circular unit disc and Y the closed circular unit disc in the plane and $f : X \to Y$ the natural injection there exists no extension of f to a continuous mapping of φX onto $\varphi Y = Y$ since φX is the one-point compactification of X.

References

1. B. Banaschewski, *Ueber nulldimensionale Räume*, Mathematische Nachrichten, *13* (1955) 129–140.

2. ———, *Ueber zwei Extremaleigenschaften topologischer Räume*, Mathematische Nachrichten, *13* (1955), 141–150.

3. ———, Report of the Summer Research Institute of the Canadian Mathematical Congress, 1957.

4. N. Bourbaki, *Topologie générale*, Actualities scientifiques et industrielles (Paris, Hermann et Cie.).

5. H. Freudenthal, *Neuaufbau der Endentheorie*, Ann. Math., *43* (1942), 261–279.

6. L. J. Heider, *A note concerning completely regular G_δ-spaces*, Proc. Amer. Math. Soc., *8* (1957), 1060–1066.

7. M. Katětov, *Ueber H-abgeschlossene und bikompakte Räume*, Casopis mat. a fys., *69* (1939–40), 36–49.

8. K. Morita, *On bicompactifications of semibicompact spaces*, Sci. Rep. Tokyo Bunrika Daigaku, Sect. A., *4* (1952), 222–229.

9. H. Wallman, *Lattices and topological spaces*. Ann. Math., *39* (1938), 112–126.

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