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RANDOMLY k-AXIAL GRAPHS

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A class of graphs called randomly k-axial graphs is introduced, which generalizes randomly traceable graphs. The problems of determining which bipartite graphs and which complete n-partite graphs are randomly k-axial are studied.

A graph G was defined to be *randomly traceable* in [1] if, for each vertex v of G, every path with initial vertex v can be extended to a hamiltonian path with initial vertex v. Equivalently, a graph of order at least 3 is randomly traceable if every path of G is contained in some hamiltonian cycle of G. It was proved in [1] that a graph G of order p is randomly traceable if and only if G is isomorphic to K_p, C_p or $K(p/2, p/2)$, where in the last case p is even. In this paper we consider a generalization of randomly traceable graphs.

DEFINITION OF RANDOMLY k-AXIAL GRAPHS. Let G be a graph and k an integer such that $1 \le k \le \delta(G)$. Let v be an arbitrary vertex of G and let $v_{11}, v_{12}, \ldots, v_{1k}$ be any k distinct vertices adjacent to v. Define the set

$$L_{1,0} = \{v, v_{11}, v_{12}, \ldots, v_{1k}\} .$$

If $L_{1,0} \ne V(G)$, let v_{21} be any vertex not in $L_{1,0}$ that is adjacent to v_{11} and define $L_{1,1} = L_{1,0} \cup \{v_{21}\}$. We now define sets $L_{m,n}$ (having

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cardinality $1 + mk + n$) inductively for certain positive integers m and nonnegative integers n for which $0 \le n \le k-1$. If a set $L_{m,n} \subseteq V(G)$ has been defined, where $0 \le n \le k-2$, and $L_{m,n} \ne V(G)$, let $v_{m+1,n+1}$ be any vertex adjacent to $v_{m,n+1}$ such that $v_{m+1,n+1} \notin L_{m,n}$ and define

$$L_{m,n+1} = L_{m,n} \cup \{v_{m+1,n+1}\} .$$

If a set $L_{m,k-1} \subseteq V(G)$ has been defined and $L_{m,k-1} \ne V(G)$, let $v_{m+1,k}$ be any vertex adjacent to $v_{m,k}$ such that $v_{m+1,k} \notin L_{m,k-1}$ and define

$$L_{m+1,0} = L_{m,k-1} \cup \{v_{m+1,k}\} .$$

If every such set $L_{m,n}$ is defined and every such sequence $L_{m,n}$ has $V(G)$ as its final term, then we say that G is *randomly* k*-axial*. If r is a positive integer for which the vertices $v_{r1}, v_{r2}, \ldots, v_{rk}$ are defined, we denote the set $\{v_{r1}, v_{r2}, \ldots, v_{rk}\}$ by L_r and refer to it as a *level set* or, more simple, as a *level*.

A more intuitive definition of randomly k-axial graphs can be given with the aid of the following terms. A random extension of a path $P : v_1, v_2, \ldots, v_n$ in a graph is a path $P' : v_1, v_2, \ldots, v_n, v_{n+1}$ where v_{n+1} is any vertex of the graph adjacent to v_n that does not belong to P . A collection of paths, each with initial vertex u , is called internally disjoint if every two paths in the collection have only the vertex u in common.

A graph G is then randomly k-axial $(1 \le k \le \delta(G))$ if for each vertex v of G , any ordered collection of k paths in G of length 1 having initial vertex v can be cyclically randomly extended to produce k internally disjoint paths whose lengths are as equal as possible and which contain all the vertices of G .

It thus follows that the randomly 1-axial graphs are precisely the randomly traceable graphs. Indeed, we also have the following.

PROPOSITION 1. *A graph* G *with* $\delta(G) \ge 2$ *is randomly* 2*-axial if and only if* G *is randomly traceable.*

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Proof. If G is randomly traceable of order p, then G is isomorphic to one of the graphs K_p $(p \geq 3)$, C_p or $K(p/2, p/2)$, where p is even and $p \geq 4$. It follows immediately that each of these graphs is randomly 2-axial.

Suppose that G is a randomly 2-axial graph, and let P be an arbitrary path of G. Then P can be labelled as

$$P : v_{r1}, v_{r-1,1}, \ldots, v_{11}, v, v_{12}, v_{22}, \ldots, v_{r2}$$

or

$$P : v_{r1}, v_{r-1,1}, \ldots, v_{11}, v, v_{12}, v_{22}, \ldots, v_{r-1,2},$$

according to whether P has even length or odd length, respectively. Since G is randomly 2-axial, the vertices of G can be listed as

$$v_{m1}, v_{m-1,1}, \ldots, v_{r1}, v_{r-1,1}, \ldots, v_{11}, v, v_{12}, v_{22}, \ldots, v_{r2}, \ldots, v_{m2}$$

or

$$v_{m1}, v_{m-1,1}, \ldots, v_{r1}, v_{r-1,1}, \ldots, v_{11}, v, v_{12}, v_{22}, \ldots, v_{r2}, \ldots, v_{m-1,2},$$

where consecutive vertices are adjacent, producing a hamiltonian path Q of G in either case. Thus P is contained in Q and, consequently, every path of G is contained in a hamiltonian path of G. By a result of Thomassen [*2*], G belongs to a class of graphs containing the randomly traceable graphs as a proper subclass. Among all these graphs, however, only the randomly traceable graphs of order at least 3 are randomly 2-axial. Thus G is randomly traceable. □

It therefore follows that the only randomly 2-axial graphs are K_p $(p \geq 3)$, C_p and $K(n, n)$, $n \geq 2$. It is obvious that K_p is randomly k-axial for every k with $1 \leq k \leq p-1$. We have already noted that the graph $K(6, 6)$ is both randomly 1-axial and randomly 2-axial. It is not difficult to verify that $K(6, 6)$ is also randomly 3-axial. However, $K(6, 6)$ is *not* randomly 4-axial; for consider the labelling of $K(6, 6)$ shown in Figure 1. Note that, as in the definition of randomly 4-axial graphs, $L_{2,2}$ is defined and $L_{2,2} \neq V(K(6, 6))$; however, there is no

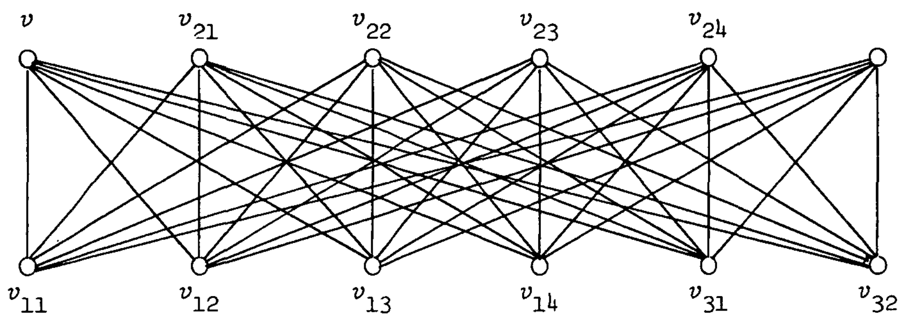

FIGURE 1

vertex $v_{33} \notin L_{2,2}$ such that v_{33} is adjacent to v_{23} ; that is, $L_{2,3}$ is not defined. Thus, the sequence $\{L_{m,n}\}$ does not have $V(K(6, 6))$ as its final term, thereby implying that $K(6, 6)$ is not randomly 4-axial. On the other hand, $K(6, 6)$ is both randomly 5-axial and randomly 6-axial. All these facts will become clear shortly as we begin our study of bipartite randomly k-axial graphs.

PROPOSITION 2. *Let G be a bipartite graph with partite sets V_1 and V_2 such that $n_1 = |V_1| \le |V_2| = n_2$. If G is randomly k-axial, $3 \le k \le n_1$, then $n_1 = n_2$ where $n_1 \equiv 0 \pmod{k}$ or $n_1 \equiv 1 \pmod{k}$.*

Proof. Assume, to the contrary, that $n_1 < n_2$. Then $n_2 = n_1 + u$, where $u \ge 1$. By the division algorithm, we can write $n_1 = ak + b$, where $a \ge 1$ and $0 \le b < k$.

Let $v \in V_2$ and apply the definition of randomly k-axial graphs to obtain a labelling of the vertices of G . For $i = 1, 2, \ldots, a$, define

$$U_i = \{v_{2i-1,1}, v_{2i-1,2}, \ldots, v_{2i-1,k}\}$$

and

$$W_i = \{v_{2i,1}, v_{2i,2}, \ldots, v_{2i,k}\} .$$

Write

$$V_1 = U_1 \cup U_2 \cup \ldots \cup U_a \cup B$$

and

$$V_2 = \{v\} \cup W_1 \cup W_2 \cup \ldots \cup W_a \cup A ,$$

where $|A| = u + b - 1$ and $|B| = b$. Since $|B| = b < k$, we must have $A = \emptyset$; otherwise, $L_{2a,b}$ is the final term in the sequence $\{L_{m,n}\}$, but $L_{2a,b} \neq V(G)$, contradicting the fact that G is randomly k-axial. Thus $u + b - 1 = 0$, implying that $u = 1$ and $b = 0$ since $u \geq 1$ and $b \geq 0$. Hence $n_2 = n_1 + 1$.

Next let $v \in V_1$ and once again apply the definition of randomly k-axial graphs to obtain a labelling of the vertices of G. For $i = 1, 2, \ldots, a$, define

$$W_i = \{v_{2i-1,1}, v_{2i-1,2}, \ldots, v_{2i-1,k}\} ,$$

and for $i = 1, 2, \ldots, a-1$, define

$$U_i = \{v_{2i,1}, v_{2i,2}, \ldots, v_{2i,k}\} .$$

Write

$$V_1 = \{v\} \cup U_1 \cup U_2 \cup \ldots \cup U_{a-1} \cup B$$

and

$$V_2 = W_1 \cup W_2 \cup \ldots \cup W_a \cup A ,$$

where $|B| = k - 1$ and $|A| = 1$. The last term in the sequence $\{L_{m,n}\}$ is then $L_{2a-1,k-1}$; however, $L_{2a-1,k-1} \neq V(G)$, contradicting the fact that G is randomly k-axial. Hence we conclude that $n_1 = n_2$.

We now show that $n_1 \equiv 0 \pmod{k}$ or $n_1 \equiv 1 \pmod{k}$. Recall that $n_1 = ak + b$, where $a \geq 0$ and $0 \leq b < k$.

Let $v \in V_1$. Since G is randomly k-axial, a labelling of $V(G)$ is produced. For $i = 1, 2, \ldots, a$, define

$$W_i = \{v_{2i-1,1}, v_{2i-1,2}, \ldots, v_{2i-1,k}\}$$

and for $i = 1, 2, \ldots, a-1$, define

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$$U_i = \{v_{2i,1}, v_{2i,2}, \ldots, v_{2i,k}\} .$$

Write

$$V_1 = \{v\} \cup U_1 \cup U_2 \cup \ldots \cup U_{a-1} \cup U_a \cup B$$

and

$$V_2 = W_1 \cup W_2 \cup \ldots \cup W_a \cup A ,$$

where $|A| = b$. If $b = 0$, then $B = \emptyset$ and $|U_a| = k - 1$; if $b \geq 1$, then $|B| = b - 1$ and $|U_a| = k$.

Suppose $b \geq 1$. Then the final term of the sequence $\{L_{m,n}\}$ is $L_{2a,b}$. Since G is randomly k-axial, $L_{2a,b} = V(G)$; hence $B = \emptyset$ and $b = 1$.

Thus $b = 0$ or $b = 1$, completing the proof. □

It therefore follows that the partite sets of a bipartite, randomly k-axial graph have the same cardinality. Further, this cardinality is either divisible by k or gives a remainder of 1 when divided by k . In the first of these cases we can say much more.

THEOREM 1. *If G is a randomly k-axial graph $(k \geq 3)$ of order p , where $2k|p$, then either $G \cong K_p$ or $G \cong K(p/2, p/2)$.*

Proof. Let $m = p/k$ and let $v_0 \in V(G)$. A-plying the definition of randomly k-axial graphs to G with $v = v_0$, we obtain a labelling of the vertices of G (as in the definition) and $L_{m-1,k-1} = V(G)$. This implies that G contains the edges indicated in Figure 2. The levels $L_1, L_2, \ldots, L_{m-1}$ are as indicated and define $L_m^* = \{v_{m1}, v_{m2}, \ldots, v_{m,k-1}\}$.

Let i be given, $1 \leq i \leq k-1$; we show that the vertex $v_{m-1,k}$ is adjacent to $v_{m,i}$. This is accomplished by a relabelling of $V(G)$. Relabel v_{ak} $(1 \leq a \leq m-1)$ as $u_{a,k-1}$, relabel v_{bi} $(1 \leq b \leq m-1)$ as u_{bk} and relabel $v_{c,k-1}$ $(1 \leq c \leq m)$ as u_{ci} . Further, relabel v_0 as

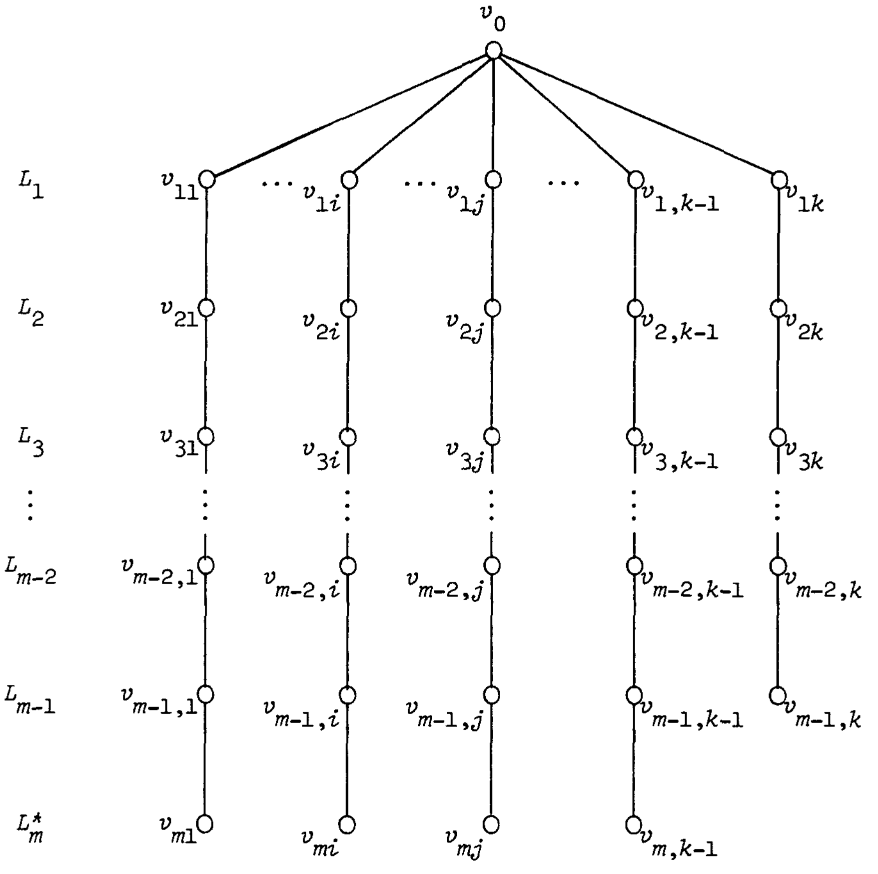

FIGURE 2

u and any v_{rs}, except v_{mi}, not already relabelled as u_{rs}. We now apply the definition of randomly k-axial graphs to G (where v and v_{rs} in the definition are replaced by u and u_{rs}). It follows that the vertex v_{mi} must now receive the label $u_{m,k-1}$ and, therefore, $u_{m-1,k-1}$ is adjacent to $u_{m,k-1}$ or, equivalently, $v_{m-1,k}$ is adjacent to v_{mi}. Since i $(1 \le i \le k-1)$ is arbitrary, $v_{m-1,k}$ is adjacent to v_{mi} for every i, $1 \le i \le k$.

Next, let j be given, $1 \le j \le k-1$. We show that $v_{m-1,j}$ is adjacent to v_{mi} for every i, $1 \le i \le k$. This is accomplished by

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another relabelling of $V(G)$. For $1 \le a \le m-1$, relabel v_{aj} as w_{ak} and v_{ak} as w_{aj} . Also, relabel v_0 as w and relabel any v_{rs} not already relabelled as w_{rs} . By the argument of the preceding paragraph, it follows that $w_{m-1,k}$ is adjacent to w_{mi} for every i , $1 \le i \le k-1$, or, equivalently, $v_{m-1,j}$ is adjacent to v_{mi} for every i , $1 \le i \le k-1$. Since j is arbitrary, we conclude that every vertex of L_{m-1} is adjacent to every vertex of L^*_m . In general, we now know that if v is any vertex of G with level L_{m-1} and set L^*_m as defined above, then every vertex of L_{m-1} is adjacent to every vertex of L^*_m .

Therefore, G contains the edges indicated in Figure 3.

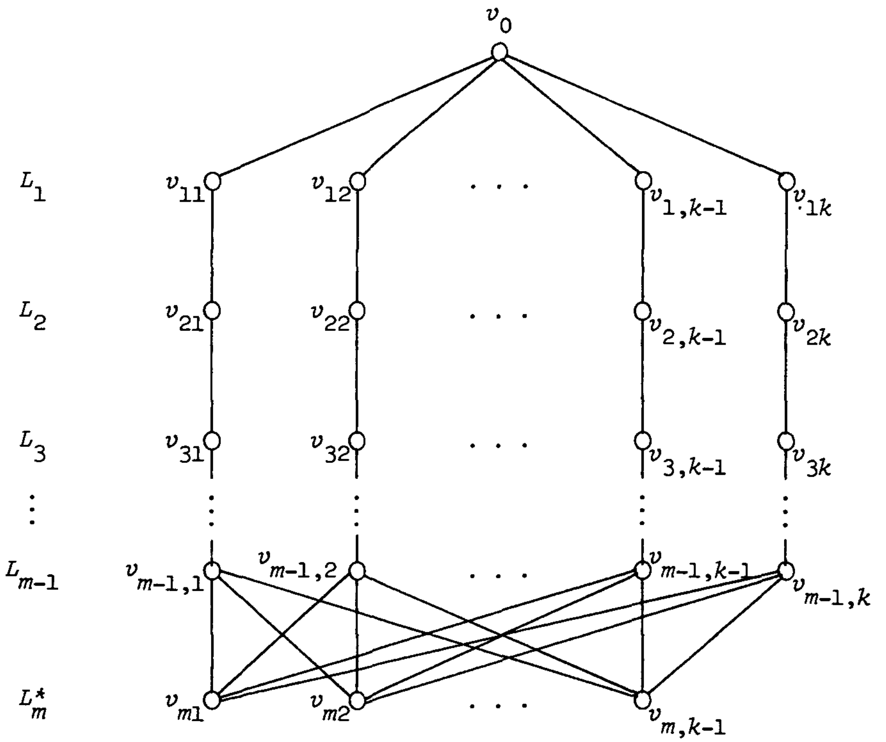

FIGURE 3

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Our next step is to show that every vertex of L_1 is adjacent to every vertex of L_2. Relabel $v_{m-1,k}$ as v' and for each j, $1 \le j \le k-1$, relabel $v_{m+1-i,j}$ as $v'_{i,j}$ for $1 \le i \le m$. Also, let $v'_{i,k} = v_{m-1-i,k}$ for $1 \le i \le m-2$ and let $v'_{m-1,k} = v_0$. Applying the definition of randomly k-axial graphs to G (with v and v_{rs} replaced by v' and v'_{rs}) we obtain the corresponding level set $L'_{m-1} = \{v_{21}, v_{22}, \ldots, v_{2,k-1}, v_0\}$ and set $(L'_m)^* = \{v_{11}, v_{12}, \ldots, v_{1,k-1}\}$. From above, we know that every vertex of L'_{m-1} is adjacent to every vertex of $(L'_m)^*$. By repeating this process twice more, say

(1) by relabelling $v_{m-1,1}$ as v' and

(2) by relabelling $v_{m-1,2}$ as v',

we conclude that every vertex of L_1 is adjacent to every vertex of L_2. The graph G now contains the edges as indicated in Figure 4.

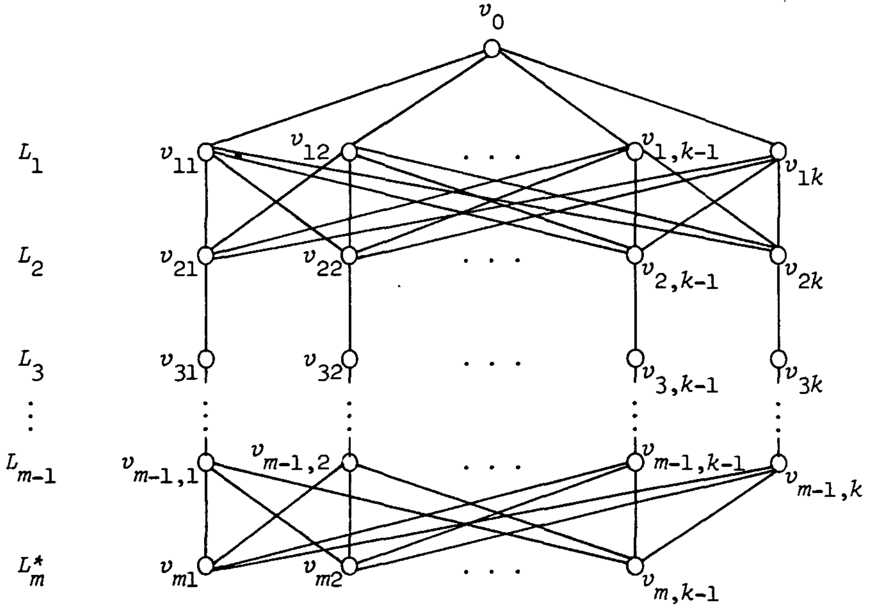

FIGURE 4

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Next we show that every vertex of level L_1 is adjacent to every vertex of set L_m^*. This can be accomplished by relabelling $v_{m,k-1}$ as v''. It is possible to relabel other vertices of G so that the corresponding levels $L_1'', L_2'', \ldots, L_{m-1}''$ are produced, where

$$L_i'' = \{v_{m-i,1}, v_{m-i,2}, \ldots, v_{m-i,k}\} = L_{m-i}$$

for $1 \le i \le m-1$. Further,

$$(L_m'')^* = \{v_{m1}, v_{m2}, \ldots, v_{m,k-2}, v_0\} .$$

From the argument given above, every vertex of L_{m-1}'' is adjacent to every vertex of $(L_m'')^*$. If we now repeat this argument, where $v_{m,l}$ $(1 \le l \le k-2)$ is relabelled as v'' and levels $L_1'', L_2'', \ldots, L_{m-1}''$ are produced exactly as above, then we see that $v_{m,k-1}$ is also adjacent to every vertex of L_{m-1}''; hence, every vertex of L_1 is adjacent to every vertex of L_m^*.

Our next step is to show that v_0 is adjacent to every vertex of L_{m-1}. Relabel $v_{m,k-1}$ as v'''. Other vertices of G can be relabelled so that corresponding levels $L_1''', L_2''', \ldots, L_{m-1}'''$ are produced, where $L_i''' = L_i$ for $1 \le i \le m-1$. Moreover,

$$(L_m''')^* = \{v_{m1}, v_{m2}, \ldots, v_{m,k-2}, v_0\} .$$

Since every vertex of L_{m-1}''' is adjacent to every vertex of $(L_m''')^*$, it follows that v_0 is adjacent to every vertex of L_{m-1}.

Define $L_m = L_m^* \cup \{v_0\}$. We have shown that every vertex of L_m is adjacent to every vertex of L_{m-1} and to every vertex of L_1. Applying the previous arguments and the definition of randomly k-axial graphs with v selected from L_1, we see that every vertex of L_1 is adjacent to every vertex of L_m and to every vertex of L_2. Continuing this procedure, it follows that every vertex of L_i is adjacent to every vertex of L_{i-1} and to every vertex of L_{i+1} $(i = 1, 2, \ldots, m)$, where the subscripts are expressed modulo m .

We now show that G contains $K(p/2, p/2)$ as a subgraph. If $m = 2$ or $m = 4$, this already follows. Thus we assume that $m \geq 6$. We show that every vertex of L_i $(1 \leq i \leq m)$ is adjacent to every vertex of L_{i+3} , where the subscripts are expressed modulo m . For convenience, let x denote any vertex of L_2 (see Figure 5). Applying the definition of randomly k-axial graphs, we can obtain the labelling of the vertices of G shown in Figure 5. Note that a vertex of G (in L_2) has not yet been labelled. Since G is randomly k-axial, this vertex must be labelled $x_{m,k-1}$. Since $x_{m,k-1}$ must be adjacent to $x_{m-1,k-1}$ and $x_{m-1,k-1} \in L_5$, it follows, because of symmetry, that for each i $(1 \leq i \leq m)$, every vertex of L_i is adjacent to every vertex of L_{i+3} , where, as always, the subscripts are expressed modulo m .

If $m = 6$ or $m = 8$, then G contains $K(p/2, p/2)$ as a subgraph. If $m \geq 10$, we use the known edges of G and the fact that G is randomly k-axial to produce yet another labelling of the vertices of G . Relabel vertex x as y , vertex $x_{m-1,k-1}$ as $y_{m-3,k-1}$ and vertex $x_{m-3,k-1}$ as $y_{m-1,k-1}$. Every other vertex x_{rs} is relabelled y_{rs} . Since G is randomly k-axial, the unlabelled vertex in L_2 must be $y_{m,k-1}$ and is adjacent to $y_{m-1,k-1}$. By symmetry, we conclude that every vertex of L_i is adjacent to every vertex of L_{i+5} .

If $m = 10$ or $m = 12$, we have now shown that G contains $K(p/2, p/2)$ as a subgraph. If $m \geq 14$, we again use the known edges of G and the fact that G is randomly k-axial to obtain a new labelling of $V(G)$. Relabel y as z , vertex $y_{m-1,k-1}$ as $z_{m-5,k-1}$ and $y_{m-5,k-1}$ as $z_{m-1,k-1}$. By the same reasoning as above, one can now show that every vertex of L_i is adjacent to every vertex of L_{i+7} for all i . Continuing this procedure, we see that every vertex of L_i is adjacent to every vertex of L_j $(1 \leq i, j \leq m)$, where i and j are of opposite

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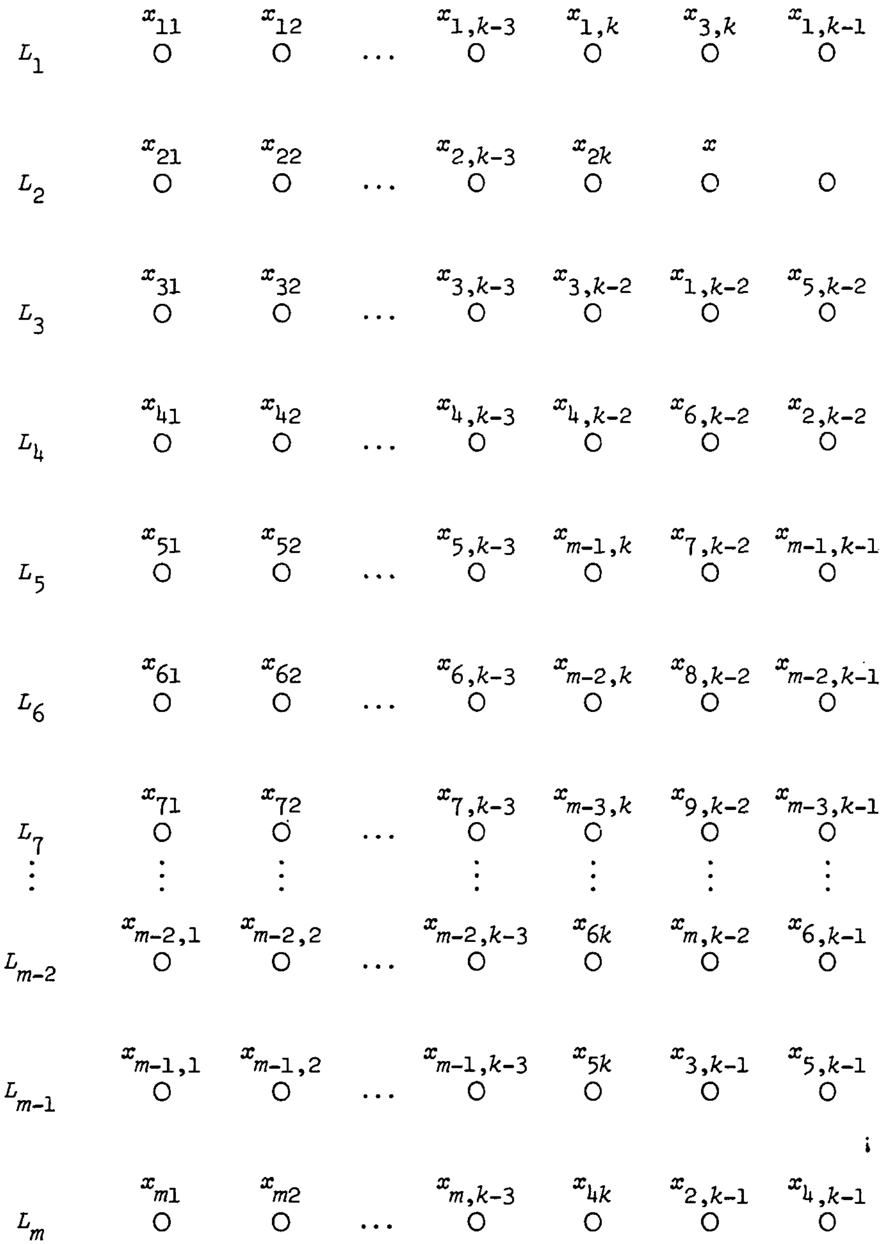

FIGURE 5

parity. Hence G contains $K(p/2,\ p/2)$ as a subgraph.

If G contains only the edges of the subgraph $K(p/2,\ p/2)$, then, of

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course, $G \cong K(p/2, p/2)$. Suppose then that G contains an edge e not belonging to the subgraph $K(p/2, p/2)$. We show that $G \cong K_p$.

Let V_1 and V_2 denote the partite sets of the subgraph $K(p/2, p/2)$, where, say, $v_0 \in V_1$. Thus

$$V_1 = \bigcup_{i=1}^{m/2} L_{2i} \quad \text{and} \quad V_2 = \bigcup_{i=1}^{m/2} L_{2i-1} .$$

Without loss of generality, we may assume that $e = ab$, where $a = v_{m-2,k}$ and $b = v_{m,k-1}$ (see Figure 2). Let $c, d \in V_2$. The proof will be complete once it is shown that $cd \in E(G)$. Again, without loss of generality, we may assume that $c = v_{m-1,k-1}$ and $d = v_{m-1,k}$. We relabel G as follows. Since $ab \in E(G)$, we can relabel b as $\bar{v}_{m-1,k}$, v_0 as $\bar{v}_0$ and all other v_{rs} except $v_{m-1,k}$ $(= d)$ as $\bar{v}_{rs}$. Since G is randomly k-axial, d must be labelled $\bar{v}_{m,k-1}$; however, $\bar{v}_{m-1,k-1}$ must be adjacent to d , but $\bar{v}_{m-1,k-1} = c$. □

Combining the previous two results, we have an immediate corollary.

COROLLARY. *Let G be a bipartite, randomly k-axial graph $(k \geq 3)$ whose partite sets have cardinality n . If $n \equiv 0 \pmod{k}$, then $G \cong K(n, n)$.*

The graph $K(n, n)$, where $n \equiv 1 \pmod{k}$ and $k \geq 3$, is readily seen to be randomly k-axial. Thus, the complete bipartite, randomly k-axial graphs are completely determined.

PROPOSITION 3. *The complete bipartite graph $K(n_1, n_2)$ is randomly k-axial $(k \geq 3)$ if and only if $n_1 = n_2$ and $n_1 \equiv 0, 1 \pmod{k}$.*

We conjecture that every bipartite, randomly k-axial graph $(k \geq 3)$ is complete bipartite.

CONJECTURE 1. Let G be a bipartite, randomly k-axial graph $(k \geq 3)$ whose partite sets have cardinality n , where $n \equiv 1 \pmod{k}$. Then $G \cong K(n, n)$.

In the case of complete tripartite graphs we have the following result. The proof, which we omit, proceeds by case study.

PROPOSITION 4. *For* $k \geq 2$, *the graph* $K(n_1, n_2, n_3)$ *is randomly* k*-axial if and only if* $n_1 = n_2 = n_3 = k/2$.

It is not difficult to verify that one of the implications of Proposition 4 can be extended, namely, for $t \geq 3$, the complete t-partite graph $K(d, d, \ldots, d)$ is randomly k-axial for all $d \geq 1$ and $k = (t-1)d$. We conjecture that the converse is also true.

CONJECTURE 2. For $k \geq 2$ and $t \geq 3$, the graph $K(n_1, n_2, \ldots, n_t)$ is randomly k-axial if and only if $n_1 = n_2 = \ldots = n_t = k/(t-1)$.

Finally, we conjecture that every randomly k-axial graph $(k \geq 3)$ is a regular complete multipartite graph.

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