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ON (n,k,l,Δ)-SYSTEMS

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The paper is devoted to studying one generalization of Steiner systems $S(n,k,l)$ closely related to packings and coverings of l-tuples by k-tuples of an n-set. One necessary and one sufficient condition for the existence of such designs are obtained.

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1. Introduction

We consider (n,k,l,Δ)-systems which are the generalization of Steiner systems $S(n,k,l)$.

Definition. A system P of k-tuples of an n element set S is called an (n,k,l,Δ)-system iff every l-tuple of S is contained in at most one k-tuple from P and every $(l-\Delta)$-tuple of S is contained in at least one k-tuple from P

Obviously, every $(n,k,l,0)$-system is a Steiner system $S(n,k,l)$, i.e. a system of k-tuples of an n-element set such that every l-tuple is contained in exactly one k-tuple from the system. It is well known that the problem of finding the values (n,k,l) such that Steiner systems $S(n,k,l)$ exist is a very difficult problem. Until now such values $n>k>l>5$ are still unknown. Only for $l=2$, $l=3$ are infinite sequences of Steiner systems known (see[**6**, **16**, **20**, **21**, **22**, **4**, **9–11**, **17**, **8**]).

On the other hand (n,k,l,Δ)-systems are also related to packings and coverings of l-tuples of an n-set by its k-tuples [**6**]. Recall that a system Q of k-tuples of an n-element set S is called an (n,k,l)-packing iff every l-tuple of S is contained in at most one k-tuple from Q, and a system P of k-tuples of an n element set S is called an (n,k,l)-covering iff every l-tuple of S is contained in at least one k-tuple from P. By definition an (n,k,l,Δ)-system is simultaneously an (n,k,l)-packing and an $(n,k,l-\Delta)$-covering.

There are well-known simple inequalities which restrict the domain of values $k-l$ for which Steiner systems can exist: for example, if $l\geqq 2$ and $n>k$ then

$$(k-l+1)(k-l+2)\leqq n-l+1 \tag{1.1}$$

and a generalized Fisher's inequality holds (see, for example, [**16**]). To prove (1.1) we can fix $l-1$ elements of an n element set and consider all k-tuples from the Steiner system that contain these $l-1$ elements. If we delete from such k-tuples these $l-1$ elements we obtain the partition of the $(n-l+1)$-set into $(k-l+1)$-subsets. We consider now the k-tuple from the Steiner system that intersects our $l-1$ elements at $l-2$ elements. The inequality (1.1) follows now from the fact that this k-tuple must intersect each $(k-l+1)$-subset of partition in at most one element.

From (1.1) it immediately follows that non-trivial Steiner systems $S(n,k,l)$ can exist only if

$$k-l<\sqrt{n} \tag{1.2}$$

holds.

The $(n,k,l-\Delta)$-systems seem a much wider class of combinatorial objects than the Steiner systems $S(n,k,l)$. However, as we show in this paper, for (n,k,l,Δ)-systems a necessary condition similar to (1.1) also holds. As we noted above it is very difficult to obtain sufficient conditions for the existence of $(n,k,l,0)$-systems for arbitrary values l because $(n,k,l,0)$-systems are simply Steiner systems $S(n,k,l)$. Using a result of S. D. Cohen on the number of solutions of one algebraic system over a finite field we obtain a sufficient condition for the existence of (n,k,l,Δ)-systems for $\Delta \geqq 2$.

The paper is organized as follows. In Section 2 we prove the necessary condition for the existence of (n,k,l,Δ)-systems. In Section 3 we give the sufficient conditions for the existence of (n,k,l,Δ)-systems for $\Delta=2$ and $\Delta \geqq 3$. For the sake of completeness in Section 4 we give the brief description of S. D. Cohen's algebraical result which is the key to obtain these sufficient conditions.

2. Necessary condition for the existence of (n,k,l,Δ)-systems

Here and in the sequel we suppose that $n>k>l\geqq \Delta+2$. In this section the following necessary condition will be formulated and proved.

Theorem 1. *If the inequality*

$$(k-l+\Delta+2)(k-l+1)>(\Delta+1)(n-l+\Delta+1) \tag{2.1}$$

holds, then (n,k,l,Δ)-systems do not exist.

Proof. We can use now the fact that an (n,k,l,Δ)-system is a packing of l-tuples of an n element set by its k-tuples. For the maximal cardinality $m(n,k,t)$ of an (n,k,t)-packing Johnson's bound [14] holds. So

$$m(n,k,t) \leqq \frac{n(k-t+1)}{n(k-t+1)-k(n-k)} \tag{2.2}$$

provided the denominator is positive; that is $k^2>(t-1)n$.

Now we will show that (n,k,l,Δ)-systems do not exist if (2.1) holds. This is a corollary from the inequality (2.2) and the recurrent inequality [**14**]:

$$m(n,k,l) \leqq \frac{n}{k} m(n-1,k-1,l-1).$$

Actually this inequality immediately imples

$$m(n,k,l)\leqq\frac{\binom{n}{l-\Delta-2}}{\binom{k}{l-\Delta-2}}m(n-l+\Delta+2,k-l+\Delta+2,\Delta+2). \tag{2.3}$$

We apply (2.2) to the last term of (2.3), i.e. to $m(n-l+\Delta+2,k-l+\Delta+2,\Delta+2)$. The denominator is positive if

$$(k-l+\Delta+2)^2>(\Delta+1)(n-l+\Delta+2) \tag{2.4}$$

holds or equivalently

$$(k-l+\Delta+2)(k-l+1)>(\Delta+1)(n-k).$$

If this inequality does not hold then nothing need be proved. So assume that (2.4) holds. Under this condition we can derive the following:

$$\begin{aligned}
&m(n-l+\Delta+2,k-l+\Delta+2,\Delta+2)\\
&\leqq\frac{(n-l+\Delta+2)(k-l+1)}{(n-l+\Delta+2)(k-l+1)-(k-l+\Delta+2)(n-k)}\\
&\leqq\frac{(n-l+\Delta+2)(k-l+1)}{(k-l+1)(k-l+2+\Delta)-(\Delta+1)(n-k)}\\
&\leqq\frac{1}{1-\dfrac{(\Delta+1)(n-k)}{(k-l+1)(k-l+\Delta+2)}}\,\frac{n-l+2+\Delta}{k-l+2+\Delta}.
\end{aligned}$$

From this inequality and (2.3) we can derive the following inequality:

$$m(n,k,l)\leqq\frac{\binom{n}{l-\Delta-1}}{\binom{k}{l-\Delta-1}}\,\frac{1}{1-\dfrac{(\Delta+1)(n-k)}{(k-l+1)(k-l+\Delta+2)}}. \tag{2.5}$$

On the other hand any (n,k,l,Δ)-system is an $(n,k,l-\Delta)$-covering. This implies that the cardinality of an (n,k,l,Δ)-system is at least $\binom{n}{l-\Delta}/\binom{k}{l-\Delta}$ and for the existence of such systems (see 2.5) the inequality

$$\frac{\binom{n}{l-\Delta-1}}{\binom{k}{l-\Delta-1}} \frac{1}{1-\frac{(\Delta+1)(n-k)}{(k-l+1)(k-l+\Delta+2)}} \geqq \frac{\binom{n}{l-\Delta}}{\binom{k}{l-\Delta}}$$

must hold. But this implies that

$$1-\frac{(\Delta+1)(n-k)}{(k-l+1)(k-l+\Delta+2)} \leqq \frac{k-l+\Delta+1}{n-l+\Delta+1}$$

and after routine simplifications we obtain the inequality

$$(k-l+\Delta+2)(k-l+1) \leqq (\Delta+1)(n-l+\Delta+1).$$

Combining this inequality with (2.4) we obtain the desired bound. The proof is complete.

To compare this result with (1.2) we can use the rougher estimate:

Corollary. *If* (n,k,l,Δ)*-systems exist then*

$$k-l \leqq \sqrt{(\Delta+1)n}. \tag{2.6}$$

This inequality is a direct generalization of the necessary condition (1.2).

3. Sufficient conditions for the existence of (n,k,l,Δ)-systems

In Section 2 we noted that if $\Delta=0$ then it is very difficult to obtain sufficient conditions for the existence of (n,k,l,Δ)-systems for arbitrary values l because (n,k,l,Δ)-systems are simply Steiner systems $S(n,k,l)$ in this case. In this section we give sufficient conditions for the existence of (n,k,l,Δ)-systems for $\Delta \geqq 3$ and $\Delta=2$. Let $s=k-l$.

Theorem 2. *Let* $k \leqq cn/(s+3)$ *and* $s \leqq c_1 \log n/\log\log n$ *for some constants* $c<1$ *and* $c_1<1/2$. *Then for all* $\Delta \geqq 3$ *and sufficiently large* n *there exist* (n,k,l,Δ)*-systems.*

Proof. Let $l=k-s$. We consider as k-tuples of an $(n,k,k-s,\Delta)$-system for $\Delta \geqq 3$ all solutions of the system of equations:

$$\sum_{i=1}^{k} x_i^t \equiv a_t \mod p, \quad t=1,\dots,s. \tag{3.1}$$

where $x_i \neq x_j$, $x_i \in \{0,1,\dots,n-1\}$, $1 \leqq i<j \leqq k$ and p is the minimal prime such that $p \geqq n$.

It is not difficult to see that the k-tuples corresponding to all the solutions of such a system form an $(n,k,k-s)$-packing (see, for example, [7, 15]). One k-tuple corresponds to $k!$ solutions because all functions in our system are symmetric. Our goal now is to prove that it is an $(n,k,k-s-3)$-covering. Let us fix the first $k-s-3$ variables in the system (3.1), say $x_i=j_i$, $i=1,\dots,k-s-3$. We obtain a system of s equations with $s+3$ variables. The number of possibilities to fix the first $k-s-3$ variables such that

$$x_i \neq x_j,\ x_i \in \{0,1,\dots,n-1\},\ 1 \leqq i < j \leqq k-s-3$$

is $(n)_{k-s-3}=n(n-1)\cdots(n-k+s+4)$. We wish to estimate now the number of solutions of the system (3.1) under fixed values $j_1,\dots,j_{k-s-3}$ of the first $k-s-3$ variables and the conditions:

$$x_i \neq x_j,\ i \neq j \text{ and } x_i \notin \{j_1,\dots,j_{k-s-3}\},\ i=k-s-2,\dots,k$$

for some fixed set $\{j_1,\dots,j_{k-s-3}\}$.

In order to do this we use the result of S. D. Cohen (see [2, 3] and the next section) which shows that for the number T of solutions of the system (3.1) without the restrictions $x_i \in \{0,1,\dots,n-1\}$ for $k-s=2$ or 3 the following inequality holds:

$$|T-p^{k-s}| \leqq \frac{k}{2} k! p^{k-s-1/2}. \tag{3.2}$$

If $k-s \geqq 4$, at worst the right side of (3.2) needs to be doubled. Using this result for $k=s+3$ we obtain that for this case the number of solutions is $p^3+(c(s+3)/2)(s+3)!p^{5/2}$ for some constant c, $|c|<1$. So the total number of solutions with the first $k-s-3$ variables arbitrarily fixed can be represented in the form

$$p^3 + c\frac{s+3}{2}(s+3)!p^{5/2}. \tag{3.3}$$

In order to obtain only solutions with restrictions:

$$x_i \neq x_j,\ 1 \leqq i < j \leqq k \text{ and } x_i \in \{0,1,\dots,n-1\}$$

we must subtract from the value (3.3) two terms corresponding to the following cases:

(1) the number of solutions satisfying the condition $x_i \in \{j_1,\dots,j_{k-s-3}\}$ for some $i \in \{k-s-2,\dots,k\}$ and fixed set $\{j_1,\dots,j_{k-s-3}\}$.

This number is at most

$$(s+3)(k-s-3)\left(p^2 + \frac{s+2}{2}(s+2)!p^{3/2}\right). \tag{3.4}$$

(2) the number of solutions satisfying the condition $x_i \in \{n, \ldots, p-1\}$ for some $i \in \{k-s-2, \ldots, k\}$.

This number is at most

$$(s+3)(p-n)\left(p^2 + \frac{s+2}{2}(s+2)!p^{3/2}\right). \tag{3.5}$$

So if the sum of the last two terms ((3.4 and (3.5)) is smaller than (3.3) then there exists at least one solution of (3.1) such that $x_i \neq x_j$, $1 \leqq i < j \leqq k$ and $x_i \in \{1, \ldots, n\}$ for $1 \leqq i \leqq k$. Because known results on the difference between consecutive primes (see, for example, [13]), imply that $p - n \leqq n^c$ for some constant $c < 1$, it is not difficult to check that this inequality holds under the conditions of Theorem 2.

This means that the set of k-tuples corresponding to all the solutions of such a system is an $(n, k, k-s-3)$-covering and so it is an (n, k, l, Δ)-system for $\Delta \geqq 3$. The proof of Theorem 2 is complete.

For the case $\Delta = 2$ we can prove the sufficient condition in the following form.

Theorem 3. *Let* $k \leqq cn/(s+2)!$ *and* $s \leqq c_1 \log n/\log\log n$ *for some constants* $c < 1$ *and* $c_1 < 1/2$. *Then for* $\Delta = 2$ *and all sufficiently large* n *there exist* (n, k, l, Δ)*-systems.*

The proof is quite similar to the proof of Theorem 2 with one difference: we fix values not of $k-s-3$ but of the first $k-s-2$ variables and for the system of s equations with $s+1$ indeterminates we use the trivial upper bound $p(s+1)!$ for the number of its solutions.

Remark 1. The assertions of Theorems 2 and 3 can be easily reformulated as sufficient conditions not for sufficiently large n only but for all n. The form of these conditions can be derived from the proof of Theorem 2.

Remark 2. For $\Delta = 1$ in [15] it was shown that $(n, k, k, -1, \Delta)$-systems exist if $k \leqq (n/2) + 1$.

4. Bounds for the number of solutions of one algebraic system

For the sake of completeness we give in this section a brief description of the result of S. D. Cohen on the number of solutions of one system of algebraic equations over a finite field. As it was shown above this result is the key to obtain the sufficient conditions for the existence of (n, k, l, Δ)-systems for the case $\Delta \geqq 2$.

Let $\mathbb{F}_p = GF(p), p$ prime. Let k, s be positive integers with $1 \leqq s \leqq k \leqq p$. Let $l = k - s$ and assume $l \leqq 2$. Write $N(k, s)$ for the number of solutions of the system:

$$\sum_{i=1}^{k} x_i^t \equiv a_t \bmod p, \; t = 1, \ldots, s, \tag{4.1}$$

where $x_i \neq x_j$, $1 \leqq i < j \leqq k$.

We shall give a brief description of the following result of S. . Cohen ([2, 3]).

Theorem 4. *Let $l=k-s\geqq 2$. Then*

$$|N(k,s)-p^l|\leqq (k/2)k!p^{l-1/2},$$

except perhaps if $l\geqq 4$ and $k^2/2<p^{1/2}<k^2$, in which case the right hand side should be doubled.

Proof. We give only a sketch of the proof which should be read along with [2], [3]. Fairly trivial estimates suffice unless $k<p^{1/4}$ which can therefore be assumed. Let s_j be the jth symmetric function of $x_1,\dots,x_k$. Then the set of all solutions of (4.1) (with distinct components) is the subset of $\mathbb{F}_p^k$ comprising those x with distinct components such that $(-1)^j s_j$ has a prescribed value b_j for $j=1,\dots,s$. Here $b_1=-a_1$, $b_2=(1/2)(a_1^2-a_2),\dots$.

Let

$$f(x)=x^k+b_1x^{k-1}+\cdots+b_{k-1}x\in\mathbb{F}_p[x],$$

where $b_1,\dots,b_s$ are the prescribed values and $b=(b_{s+1},\dots,b_{k-1})\in\mathbb{F}_p^{l-1}$ is arbitrary. Then

$$N(k,s)=k!\sum_{b\in\mathbb{F}_p^{l-1}}M(b) \tag{4.2}$$

where $M(b)$ denotes the number of a in $\mathbb{F}_p$ such that $f(x)+a$ splits completely into a product of k distinct linear factors over $\mathbb{F}_p$.

Rather than estimate $M(b)$ in every case, we restrict ourselves to those b in the set

$$\mathbf{B}=\{b\in\mathbb{F}_p^{l-1}\colon f' \text{ has } k-1 \text{ distinct roots in } \bar{\mathbb{F}}_p \text{ all giving rise to distinct values}\}.$$

Here $\bar{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$.

By Lemma 5 of [2], $|\mathbf{B}|\geqq p^{l-1}-cp^{l-2}$, where $c=c(k,s)$ is independent of p. The arguments of Lemmas 6 and 7 of [2] show that none of the polynomial equations which arise are identities and we can routinely bound their number of solutions. More specifically, but briefly, as regards Lemma 6 of [2], the relevant polynomials can be totally composite only if they are polynomials in x^d for some $d>1$ and this excludes at most lp^{l-2} elements ($(l-1)$-tuples) of $\mathbf{B}$. On the other hand, solutions of (5.4) and (5.6) exclude, between them, for each j (with $1\leqq j\leqq l-1$) at most $3kp^{l-2}$ elements, and so a total of at most $3lkp^{l-2}$ elements. Further, using the bound in Bezout's theorem, for each $j\leqq l-1$, (5.8) of [2] excludes $(k+j-3)(k-1)$ elements from $\mathbf{B}$ and so $(l-1)(k-1)(k+l/2-3)$ altogether. This gives the following lemma;

Lemma. *The size of $\mathbf{B}$ satisfies*

$$|\mathbf{B}|\geqq p^{l-1}-c(k,l)p^{l-2},$$

where

$$c(k,2)=k^2+3k+4,$$
$$c(k,3)=2k^2+4k+6,$$
$$c(k,l)\leqq lk\left(k+\frac{l}{2}\right),\ l\geqq 4.$$

Now for $b\in\mathbf{B}$, $M(b)$ can be interpreted as the number of a in $\mathbb{F}_p$ such that $t+a$ is unramified and splits completely into first degree primes in E, the splitting field of $f(x)+t$ over $\mathbb{F}_p(t)$. Since the Galois group of $f(x)+t$ over $\mathbb{F}_p(t)$ is S_k [**1**] and so has order $k!$ we conclude that $k!M(b)$ is exactly the number of first degree prime divisors of E which divide a finite unramified first degree prime $t+a$ of $\mathbb{F}_p(t)$. On the other hand, by Weil's theorem (which applies since $\mathbb{F}_p$ is algebraically closed in E), the total number of first degree prime divisors of E differs from p by at most $2g\sqrt{p}$, where g is the genus of E. So we obtain for $b\in\mathbf{B}$

$$|k!M(b)+T-p|\leqq 2g\sqrt{p}, \tag{4.3}$$

where T is the number of first degree prime divisors in E which are infinite or ramified.

Moreover, by Proposition 5.15 of [**5**], the ramification index of every finite ramified prime in E is 2 and the ramification index of the infinite prime is k. Using the definition of $\mathbf{B}$, this means that the relative different of E over $\mathbb{F}_p(t)$ has degree

$$d=\frac{(k-1)k!}{2}+(k-1)(k-1)!. \tag{4.4}$$

Let g be the genus of E. By the Hurwitz formula and (4.4)

$$2g-2=-2k!+d=\tfrac{1}{2}(k^2-3k-2)(k-1)!$$

From the above there are at most $((k-1)k!/2)$ finite ramified first degree prime divisors of E and at most $(k-1)!$ infinite first degree prime divisors of E. Thus

$$T\leqq\tfrac{1}{2}(k^2-k+2)(k-1)!$$

and from (4.3)

$$|k!M(b)-p|\leqq(\tfrac{1}{2}(k^3-3k-2)(k-1)!+2)\sqrt{p}+\tfrac{1}{2}(k^2-k+2)(k-1)!, \tag{4.5}$$

where for the upper bound for $k!M(b)$, we can disregard the last term.

For $b\notin\mathbf{B}$ we can use "almost" trivial estimate

$$M(b)\leqq\frac{p}{k}$$

which is arrived at by assuming, in the worst case, that (all but one of) the members of

$\mathbb{F}_p$ can be grouped in classes of size k, all giving the same value to f. Combining this bound with (4.5) and the lemma we can obtain the result of Theorem 4.

5. Resume

For (n,k,l,Δ)-systems the notion of Δ is similar to the notion of a covering radius of a code with given distance (or packing radius) [**12**, **16**, **18**, **19**]. It is known that for BCH-codes the covering radius is roughly speaking twice the packing radius [**12**, **18**, **19**]. In contrast with these results, for our case the covering radius (i.e. the value of $k-l+\Delta$) is equal to the packing radius plus a constant (2 or 3).

One interesting question arises if we compare the necessary condition with the sufficient one. Roughly speaking the necessary condition is: $k-l<\sqrt{n(\Delta+1)}$ but the sufficient one is: $k-l<c\log n/\log\log n$ (with some additional restriction on the size k). It is not difficult to see that the bound for $k-l$ is determined by the value of coefficient K in S. D. Cohen's bound (5.1) (see Theorem 4) for the number of solutions of the above system with k indeterminates and s equations (in S. D. Cohen's formula $K=(k/2)k!$):

$$|N(k,s)-p^{k-s}|\leqq Kp^{k-s-1/2}. \tag{5.1}$$

From the necessary condition (Theorem 1) it is not difficult to prove that $K>ck$. If anybody can decrease the value of K in (5.1) then we can increase the upper bound for $k-l$ in our sufficient condition.

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