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AN INEQUALITY FOR POSITIVE SEMIDEFINITE HERMITIAN MATRICES[1]

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Let A and B be positive semidefinite Hermitian n-square matrices. If $A-B$ is positive semidefinite, write $A \geq B$. Haynsworth [1] has proved that if $A \geq B$ then $\det(A+B) \geq \det A + n \det B$.

Let G be a subgroup of the symmetric group, S_n, and let λ be a character on G. Let

$$e_r(A) = \sum_{g \in G} \lambda(g) E_r(a_{1g(1)}, \ldots, a_{ng(n)})$$

where $A=(a_{ij})$ and E_r is the rth elementary symmetric function.

THEOREM. *Let* $A \geq B$. *Then* $e_r(A+B) \geq e_r(A) + (2^r - 1) e_r(B)$. In particular, if $G=S_n$ *and* $\lambda = sgn$, $\det(A+B) \geq \det A + (2^n - 1) \det B$.

Proof. Let $K_r(X)$ be the rth Kronecker power of n-square X. Observe that $(A-B) \otimes B \geq 0$, so $A \otimes B \geq B \otimes B \equiv K_2(B)$. It follows by induction that

$$K_r(A+B) \geq K_r(A) + (2^r - 1) K_r(B). \tag{1}$$

Let Γ be the set of integer sequences of length r chosen from $1, 2, \ldots, n$. Then, $K_r(X)$ is indexed by the set Γ ordered lexicographically.

Let Ω be the subset of Γ consisting of the $(n!/(n-r)!)$ sequences in which no integer is repeated.

Let $k_r(X)$ be the $(n!/(n-r)!)$-square principal submatrix of $K_r(X)$ corresponding to Ω.

Let $Q(g)$ be the n-square permutation matrix defined by g ($\in S_n$). It follows from the orthogonality relations for characters that

$$C_r = \sum_{g \in G} \lambda(g) K_r(Q(g))$$

is a positive multiple of a projection. Since $\lambda(g^{-1}) = \overline{\lambda(g)}$, C_r is hermitian. Thus, $C_r \geq 0$. Let c_r be the principal submatrix of C_r^T corresponding to Ω, i.e.,

$$c_r = \sum_{g \in G} \lambda(g) k_r(Q(g^{-1})).$$

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Certainly, $c_r \geq 0$. Let $\sigma(X)$ be the sum of the elements of X. Finally, let $A \circ B = (a_{ij}b_{ij})$ be the Hadamard product of A and B. The straight forward observation that $r!e_r(A) = \sigma(c_r \circ k_r(A))$ has been made in [2] and [3]. The Theorem now follows from (1) and the linearity of σ and Hadamard product.

References

1. Emilie V. Haynsworth, *Applications of an inequality for the Schur complement*, Proc. Amer. Math. Soc. **24** (1970) 512–516.

2. Russell Merris, *A dominance theorem for partitioned hermitian matrices*, Trans. Amer. Math. Soc. **164** (1972) 341–352..

3. ———, *Inequalities for matrix functions*, J. Algebra, **22** (1972) 451–460.

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