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STRONGLY E-REFLEXIVE INVERSE SEMIGROUPS II

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In a recent paper (**13**), we introduced the class of strongly E-reflexive inverse semigroups. This class was shown to coincide with the class of those inverse semigroups which are semilattices of E-unitary inverse semigroups. In particular, therefore, E-unitary inverse semigroups and semilattices of groups are strongly E-reflexive, and in fact so are subdirect products of these two types of semigroups.

J. Mills (**1**) considers orthodox semigroups which are subdirect products of an E-unitary regular semigroup and a semilattice of groups, and of course there are strong connections between the two papers.

In this communication we wish in part to specialise Mills' results to inverse semigroups, and in doing so, to consider them in the context of the theory of strongly E-reflexive inverse semigroups. As well, we give yet another characterisation of these semigroups in terms of E-unitary inverse semigroups, and show that an inverse semigroup which is a semilattice of strongly E-reflexive inverse semigroups is again strongly E-reflexive.

1

In this section, some results on congruences which will be needed below are collected together.

Let S be an inverse semigroup with semilattice of idempotents E. Let $\sigma = \{(x, y) \in S \times S \mid ex = ey \text{ for some } e \in E\}$; then σ is the minimum group congruence on S (**10**). Moreover, S is said to be *E-unitary* if $E\sigma = E$. In general, there exists a minimum E-unitary congruence on S, which we shall denote by κ.

Proposition 1. (See **12**) *κ is the congruence on S generated by $\sigma \cap \mathcal{R}$.*

Let $\mu = \{(a, b) \in S \times S \mid a^{-1}ea = b^{-1}eb \text{ for all } e \in E\}$; then μ is the maximum idempotent-separating congruence on S (**7**). Recall that a congruence ν on S is called *idempotent-determined* (**5**) if $(e, x) \in \nu$ and $e \in E$ imply that $x \in E$; as is natural, a homomorphism on S will be called *idempotent-determined* when the associated congruence on S is so. The minimum semilattice of groups congruence on S will be denoted by ξ, the minimum semilattice congruence by η, and the identity congruence by i.

Note that $\kappa \subseteq \eta \cap \sigma$ and that $\xi \subseteq \eta \cap \sigma$.

Proposition 2. (See **1**) $\xi = \{(a, b) \in \eta \mid ea = eb \text{ for some } e^2 = e\eta a\}$.

Remark. Proposition 2 follows, by the usual type of argument based on abstract nonsense, from the result due to Hardy and Tirasupa (**6**, Lemma 1) quoted in (**13**, §1).

The next result is a formulation of the well-known theorem due to Tamura.

Proposition 3. (See **15**) *Let A be an η-class of S. Then the universal congruence on the inverse subsemigroup A is the only semilattice congruence on A.*

As usual, we adhere to the notation and terminology of (**2**, **14**), and we assume familiarity with the basic theory of inverse semigroups contained in (**2**). Finally, the symbol '$\subset$' means 'properly contained in'.

2

This section is devoted to proving two theorems on strongly E-reflexive inverse semigroups in general. First we recall some theory from (**13**).

The inverse semigroup S is said to be *strongly E-reflexive* if, given $e \in E^1$ and x and y in S, $exy \in E$ implies that $eyx \in E$, where the element 1 is the identity of S^1.

Note that the subdirect product of a family of strongly E-reflexive inverse semigroups is again strongly E-reflexive.

Proposition 4. (See **13**) *Let S be an inverse semigroup. Then the following are equivalent*:

(i) *S is strongly E-reflexive.*
(ii) *S is a semilattice of E-unitary inverse semigroups.*
(iii) *The congruence ξ is idempotent-determined.*

Now follows another characterisation of these semigroups.

Theorem 1. *Let S be an inverse semigroup. Then S is strongly E-reflexive if and only if S is a subdirect product of E-unitary inverse semigroups and E-unitary inverse semigroups with zero.*

Proof. Suppose that S is strongly E-reflexive. By Proposition 4, S is the semilattice W of E-unitary inverse semigroups S_α, $\alpha \in W$, say. The theory of (**13**, §3) – see Theorem 11 there – shows that there exists an inverse semigroup T which is a strong semilattice of E-unitary inverse semigroups T_α, $\alpha \in W$, and an embedding of S in T which embeds each S_α in T_α. (In fact each T_α is a semidirect product of a semilattice and a group, but we do not need this result here.) Thus we may consider $S \subseteq T$, with $S_\alpha \subseteq T_\alpha$ for each $\alpha \in W$.

Given $\alpha \geqslant \beta$ in W, let $\phi_{\alpha\beta}$ denote the structural homomorphism of T going from T_α to T_β. For each $\delta \in W$ define the map ϕ_δ: $S \to T_\delta$ or ϕ_δ: $S \to T_\delta^0$, depending on whether δ is or is not the minimum member of W, respectively, as follows:

if $x \in S_\alpha$ with $\alpha \geqslant \delta$, let $x\phi_\delta = x\phi_{\alpha\delta}$;
if δ is not the minimum member of W, and
if $x \in S_\alpha$ with $\alpha \ngeqslant \delta$, let $x\phi_\delta = 0$.

Note that $0 \in \operatorname{Im}\phi_\delta$ in the latter case, and that each $\phi_\delta | S_\delta$ is the identity map.

It is easily checked that each ϕ_δ is a homomorphism, so that $\operatorname{Im}\phi_\delta$ is an E-unitary inverse semigroup with zero added possibly. Clearly the ϕ_δ separate the elements of S, so that the result follows.

The converse is immediate in view of Proposition 4 and the remark which precedes it.

The final result of the section indicates to some extent how extensive is the class of strongly E-reflexive inverse semigroups. It generalises part of (**13**, Theorem 1).

Theorem 2. *Let S be an inverse semigroup which is a semilattice of strongly E-reflexive inverse semigroups. Then S is strongly E-reflexive.*

Proof. Let S be the semilattice W of strongly E-reflexive inverse semigroups S_α, $\alpha \in W$, and let ν be the semilattice congruence on S induced by the canonical homomorphism from S onto W. Then $\eta \subseteq \nu$, so that for each $\alpha \in W$, $\eta_\alpha \equiv \eta \cap (S_\alpha \times S_\alpha)$ is a semilattice congruence on S_α.

Pick $\alpha \in W$, and let η'_α denote the minimum semilattice congruence on S_α. Then $\eta'_\alpha \subseteq \eta_\alpha$; suppose that $\eta'_\alpha \subset \eta_\alpha$. Then for some $x \in S_\alpha$, $x\eta'_\alpha \subset x\eta_\alpha$, so that $\eta''_\alpha \equiv \eta'_\alpha \cap (x\eta_\alpha \times x\eta_\alpha)$ is a semilattice congruence on the inverse subsemigroup $x\eta_\alpha$ of S which is not the universal congruence. This contradicts Tamura's theorem (Proposition 3 above).

Hence for each $\alpha \in W$, η_α is the minimum semilattice congruence on S_α. Setting

$$\xi_\alpha = \{(a, b) \in \eta_\alpha | ea = eb \quad \text{for some} \quad e^2 = e\eta_\alpha a\},$$

it follows from Proposition 2 that ξ_α is the minimum semilattice of groups congruence on S_α. Clearly $\xi = \cup \xi_\alpha$, since $\eta \subseteq \nu$. However, by Proposition 4, each ξ_α is idempotent-determined, and it follows that ξ itself is idempotent-determined. Hence, by Proposition 4 again, S itself is strongly E-reflexive.

3

The final section is devoted to specialising the theory of (**1**) to inverse semigroups, and to placing it in the context of the theory of strongly E-reflexive inverse semigroups.

First of all we have a result which is of independent interest.

Proposition 5. *Let S be an inverse semigroup. Then the following are equivalent:*

(i) *$\mathcal{R} \cap \sigma$ is a congruence.*

(i)′ *$\mathcal{L} \cap \sigma$ is a congruence.*

(ii) *$\mathcal{R} \cap \sigma = \mathcal{L} \cap \sigma$.* (A)

(iii) *$\mathcal{R} \cap \sigma = \mu \cap \sigma$.*

(iv) *There exists an idempotent-separating E-unitary congruence ν on S.*

In this case $\nu \cap \sigma$ is unique; in fact, $\nu \cap \sigma = \mu \cap \sigma = \kappa$.

Proof. (ii)$\Rightarrow$(i). $\mathcal{R} \cap \sigma$ is a left congruence, and $\mathcal{L} \cap \sigma$ is a right congruence.

(i)$\Rightarrow$(iii). $\mathcal{R} \cap \sigma$ is certainly idempotent-separating, so that $\mathcal{R} \cap \sigma \subseteq \mu$. Hence $\mathcal{R} \cap \sigma \subseteq \mu \cap \sigma$. But $\mu \subseteq \mathcal{R}$, whence $\mu \cap \sigma \subseteq \mathcal{R} \cap \sigma$.

(iii)$\Rightarrow$(iv). Let $\nu = \mu \cap \sigma$. Then ν is an idempotent-separating congruence, which moreover is the congruence generated by $\mathcal{R} \cap \sigma$; in fact it is equal to $\mathcal{R} \cap \sigma$. Hence ν is an E-unitary congruence, being the minimum E-unitary congruence on S (see Proposition 1).

(iv)$\Rightarrow$(ii). By Proposition 1, $\mathcal{R} \cap \sigma \subseteq \nu$ and by supposition $\nu \subseteq \mu$. Hence $\mathcal{R} \cap \sigma \subseteq \nu \subseteq \mu$, so that as before $\nu \cap \sigma = \mathcal{R} \cap \sigma = \mu \cap \sigma$. By duality, $\nu \cap \sigma = \mathcal{L} \cap \sigma = \nu \cap \sigma$, and the result follows.

Now conditions (ii), (iii), and (iv) are self-dual, while (i)′ is the dual of (i), whence the full result.

The next result is the analogue of (**1**, Theorem 3.5), and follows to some extent the proof of that result.

Theorem 3. *Let S be an inverse semigroup. Then S is a subdirect product of an E-unitary inverse semigroup and a semilattice of groups if and only if S is strongly E-reflexive and satisfies any one of the mutually equivalent conditions* (A).

Proof. Suppose that S is a subdirect product of an E-unitary inverse semigroup and a semilattice of groups. In view of Proposition 4 and the remark which precedes it, S is strongly E-reflexive, and moreover $\xi \cap \kappa = i$. Following the exact argument used in (**1**, Theorem 3.5) we show that κ is idempotent-separating.

Let e, f be idempotents with $e\kappa f$. Then $e\eta f$, so that by Proposition 2, $e\xi f$. Hence $(e, f) \in \kappa \cap \xi = i$, so that $e = f$.

Hence S satisfies condition (iv) of Proposition 5.

Conversely, suppose that S is strongly E-reflexive and satisfies condition (iv) of Proposition 5, say. Then κ is idempotent-separating, and by Proposition 4, ξ is idempotent-determined. Hence $\kappa \cap \xi$ is both idempotent-separating and idempotent-determined, and it follows that $\kappa \cap \xi = i$, whence the result.

Remarks. Consider B^0, where B is the bicyclic semigroup. Then B^0 is strongly E-reflexive. However, by condition (ii) of (A) say, B^0 is not a subdirect product of an E-unitary inverse semigroup and a semilattice of groups. Note that the equivalence $\mathcal{H} \cap \sigma$ on B^0 is a congruence; in fact, $\mathcal{H} \cap \sigma = i$.

Now consider the bisimple inverse ω-semigroup $S(G, \alpha)$ where the endomorphism α of G is injective. Then $S(G, \alpha)$ is E-unitary and the two distinct congruences $\mathcal{H}$ and i are both idempotent-separating and E-unitary; note that $\mathcal{H} \not\subseteq \sigma$ (see (**11**)).

Finally consider $S(G, \alpha)$ whenever α is *not* injective. As shown in (**13**, §1), $S(G, \alpha)$ is *not* strongly E-reflexive. However, $\mathcal{H}$ is an idempotent-separating E-unitary congruence on $S(G, \alpha)$ (see (**11**) again).

Following the theory of (**3, 4, 8**), whenever an inverse semigroup S satisfies condition (iv) of Proposition 5, we may regard the structure of S as being known, albeit in a highly abstract form, in terms of an E-unitary inverse semigroup and a semilattice of groups. Note also that this situation is in a sense the opposite to that considered by McAlister in his theory of "E-unitary covers" (see (**9**), for example).

We now wish to consider what Theorem 3 means in terms of the structure theory for strongly E-reflexive inverse semigroups developed in (**13**, §3). A brief sketch of this theory now follows.

Let X be a down-directed partially ordered set, and let Y be a non-empty subsemilattice and order-ideal of X. Suppose that an inverse semigroup T is given together with a homomorphism $\phi: T \to \mathcal{I}_X$, by means of which we consider T acting on X on the left; as usual, given $t \in T$, $t\phi$ will be denoted by t. Suppose further that $X = TY$ and that for each $t \in T$, the domain Δt and range ∇t of (the action of)t are non-empty order ideals of X, and $t: \Delta t \to \nabla t$ is an order-isomorphism. Define $L = L(T, X, Y)$ to be

$$\{(a, t) | t \in T, \quad a \in Y \cap \Delta t^{-1}, \quad t^{-1}a \in Y\}$$

under the multiplication $(a, t)(b, s) = (t(t^{-1}a \wedge b), ts)$. Then L is an inverse semigroup with semilattice of idempotents $\mathscr{E} \equiv \{(a, e) | e^2 = e \in T, a \in Y \cap \Delta e\}$, and (T, X, Y) is said to be an *L-triple*. If $(a, t) \in L$, then $(a, t)^{-1} = (t^{-1}a, t^{-1})$ and $(a, t)(a, t)^{-1} = (a, tt^{-1})$; moreover if (a, e) and (b, f) are in $\mathscr{E}$, then $(a, e)(b, f) = (a \wedge b, ef)$. We also note that the given hypotheses imply that for each $t \in T$ there exists $a \in Y$ such that $(a, t) \in L$.

If, further, T has the property that

> for each $a \in Y$ there is a minimum idempotent $e(a)$ in T such that $a \in \Delta e(a)$, where for each a, b in Y, $e(a \wedge b) = e(a) \cdot e(b)$,

then (T, X, Y) is said to be *strict L-triple.*

If (T, X, Y) is a strict L-triple, define $L_m = L_m(T, X, Y)$ to be $\{(a, t) \in L | tt^{-1} = e(a)\}$. Then L_m is an inverse subsemigroup of L with semilattice of idempotents $\mathscr{E}_m = \{(a, e(a)) | a \in Y\}$, and the map $(a, t) \mapsto a$ is a homomorphism from $\mathscr{E}_m$ onto Y. If, further, the homomorphism $(a, t) \mapsto t$ from L_m into T is surjective, then (T, X, Y) is called a *fully strict L-triple.*

We now summarise the main results which connect the theory of strongly E-reflexive inverse semigroups with that of L-triples.

Theorem 4. (See **13**) (i) *Let S be a strongly E-reflexive inverse semigroup. Then there exist a fully strict L-triple (T, X, Y), with T a semilattice of groups, and an isomorphism from S onto L_m.*

(ii) *Let (T, X, Y) be an L-triple with T a semilattice of groups. Then the inverse semigroup $L(T, X, Y)$ is strongly E-reflexive. Moreover, X can be embedded in a semilattice $\bar{X}$ such that $(T, \bar{X}, \bar{X})$ is an L-triple with $L(T, X, Y)$ embedded in $L(T, \bar{X}, \bar{X})$.*

In fact the inverse semigroup $L(T, \bar{X}, \bar{X})$ of Theorem 4 (ii) is a strong semilattice of inverse semigroups each of which is a semidirect product of a semilattice and a group. Each of the semigroups $L_m(T, X, Y)$, $L(T, X, Y)$ and $L(T, \bar{X}, \bar{X})$ can be presented explicitly as a semilattice of E-unitary inverse semigroups (see (**13**) for further details).

Our final theorems can be viewed as concrete analogues of (**1**, Theorem 3.9). Before stating them, we fix some notation and prove a technical lemma.

Let T be the semilattice W of groups G_α, $\alpha \in W$, with connecting homomorphisms

$\psi_{\alpha\beta}$, $\alpha \geqslant \beta$, and let (T, X, Y) be an L-triple specified by the homomorphism $\phi: T \to \mathcal{I}_X$. Denote the identity of G_α by ϵ_α, and recall that for each $t \in G_\alpha$, t is an order automorphism with domain and range $\Delta\epsilon_\alpha$. Further, for each $\alpha \in W$, let $K_\alpha = \bigcup\{\operatorname{Ker} \psi_{\alpha\beta} \mid \beta \leqslant \alpha\}$. Then K_α is a normal subgroup of G_α. Let $\bar{G}_\alpha = G_\alpha/K_\alpha$ and let ν_α: $G_\alpha \to \bar{G}_\alpha$ denote the canonical homomorphism. Given $\alpha \geqslant \beta$ in W, $\psi_{\alpha\beta}: G_\alpha \to G_\beta$ induces a unique injective homomorphism $\bar{\psi}_{\alpha\beta}$: $\bar{G}_\alpha \to \bar{G}_\beta$ such that $\nu_\alpha \bar{\psi}_{\alpha\beta} = \psi_{\alpha\beta}\nu_\beta$. The $\bar{\psi}_{\alpha\beta}$ form a family of connecting homomorphisms under which $\bar{T} \equiv \bigcup\{\bar{G}_\alpha \mid \alpha \in W\}$ becomes an E-unitary semilattice of groups on W. In fact, it follows easily from Proposition 1 that $\{K_\alpha \mid \alpha \in W\}$ is the kernel normal system on T for κ, so that $\bar{T} \approx T/\kappa$.

We denote by ν the canonical homomorphism from T onto $\bar{T}$.

Lemma 1. *Let S be an inverse semigroup with semilattice of idempotents E. Let ν_1 be a left congruence on S and let ν_2 be a congruence on S. Then $\nu_1 \subseteq \nu_2$ if and only if $e\nu_1 \subseteq e\nu_2$ for all $e \in E$.*

Proof. Suppose that $e\nu_1 \subseteq e\nu_2$ for all $e \in E$ and let $(a, b) \in \nu_1$. Then $(a^{-1}a, a^{-1}b) \in \nu_1$, so that $(a^{-1}a, a^{-1}b) \in \nu_2$. Hence $(a, aa^{-1}b) \in \nu_2$, and it follows that $a\nu_2^{\natural} \leqslant b\nu_2^{\natural}$. Similarly, $b\nu_2^{\natural} \leqslant a\nu_2^{\natural}$, whence $(a, b) \in \nu_2$. Thus $\nu_1 \subseteq \nu_2$.

The converse is immediate.

For brevity, $L(T, X, Y)$ will be denoted by L, and whenever (T, X, Y) is a strict L-triple, $L_m(T, X, Y)$ will be denoted by L_m.

Theorem 5. *The following are equivalent*:

(i) *L is the subdirect product of an E-unitary inverse semigroup and a semilattice of groups.*

(ii) *Given $\alpha \in W$, $t \in K_\alpha$, and $a \in Y \cap \Delta\epsilon_\alpha$ such that $ta \in Y$, then $tb = b$ for all $b \leqslant a$.* (1)

Proof. (i)$\Rightarrow$(ii) Suppose that we are given $\alpha \in W$, $t \in K_\alpha$, and $a \in Y \cap \Delta\epsilon_\alpha$ such that $ta \in Y$. Let t^{-1} be denoted by s. Then $(a, s) \in L$ and $s \in K_\alpha$, so that $s \in \operatorname{Ker} \psi_{\alpha\beta}$ for some $\beta \leqslant \alpha$. Recall that there exists $c \in Y$ such that $(c, \epsilon_\beta) \in \mathscr{E}$. Now $(a, \epsilon_\alpha) \in \mathscr{E}$ and $(c, \epsilon_\beta)(a, \epsilon_\alpha) = (c, \epsilon_\beta)(a, s) = (c \wedge a, \epsilon_\beta)$. Hence $(a, \epsilon_\alpha)\sigma(a, s)$; moreover, $(a, \epsilon_\alpha)\mathscr{R}(a, s)$.

Now by Theorem 4, L is strongly E-reflexive, so that by Theorem 3, $\mathscr{R} \cap \sigma \subseteq \mu$ on L. Hence $(a, \epsilon_\alpha)\mu(a, s)$. Let $b \leqslant a$. Then $b \in \Delta\epsilon_\alpha$, so that $(b, \epsilon_\alpha) \in \mathscr{E}$, and it follows from the definition of μ that $(a, \epsilon_\alpha)^{-1}(b, \epsilon_\alpha)(a, \epsilon_\alpha) = (a, s)^{-1}(b, \epsilon_\alpha)(a, s)$. Now the left hand side of this equation equals (b, ϵ_α) while on the right, $(a, s)^{-1}(b, \epsilon_\alpha)(a, s) = (s^{-1}a, s^{-1})(b, s) = (s^{-1}b, \epsilon_\alpha)$. Hence $tb = s^{-1}b = b$.

(ii) $\Rightarrow$ (i). In view of Theorems 3 and 4, it suffices to prove that $\mathscr{R} \cap \sigma = \mu \cap \sigma$ on L, or what is equivalent, that $\mathscr{R} \cap \sigma \subseteq \mu$. By Lemma 1, then this amounts to showing that for $(a, \epsilon_\alpha) \in \mathscr{E}$ and $(d, s) \in L$, $(a, \epsilon_\alpha)\mathscr{R} \cap \sigma(d, s)$ implies that $(a, \epsilon_\alpha)\mu(d, s)$.

Now $(a, \epsilon_\alpha)\mathscr{R}(d, s)$ if and only if $d = a$ and $s \in G_\alpha$, and it is easy to see that

$(a, \epsilon_\alpha)\sigma(a, s)$ implies that $s \in K_\alpha$. Thus $s^{-1} \in K_\alpha$. Hence for all $(c, \epsilon_\gamma) \in \mathscr{E}$,

$$\begin{aligned}(a, s)^{-1}(c, \epsilon_\gamma)(a, s) &= (s^{-1}a, s^{-1})(c \wedge a, \epsilon_\gamma s)\\ &= (s^{-1}(c \wedge a), \epsilon_{\alpha\gamma})\\ &= (c \wedge a, \epsilon_{\alpha\gamma}),\end{aligned}$$

using the hypotheses with s^{-1} in place of t and $c \wedge a$ in place of b. But $(c \wedge a, \epsilon_{\alpha\gamma}) = (a, \epsilon_\alpha)^{-1}(c, \epsilon_\gamma)(a, \epsilon_\alpha)$, so that $(a, \epsilon_\alpha)\mu(a, s)$, and the result follows.

A particularly favourable circumstance in which L becomes the subdirect product of an E-unitary inverse semigroup and a semilattice of groups is described in the next result.

Theorem 6. *The homomorphism $\phi: T \to \mathscr{I}_X$ induces a homomorphism $\bar{\phi}: \bar{T} \to \mathscr{I}_X$ such that $\phi = \nu\bar{\phi}$ if and only if*

$$\textit{each} \quad t \in \cup\{K_\alpha | \alpha \in W\} \quad \textit{is the identity map on} \quad \Delta t. \tag{2}$$

In this case, $(\bar{T}, X, Y)$ becomes an L-triple under the induced action, and $L(T, X, Y)$ is the subdirect product of the E-unitary inverse semigroup $L(\bar{T}, X, Y)$ and the semilattice of groups T.

Proof. The first part is immediate, as is the fact that $(\bar{T}, X, Y)$ becomes an L-triple under the induced action. Suppose then that each $t \in \cup\{K_\alpha | \alpha \in W\}$ is the identity map on Δt, and let $\bar{L} = L(\bar{T}, X, Y)$. The map $\theta_1: L \to \bar{L}$ defined by the rule θ_1: $(a, t) \mapsto (a, t\nu)$ is clearly a surjective homomorphism, which together with the surjective homomorphism θ_2: $(a, t) \mapsto t$ from L onto T separates the elements of L.

We claim that $\bar{L}$ is E-unitary. To see this, recall from **(13)** that the second projection map $(a, t\nu) \mapsto t\nu$ from $\bar{L}$ onto $\bar{T}$ is an idempotent-determined homomorphism. Since $\bar{T}$ is E-unitary, it follows easily that $\bar{L}$ is E-unitary.

Whenever (T, X, Y) is a fully strict L-triple we have analogous results for L_m, the proofs being slightly more complicated.

Theorem 7. *Suppose that (T, X, Y) is a fully strict L-triple. Then the following are equivalent*:

(i) *L_m is a subdirect product of an E-unitary inverse semigroup and a semilattice of groups.*

(ii) *Given $\alpha \in W$, $t \in K_\alpha$, and $a \in Y \cap \Delta\epsilon_\alpha$ such that $ta \in Y$, then $tb = b$ for all $b \leqslant a$.*

Proof. (i)$\Rightarrow$(ii) Suppose that we are given $\alpha \in W$, $t \in K_\alpha$, and $a \in Y \cap \Delta\epsilon_\alpha$ such that $ta \in Y$. Let $\epsilon_\beta = e(a)$, and let $u = t\epsilon_\beta$. Then $\epsilon_\beta \leqslant \epsilon_\alpha$, $u \in G_\beta$, and $ua = t\epsilon_\beta a = ta \in Y$. Denote u^{-1} by v. Then $(a, v) \in L_m$, and it is easily seen that $v \in K_\beta$, so that $v \in \operatorname{Ker} \psi_{\beta\gamma}$ for some $\gamma \leqslant \beta$. Recall that since (T, X, Y) is a fully strict L-triple there exists $c \in Y$ such that $(c, \epsilon_\gamma) \in \mathscr{E}_m$. Now $(a, \epsilon_\beta) \in \mathscr{E}_m$.

Proceeding exactly as in the proof of the first part of Theorem 5 with (a, ϵ_β) in place of (a, ϵ_α), and (a, v) in place of (a, s), we deduce that $(a, \epsilon_\beta)\mu(a, v)$. Let $b \leqslant a$, and let $\epsilon_\delta = e(b)$. Then $\epsilon_\delta \leq \epsilon_\beta$, and $(b, \epsilon_\delta) \in \mathscr{E}_m$. It follows that $(a, \epsilon_\beta)^{-1}(b, \epsilon_\delta)(a, \epsilon_\beta) = (a, v)^{-1}(b, \epsilon_\delta)(a, v)$. The left hand side of the equation equals (b, ϵ_δ), while on the right, $(a, v)^{-1}(b, \epsilon_\delta)(a, v) = (v^{-1}a, v^{-1})(b, \epsilon_\delta v) = (v^{-1}b, \epsilon_\delta)$. Hence $tb = t\epsilon_\beta b = ub = v^{-1}b = b$.

(ii)$\Rightarrow$(i) The proof is almost a replica of that for the second part of Theorem 5.

Corollary. *If (T, X, Y) is a fully strict L-triple, then L_m is a subdirect product of an E-unitary inverse semigroup and a semilattice of groups if and only if L is also.*

Proof. This follows immediately from Theorems 5 and 7.

The analogue of Theorem 6 holds with an entirely similar proof.

Theorem 8. *Suppose that (T, X, Y) is a fully strict L-triple. Then the homomorphism ϕ: $T \to \mathscr{I}_X$ induces a homomorphism $\bar{\phi}$: $\bar{T} \to \mathscr{I}_X$ such that $\phi = \nu\bar{\phi}$ if and only if each $t \in \cup\{K_\alpha | \alpha \in W\}$ is the identity map on Δt. In this case, $(\bar{T}, X, Y)$ becomes a fully strict L-triple under the induced action, and $L_m(T, X, Y)$ is the subdirect product of the E-unitary inverse semigroup $L_m(\bar{T}, X, Y)$ and the semilattice of groups T.*

Given our L-triple (T, X, Y), let $\bar{X}$ be the set of order-ideals A of X such that $A \subseteq g(Y \cap \Delta\epsilon_\alpha)$ for some $g \in G_\alpha$, $\alpha \in W$, the order on $\bar{X}$ being that of inclusion. For each $\alpha \in W$, let $\Delta\epsilon_\beta = \{A \in \bar{X} | A \subseteq \Delta\epsilon_\beta\}$. Following the preamble to (**13**, Theorem 11), $(T, \bar{X}, \bar{X})$ is an L-triple, $g \in G_\beta$ having domain $\Delta\epsilon_\beta$. In fact, as shown there, $L(T, \bar{X}, \bar{X})$ is a strong semilattice of inverse semigroups each of which is a semidirect product of a semilattice and a group, and there is a natural embedding of $L(T, X, Y)$ in $L(T, \bar{X}, \bar{X})$ (see Theorem 4 (ii) and subsequent remarks). Only under the favourable circumstances of Theorems 6 and 8 however, viz. that condition (2) holds, is it the case that $L(T, \bar{X}, \bar{X})$ is also the subdirect product of an E-unitary inverse semigroup and a semilattice of groups. (This follows on applying Theorem 6 to $L(T, \bar{X}, \bar{X})$.)

Note that B^0 is a strong semilattice of E-unitary inverse semigroups which is not the subdirect product of an E-unitary inverse semigroup and a semilattice of groups, B being the bicyclic semigroup (see the remarks after Theorem 3).

We now consider a slightly more complicated example. This example shows, firstly, that the subdirect product of an E-unitary inverse semigroup and a semilattice of groups need not be a strong semilattice of E-unitary inverse semigroups. In the second place, it shows that the condition (1) does not imply the condition (2) (of course, the latter does imply the former), even for a fully strict L-triple (T, X, Y). Thirdly, this example also shows that $L(T, \bar{X}, \bar{X})$ need not be a subdirect product of an E-unitary inverse semigroup and a semilattice of groups even though $L_m(T, X, Y)$ and $L(T, X, Y)$ are.

Example. Let $X = \{a, b, c, d, e, f\}$ where $a = b \wedge c$, $b \vee c = d = e \wedge f$, b and c are incomparable, and e and f are also incomparable. Let $Y = \{a, b, c, d, e\}$, let W be the chain $\{\alpha, \beta\}$ with $\alpha > \beta$, and let T be the semilattice W of groups $G =$ the Klein 4-group and $G_\beta =$ the cyclic 2-group, the linking map from G_α to G_β being surjective. Then, in an obvious way, (T, X, Y) is a fully strict L-triple, where T has a faithful action on X with $\Delta\epsilon_\alpha = X$ and $\Delta\epsilon_\beta = \{a, b, c\}$. Let $t \in T$ be the element which permutes e and f and leaves the rest of X invariant. Then $K_\alpha = \{\epsilon_\alpha, t\}$, and $K_\beta = \{\epsilon_\beta\}$.

It is clear, therefore, that condition (1) is satisfied, but not condition (2), for the triple (T, X, Y). However, not even condition (1) is satisfied for the triple $(T, \bar{X}, \bar{X})$. For if we consider element $Y \in \Delta\epsilon_\alpha$, $tY \neq Y$ (of course, $Y \in \bar{X}$ and $tY \in \bar{X}$).

It now remains to show that $L_m(T, X, Y)$ is not a strong semilattice of E-unitary inverse semigroups. Suppose that L_m is the semilattice Λ of E-unitary inverse semigroups S_λ, $\lambda \in \Lambda$, where $(b, g) \in S_\mu$ say, g being the non-identity element of G_β. Then $(c, g) = (b, g)^{-1} \in S_\mu$ so that $(c, \epsilon_\beta) = (c, g)(c, g)^{-1} \in S_\mu$ and $(b, \epsilon_\beta) = (b, g)(b, g)^{-1} \in S_\mu$. Hence $(a, \epsilon_\beta) = (b, \epsilon_\beta)(c, \epsilon_\beta) \in S_\mu$. It follows that $\{a, b, c\} \times G_\beta \subseteq S_\mu$. A similar argument establishes that $\{d\} \times G_\alpha \subseteq S_\lambda$ for some $\lambda \geq \mu$. In fact $\lambda > \mu$, since $(a, \epsilon_\beta)(d, t) = (a, \epsilon_\beta)$ where (d, t) is non-idempotent. Let s be the element of G_α which permutes b and c and leaves the rest of X invariant. The only remaining elements of L_m are then (e, ϵ_α) and (e, s). Together these form the group of units U of L_m. Thus $U \subseteq S_{\lambda'}$, where $\lambda' \geq \lambda$.

Note that $\{a, b, c\} \times \epsilon_\beta$ is not a p-ideal of $\mathscr{E}_m$, see (**13**), since $(b, \epsilon_\beta) \leq (d, \epsilon_\alpha)$ and $(c, \epsilon_\beta) \leq (d, \epsilon_\alpha)$, and desired result follows from (**13**, Theorem 8) and the above remarks.

This example is a variant on (**13**, Example 5.2).

REFERENCES

(**1**) J. E. MILLS, Certain congruences on orthodox semigroups, *Pacific J. Math.* **64** (1976), 217–226.

(**2**) A. H. CLIFFORD and G. B. PRESTON, *The Algebraic Theory of Semigroups*, (Math. Surveys No. 7, Amer. Math. Soc., Providence, Vol. 1, 1961 and Vol. 2, 1967).

(**3**) A. COUDRON, Sur les extensions des demigroupes réciproques, *Bull. Soc. Roy. Sci. Liège* **37** (1968), 409–419.

(**4**) H. D. d'ALARCAO, Idempotent-separating extensions of inverse semigroups, *J. Austral. Math. Soc.* **9** (1969), 211–217.

(**5**) D. G. GREEN, Extensions of a semilattice by an inverse semigroup, *Bull. Austral. Math. Soc.* **9** (1973), 21–31.

(**6**) D. W. HARDY and Y. TIRASUPA, Semilattices of proper inverse semigroups, *Semigroup Forum* **13** (1976), 29–36.

(**7**) J. M. HOWIE, The maximum idempotent-separating congruence on an inverse semigroup, *Proc. Edinburgh Math. Soc.* (2) **14** (1964), 71–79.

(**8**) H. LAUSCH, Cohomology of inverse semigroups, *J. Algebra* **35** (1975), 273–303.

(**9**) D. B. MCALISTER, Some covering and embedding theorems for inverse semigroups, *J. Austral. Math. Soc.*, **22** (1976), 188–211.

(**10**) W. D. MUNN, A class of irreducible matrix representations of an arbitrary inverse semigroup, *Proc. Glasgow Math. Assoc.* **5** (1961), 41–48.

(**11**) W. D. MUNN and N. R. REILLY, Congruences on a bisimple ω-semigroup, *Proc. Glasgow Math. Assoc.* **7** (1966), 184–192.

(**12**) L. O'CARROLL, Reduced inverse and partially ordered semigroups, *J. London Math. Soc.* (2) **9** (1974), 293–301.

(13) L. O'CARROLL, Strongly E-reflexive inverse semigroups, *Proc. Edinburgh Math. Soc.*, (2) **20** (1977), 339–354.

(14 M. PETRICH, *Introduction to Semigroups*, (Merrill, Columbus, Ohio, 1973).

(15) T. TAMURA, The theory of construction of finite semigroups I, *Osaka Math. J.* **8** (1956), 243–261.

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