Infinitesimal Analysis of an arc in n-space.

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1. EXTENSION OF SERRET-FRENET FORMULAE.

We may develop the idea of principal lines at any point on a curve of $(n-1)$-triple curvature geometrically in the following way:

Two consecutive points on the curve determine the tangent, three consecutive points the osculating points, four consecutive points the osculating 3-space and so on, at any point on the curve. At the same point we have an $(n-1)$-space perpendicular to the tangent and we shall call this space the first normal space at the point; the intersection of the first normal space with the osculating plane is a line[1] which we shall name as the first normal at the point. Similarly all lines perpendicular to the osculating plane determine an $(n-2)$-space, the second normal space at the point, and the intersection of this space with the osculating 3-space is the second normal at the point. Proceeding thus we have lastly the $(n-1)$th normal which is perpendicular to the osculating $(n-1)$-space at the point. We thus see that the rth normal lies in the osculating $(r+1)$-space and is perpendicular to r consecutive tangents. These $n-1$ normals with the tangent constitute the n principal lines at the point which are mutually orthogonal.

Secondly, let us define the positive directions of these lines. Let the coordinates of any point on the curve be given as functions of a variable parameter:

$$x_1 = f_1(s),\ x_2 = f_2(s),\ \ldots\ldots,\ x_n = f_n(s),$$

where s denotes the length of an arc of the curve measured from some fixed point on it. We assume, as in the ordinary geometry, that the positive direction of currency along the curve to be that as given by increasing the values of s; we shall assume, moreover, the functions $f_1, f_2, \ldots, f_n$ with their derivatives up to the required order to be regular, continuous and finite throughout the range of the par-

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[1] Cayley :—A Memoir on Abstract Geometry : *Phil. Trans. Royal Soc., London*, **160** (1870) :—"an $(n-r)$-fold linear relation determines an r-omal."

ameter considered. Then the positive direction of the tangent is taken to be that in which s increases; the positive direction of the rth normal ($r = 2, 3, \ldots n-1$) is from the centre of the osculating r-spheric to the centre of the osculating $(r+1)$-spheric; (since the osculating r-space intersects the osculating $(r+1)$-spheric in the osculating r-spheric,[1] therefore the line joining the centres of the two spherics is perpendicular to this r-space and is, therefore, parallel to the rth normal); and the positive direction of the $(n-1)$th normal is taken to be that in which this with the positive directions of the other principal lines can be brought into coincidence, by orientation in space, with the positive directions of the coordinate axes.

Lastly, let us define curvatures and find the inclinations of two sets of principal lines at two consecutive points P, Q on the curve. Two consecutive osculating r-spaces lie in the osculating $(r+1)$-space and the angle between these two r-spaces will be taken as the angle between their normals in the same $(r+1)$-space. Suppose, then, $d\psi_1, d\psi_2, \ldots . d\psi_{n-1}$ are the angles between two consecutive tangents, two consecutive osculating planes, ..., two consecutive osculating $(n-1)$-spaces, and the successive curvatures are defined as:

$$\frac{1}{\rho_1} = \frac{d\psi_1}{ds}, \quad \frac{1}{\rho_2} = \frac{d\psi_2}{ds}, \quad \ldots\ldots, \quad \frac{1}{\rho_{n-1}} = \frac{d\psi_{n-1}}{ds}{}^{2}.$$

This is an extension of the ordinary idea of curvature, viz., the rate of deflection of an osculating space.

The normal to the osculating r-space at P in the osculating $(r+1)$-space at the same point is the rth normal at the point, and the rth normal space at Q contains all lines perpendicular to the osculating r-space at this point; the intersection of the osculating $(r+1)$-space at P with the rth normal space at Q is a line m_r say, at Q. Hence the angle between the rth normal at P and m_r measures the angle between the consecutive osculating r-spaces. Again the osculating r-space at Q, the $(r-1)$th normal at Q, the rth normal at P and m_r all lie in the osculating $(r+1)$-space at P, and in this space the latter three lines are perpendicular to the osculating $(r-1)$-space at Q. Hence, since in an $(r+1)$-space there can not be more than two independent perpendiculars to an $(r-1)$-space, the three lines lie in a plane. Therefore, remembering the positive directions as defined

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[1] Veronese:—"Fondamenti di Geometria etc.", translated into German by Adolf Schepp, "Grundzüge der Geometrie etc.", §174, Stz. III.

[2] Pirondini:—"Sulle linee a tripla curvatura etc.", *Giorn. di Battaglini* (1890).

before, the angle between the rth normal at P and $(r-1)$th normal at Q is $\frac{\pi}{2}-d\psi_r$. In a similar way it may be seen that the angle between the $(r-1)$th normal at P and the rth normal at Q is $\frac{\pi}{2}-d\psi_r$. Thus the direction-cosines of the principal lines at Q referred to those at P may be given by the following table:

tangent	1	$d\psi_1$	0	0	0	. .	0	0	0	0	0
1st normal	$-d\psi_1$	1	$d\psi_2$	0	0	. .	0	0	0	0	0
2nd normal	0	$-d\psi_2$	1	$d\psi_3$	0	. .	0	0	0	0	0
. . . . . .											
$(n-3)$th normal	0	0	0	0	0	. .	0	$-d\psi_{n-3}$	1	$d\psi_{n-2}$	0
$(n-2)$th normal	0	0	0	0	0	. .	0	0	$-d\psi_{n-2}$	1	$d\psi_{n-1}$
$(n-1)$th normal	0	0	0	0	0	. .	0	0	0	$-d\psi_{n-1}$	$1.$

Therefore, if l_{ij} $(i=1, 2, \ldots n;\ j=1, 2, \ldots n)$ be the direction-cosines of the principal lines at P referred to the coordinate axes, we have

$$\begin{aligned}
l_{1i} &+ dl_{1i} &&= l_{1i}+d\psi_1 l_{2i}\\
l_{2i} &+ dl_{2i} &&= -d\psi_1 l_{1i}+l_{2i}+d\psi_2 l_{3i}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
l_{(n-1)i} &+ dl_{(n+1)i} &&= -d\psi_{n-2}\, l_{(n-2)i}+l_{(n-1)i}+d\psi_{n-1}\, l_{ni}\\
l_{ni} &+ dl_{ni} &&= -d\psi_{n-1}\, l_{(n-1)i}+l_{ni}.
\end{aligned}$$

Accordingly,

$$\left.\begin{aligned}
&\frac{dl_{1i}}{ds}=\frac{l_{2i}}{\rho_1},\quad \frac{dl_{2i}}{ds}=\frac{l_{3i}}{\rho_2}-\frac{l_{1i}}{\rho_1},\ \ldots\ldots\ldots\ldots,\\
&\frac{dl_{(n-1)i}}{ds}=\frac{l_{ni}}{\rho_{n-1}}-\frac{l_{(n-2)i}}{\rho_{n-2}},\quad \frac{dl_{ni}}{ds}=-\frac{l_{(n-1)i}}{\rho_{n-1}}.^{1}
\end{aligned}\right\} \qquad (1)$$

$$(i=1, 2, \ldots n)$$

2. Radii of Curvature.

It will be advantageous to employ the following notations[2]:

Let $D^r \equiv \dfrac{d^r}{ds^r}$, where r is any positive integer,

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[1] These formulae have been deduced for curves in four dimensional space, by Prof. J. G. Hardie, in the *American Journal of Math.*, **24**, and also, from a different standpoint, by Prof. S. D. Mookerjee in the *Bulletin of the Calcutta Math. Soc.*, **1**, 1909.

[2] These notations have been introduced by Prof. Mookerjee in paper I. on "Parametric Coefficients, etc." in the above volume, and in a treatise published by the Calcutta University.

and $$\Sigma D^{m_1} x_r D^{m_2} x_r \equiv (m_1 m_2),$$

$$\left\{ \Sigma \begin{vmatrix} D^{m_1} x_{r_1} & D^{m_1} x_{r_2} \\ D^{m_2} x_{r_1} & D^{m_2} x_{r_2} \end{vmatrix}^2 \right\}^{\frac{1}{2}} \equiv [m_1 m_2],$$

$$\left\{ \Sigma \begin{vmatrix} D^{m_1} x_{r_1} & D^{m_1} x_{r_2} & D^{m_1} x_{r_3} \\ D^{m_2} x_{r_1} & D^{m_2} x_{r_2} & D^{m_2} x_{r_3} \\ D^{m_3} x_{r_1} & D^{m_3} x_{r_2} & D^{m_3} x_{r_3} \end{vmatrix}^2 \right\}^{\frac{1}{2}} \equiv [m_1 m_2 m_3]; \text{ and so on.}$$

Further, let

$$\Sigma \begin{vmatrix} D^{m_1} x_{r_1} & D^{m_1} x_{r_2} & \dots & D^{m_1} x_{r_p} \\ D^{m_2} x_{r_1} & D^{m_2} x_{r_2} & \dots & D^{m_2} x_{r_p} \\ \dots & \dots & \dots & \dots \\ D^{m_p} x_{r_1} & D^{m_p} x_{r_2} & \dots & D^{m_p} x_{r_p} \end{vmatrix} \begin{vmatrix} D^{n_1} x_{r_1} & D^{n_1} x_{r_2} & \dots & D^{n_1} x_{r_p} \\ D^{n_2} x_{r_1} & D^{n_2} x_{r_2} & \dots & D^{n_2} x_{r_p} \\ \dots & \dots & \dots & \dots \\ D^{n_p} x_{r_1} & D^{n_p} x_{r_2} & \dots & D^{n_p} x_{r_p} \end{vmatrix}$$

$$\equiv [m_1 m_2 \dots m_p \mid n_1 n_2 \dots n_p].$$

The following relations may be seen to exist among these quantities:

$$[m_1 m_2]^2 = (m_1 m_1)(m_2 m_2) - (m_1 m_2)^2$$

$$[m_1 m_2 m_3]^2 (m_1 m_2) = [m_1 m_2]^2 [m_1 m_3]^2 - [m_1 m_2 \mid m_1 m_3]^2.$$

$$[m_1 m_2 m_3 m_4]^2 [m_1 m_2]^2 = [m_1 m_2 m_3]^2 [m_1 m_2 m_4]^2 - [m_1 m_2 m_3 \mid m_1 m_2 m_4]^2; \text{ and so on.}$$

Also,

$$[m_1 m_2, \dots m_r \mid n_1 n_2 \dots n_r] = \begin{vmatrix} (m_1 n_1) & (m_1 n_2) & \dots & (m_1 n_r) \\ (m_2 n_1) & (m_2 n_2) & \dots & (m_2 n_r) \\ \dots & \dots & \dots & \dots \\ (m_r n_1) & (m_r n_2) & \dots & (m_r n_r) \end{vmatrix}.$$

If $x_1, x_2, \dots x_n$ be the coordinates of any point on the curve, the content V_p of the simplex of the pth order, formed by the given point and p points consecutive to it, is given by

$$V_p{}^2 = \frac{1}{(p!)^2} \Sigma \begin{vmatrix} dx_1 & dx_2 & \dots & dx_p \\ d^2 x_1 & d^2 x_2 & \dots & d^2 x_p \\ \dots & \dots & \dots & \dots \\ d^p x_1 & d^p x_2 & \dots & d^p x_p \end{vmatrix}^2 = \frac{1}{(p!)^2} [1\, 2 \dots p]^2 ds^{p(p+1)}$$

or, $$[1\, 2 \dots p] = \frac{p!\, V_p}{V_1^{\frac{p(p+1)}{2}}}.$$

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Now, from the formulae of the last article, we have

$$D^2 x_i = \frac{l_{2i}}{\rho_1}, \quad D^3 x_i = -\frac{l_{1i}}{\rho_1^2} - \frac{l_{2i}}{\rho_1^2}\rho'_1 + \frac{l_{3i}}{\rho_1 \rho_2},$$

$$D^4 x_i = \frac{3l_{1i}}{\rho_1^3}\rho'_1 - \frac{l_{2i}}{\rho_1}\left(\frac{1-2\rho'^2_1}{\rho_1^2} + \frac{\rho_1''}{\rho_1} + \frac{1}{\rho_2^2}\right) - \frac{l_{3i}}{\rho_1\rho_2}\left(\frac{2\rho'_1}{\rho_1} + \frac{\rho'_2}{\rho_2}\right) + \frac{l_{4i}}{\rho_1\rho_2\rho_3}$$

and so on, when the accents will always indicate differentiation with respect to s.

Substituting these values, it may be seen that

$$\frac{1}{\rho_1} = [1\ 2], \qquad \frac{1}{\rho_1^2\rho_2} = [1\ 2\ 3], \qquad \frac{1}{\rho_1^3\rho_2^2\rho_1} = [1\ 2\ 3\ 4], \ldots,$$

$$\frac{1}{\rho_1^{n-1}\rho_2^{n-2}\ldots\rho_{n-2}^2\,\rho^{n-1}} = [1\ 2\ldots n].$$

Thus,
$$\rho_1 = \frac{1}{[1\ 2]}, \qquad \rho_2 = \frac{[1\ 2]^2}{[1\ 2\ 3]}, \qquad \rho_3 = \frac{[1\ 2\ 3]^2}{[1\ 2][1\ 2\ 3\ 4]}, \ldots,$$

$$\rho_{n-1} = \frac{[1\ 2\ldots(n-1)]^2}{[1\ 2\ldots(n-2)][1\ 2\ldots n]}.$$

Therefore, $\rho_1\rho_2\ldots\rho_{n-1} = \dfrac{[1\ 2\ldots n-1]}{[1\ 2\ldots n]}$.

It may also be seen that

$$d\psi_1 = \frac{2!\ V_2}{V_1^2}, \qquad d\psi_2 = \frac{V_1 . 3!\ V_3}{(2!\ V_2)^2}, \qquad d\psi_3 = \frac{2!\ V_2 . 4!\ V_4}{(3!\ V_3)^2},$$

and generally

$$d\psi_{n-1} = \frac{(n-2)!\ V_{n-2} . n!\ V_n}{\{(n-1)!\ V_{n-1}\}^2}.$$

Therefore, $d\psi_1\, d\psi_2 \ldots d\psi_{n-1} = \dfrac{n!\ V_n}{(n-1)!\ V_1\, V_{n-1}}$.

3. Spherics of Closest Contact.

Let the equation to the osculating n-spheric at a point on the curve be $\Sigma(x_i - a_i)^2 = R^2$. This spheric, of $n-1$ dimensional boundary, passes through $n+1$ consecutive points on the curve. Differentiating the equation n times:

$$\Sigma\, l_{1i}(x_i - a_i) = 0 \qquad (1)$$

$$\Sigma\, l_{2i}(x_i - a_i) = -\rho_1 \qquad (2)$$

$$\Sigma\, l_{3i}(x_i - a_i) = -\rho_2\rho'_1 \qquad (3)$$

$$\Sigma\, l_{4i}(x_i - a_i) = -\rho_3\left((\rho_2\rho'_1)' + \frac{\rho_1}{\rho_2}\right) \qquad (4)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Sigma\, lx_i(x_i - a_i) = -\rho_{n-1}((\rho_{n-2}((\rho_{n-3}\ldots((\rho_3((\rho_2\rho'_1)' + \rho_1/\rho_2))' + \ldots)))\ldots) \quad (n).$$

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There are, lastly, $\frac{n}{2} - 1$ or $\frac{n-3}{2}$ brackets at the end according as n is even or odd. Or, if we denote the expressions on the right-hand sides of (1) to (n) by $a_1, a_2, \ldots a_n$,

$$a_n = \rho_{n-1}\left(a'_{n-1} + \frac{a_{n-2}}{\rho_{n-2}}\right) = \rho_{n-1}\left(\left(\rho_{n-2}\left(a'_{n-2} + \frac{a_{n-3}}{\rho_{n-3}}\right)\right)' + \frac{a_{n-2}}{\rho_{n-2}}\right) = \ldots$$

$$= \rho_{n-1}\left(\left(\rho_{n-2} : \ldots \left(\left(\rho_3\left(\left(\rho_2\rho'_1\right)' + \frac{a_2}{\rho_2}\right)\right)' + \frac{a_3}{\rho_3}\right)\right)' + \ldots \frac{a_{n-3}}{\rho_{n-3}}\right)\right)' + \frac{a_{n-2}}{\rho_{n-2}}\right).$$

Squaring and adding, $R^2 = \Sigma a_i{}^2$;

also, $a_i = x_i - \sum_{j=2}^{j=n} l_{ji} a_j$.

Thus, for the osculating r-spheric $a_{r+1} = a_{r+2} = \ldots = a_n = 0$, since its centre lies in the osculating r-space. This shews that *the centre of the osculating* $(r+1)$*-spheric lies on a line drawn through the centre of the osculating* r*-spheric parallel to the* r*th normal, and the length of the line joining the centres of the two spherics is given, in magnitude, by* $-a_{r+1}$.

(i) Let us denote the radius of the r-spheric of closest contact by R_r. We have

$$R_2{}^2 = a_2{}^2 \text{ and } a_3 = -\frac{dR_2}{d\psi_2}.$$

$$R_3{}^2 = a_2{}^2 + a_3{}^2,\ R_3R'_3 = a_2a'_2 + a_3a'_3 = \frac{a_3a_4}{\rho_3},$$

$$\text{whence} \quad a_4 = \frac{R_3}{a_3} \cdot \frac{dR_3}{d\psi_3} = -R_3\frac{dR_3}{d\psi_3} \cdot \frac{d\psi_2}{dR_2}.$$

Again, $R_4{}^2 = a_2{}^2 + a_3{}^2 + a_4{}^2, \quad R_4R'_4 = a_2a'_2 + a_3a'_3 + a_4a'_4 = \frac{a_4a_5}{\rho_4}$,

$$\text{whence} \qquad a_5 = \frac{R_4}{a_4} \cdot \frac{dR_4}{d\psi_4} = -R_4\frac{dR_4}{d\psi_4} \cdot \frac{d\psi_3}{dR_3} \cdot \frac{dR_2}{d\psi_2}.$$

Similarly $a_6 = -R_5\frac{dR_5}{d\psi_5} \cdot \frac{d\psi_4}{dR_4} \cdot \frac{dR_3}{d\psi_3} \cdot \frac{d\psi_2}{dR_2}$; and so on.

Hence, $R_3{}^2 = R_2{}^2 + \left(\frac{dR}{d\psi_2}\right)^2$

$$R_4{}^2 = R_3{}^2 + \left\{R_2{}^2 + \left(\frac{dR_2}{d\psi_2}\right)^2\right\}\left(\frac{dR_3}{d\psi_3} \cdot \frac{d\psi_2}{dR_2}\right)^2$$

$$= R_2{}^2 + \left(\frac{dR_2}{d\psi_2}\right)^2 + \left(\frac{dR_3}{d\psi_3}\right)^2 + \left(R_2\frac{dR_3}{d\psi_3} \cdot \frac{d\psi_2}{dR_2}\right)^2, \text{ and so on.}$$

We may also express these radii in terms of the quantities introduced in the last article. We have $(1\ 1) = 1$, differentiating $(1\ 2) = 0$, $(2\ 2) = -(1\ 3) = [1\ 2]^2$. Also

$$3\,(2\ 3) = -(1\ 4), \quad [1\ 2\ 3]^2 = [2\ 3]^2 - [1\ 2]^6.$$

$$R_2 = \frac{1}{[1\ 2]}, \quad R'_2 = \frac{-[1\ 2 \mid 1\ 3]}{[1\ 2]^3}, \quad \frac{dR_2}{d\psi_2} = \frac{-[1\ 2 \mid 1\ 3]}{[1\ 2]\,[1\ 2\ 3]}.$$

Hence, $R_3{}^2 = \dfrac{1}{[1\ 2]^2} + \dfrac{[1\ 2 \mid 1\ 3]^2}{[1\ 2]^2[1\ 2\ 3]^2} = \dfrac{[1\ 3]^2}{[1\ 2\ 3]^2}$.

Similarly it may be seen that

$$\frac{dR_3}{d\psi_3} = \frac{(2\ 3)\,\{[1\ 2]^2[1\ 2\ 3]^2 - [1\ 2\ 3 \mid 1\ 3\ 4]\}}{[1\ 2]\,[1\ 3]\,[1\ 2\ 3]\,[1\ 2\ 3\ 4]}.$$

Thus $\quad R_4{}^2 = \dfrac{[1\ 3\ 4]^2 + [1\ 2]^4[1\ 2\ 3]^2 - 2[1\ 2]^2[1\ 2\ 3 \mid 1\ 3\ 4]}{[1\ 2\ 3\ 4]^2}$; and so on.

(ii) Let ds_r be the differential of the arc of the locus of the centres of consecutive R_r's.

Then $\left(\dfrac{ds_2}{d\psi_2}\right)^2 = R_3{}^2$, as in the ordinary geometry.

And, for the centre of the osculating r-spheric

$$\alpha_i = x_i - (l_{2i}\, a_2 + l_{3i}\, a_3 + \dots + l_{ri}\, a_r).$$

Therefore $\quad \Sigma\, \alpha'_i{}^2 = \dfrac{a^2{}_r + a^2{}_{r+1}}{\rho_r{}^2}$, by differentiating α_i, using § 1 (1) and the relations $a'_r = \dfrac{a_{r+1}}{\rho_r} - \dfrac{a_{r-1}}{\rho_{r-1}}$.

Hence $\quad \left(\dfrac{ds_r}{d\psi_r}\right)^2 = R^2{}_{r+1} - R^2{}_{r-1}$, for $r = 3, 4, \dots\ n-1$.

Lastly, for the centre of the osculating n-spheric,

$$\alpha'_i = -l_{ni}\left(a'_n + \frac{a_{n-1}}{\rho_{n-1}}\right)$$

$$= -\frac{l_{ni}}{a_n}\left\{a_n\, a'_n + \left(a'_{n-1} + \frac{a_{n-2}}{\rho_{n-2}}\right) a_{n-1}\right\}$$

$$= -\frac{l_{ni}}{a_n}\left\{a_n\, a'_n + a_{n-1}\, a'_{n-1} + \left(a'_{n-2} + \frac{a_{n-3}}{\rho_{n-3}}\right) a_{n-2}\right\}$$

$$= \dots\dots$$

$$= -\frac{l_{ni}}{a_n}\,(a_n\, a'_n + a_{n-1}\, a'_{n-1} + \dots + a_3\, a'_3 + a_2\, a'_2)$$

$$= -l_{ni}\,\frac{R_n\, R'_n}{a_n}.$$

Hence, $ds_n{}^2 = \dfrac{(R_n\, dR_n)^2}{R_n{}^2 - R^2_{n-1}}$.

(iii) Let $d\epsilon_r$ be the angle between two consecutive R_r's.

Then $(R_2\, d\epsilon_2)^2 = ds^2 + \left(\dfrac{R_2}{R_3}\right)^2 ds_2{}^2$, as in the ordinary geometry.

The direction-cosines m_{ri} of R_r at the point is given by $-\sum\limits_{j=1}^{j=r} \dfrac{l_{ji}\, a_j}{R_r}$.

Or, $m'_{ri} = -\dfrac{1}{R_r}\left(l_{1i} + \dfrac{l_{(r+1)\,i}\, a_r + l_{ri}\, a_{r+1}}{\rho_r}\right) + \dfrac{R'_r}{R_r{}^2} \cdot \sum\limits_{j=2}^{j=r} l_{ji}\, a_j$,

for $r = 3, 4, \ldots .\ n-1$;

and $\quad = -\dfrac{1}{R_n}\left(l_{1i} + \dfrac{l_{ni}\, R_n R'_n}{a_n}\right) + \dfrac{R'_n}{R_n{}^2} \sum\limits_{j=2}^{j=n} l_{ji}\, a_j$, for $r = n$.

Therefore, $\Sigma\, m'_{ri}{}^2 = \dfrac{1 + s'_r{}^2 - R'_r{}^2}{R_r{}^2}$.

Hence[1] $(R_r\, d\epsilon_r)^2 = ds^2 + ds^2{}_r - dR_r{}^2$ for $r = 3, 4 \ldots .\ n$.

4. Osculating Cones.

As we have considered osculating spherics of different dimensions determined by consecutive points of the system, we may consider osculating right cones of different dimensions determined by consecutive osculating spaces of the system. The right cone of the nth order[2] having as its vertex the point of intersection of n consecutive osculating $(n-1)$-spaces and which is generated by these spaces will evidently osculate the given curve. We may, otherwise, imagine that an n-spheric is described having as its centre the point of intersection of these osculating spaces; these spaces will intersect the spheric in a spherical simplex of the nth order,[3] and we may imagine

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[1] A number of formulae of similar kind for curves of double curvature are given in a memoir by M. Saint-Venant, *Journal de l'Ecole Polytechnique*, Cahier XXX.

[2] Veronese. *loc. cit.*, secs. 179, 180.

[3] A remarkable treatment of the subject is to be found in Theorie der Vielfachen Kontinuitat by L. Schläfli, where we have the following definition of a spherical simplex: "Das $(n-1)$-fache höhere Kontinuum, welches alle auf der Polysphäre befindlichen Lösungen enthält, ... heisst totales sphärisches Kontinuum; ein Stück desselben, welches von $(n-1)$-fachen durchs Centrum gehenden linearen Kontinuen begrenzt wird, sphärisches Polyschem, und in Besondern Plagioschem, wenn die Zahl der begrenzenden Kontinuen n ist," § 19.

an $(n-1)$-spheric inscribed within the simplex. Then the cone under consideration has its vertex at the centre of the n-spheric and stands on the $(n-1)$-inscribed spheric.

Since the $(n-1)$th normal is normal to the osculating $(n-1)$-space, the axis of the cone will be equally inclined to n consecutive $(n-1)$th normals. If λ_i be the direction-cosines of the axis and ϕ the vertical angle of the cone, we shall have

$$\Sigma \lambda_i l_{ni} = \sin \phi. \tag{1}$$

Differentiating $n-1$ times

$$\Sigma \lambda_i l_{(n-1)i} = 0 \tag{2}$$

$$\Sigma \lambda_i l_{(n-2)i} = \frac{\rho_{n-2}}{\rho_{n-1}} \sin \phi \tag{3}$$

$$\Sigma \lambda_i l_{(n-3)i} = -\rho_{n-3} \left(\frac{\rho_{n-2}}{\rho_{n-1}}\right)' . \sin \phi \tag{4}$$

$$\Sigma \lambda_i l_{(n-4)i} = \rho_{n-4} \left(\left(\rho_{n-3}\left(\frac{\rho_{n-2}}{\rho_{n-1}}\right)'\right)' + \frac{1}{\rho_{n-3}} . \frac{\rho_{n-2}}{\rho_{n-1}}\right) \sin \phi, \tag{5}$$

and so on.

If we denote the coefficients of $\sin \phi$ on the right-hand side expressions of (1) to (n) by $b_1, b_2, \ldots . b_n$ we shall have

$$b_n = \rho_1\left(-b'_{n-1} + \frac{b_{n-2}}{\rho_2}\right) = \rho_1\left(-\rho_2\left(-b'_{n-2} + \frac{b_{n-3}}{\rho_3}\right)\right)' + \frac{b_{n-2}}{\rho_2}\Big) = \ldots .$$

$$= \rho_1\left(-\rho_2\left(-\rho_3\left(\ldots\left(-\rho_{n-3}\left(-\left(\frac{\rho_{n-2}}{\rho_{n-1}}\right)' + \frac{b_2}{\rho_{n-2}}\right)\right)' + \frac{b_3}{\rho_{n-3}}\right)\right)' + \ldots + \frac{b_{n-2}}{\rho_2}\right).$$

Squaring and adding, $\cot^2 \phi = b_3^2 + b_4^2 + \ldots . + b_n^2$.

Also $\lambda_i = (l_{ni} + l_{(n-2\,i} b_3 + \ldots . l_{1i} b_n) \sin \phi$.

(i) Let ϕ_r be the vertical angle of the osculating right cone of the rth order for a curve of $(r-1)$ tuple curvature.[1]

Then it may be seen, on reduction, that

$$\cot^2 \phi_3 = \left(\frac{\rho_1}{\rho_2}\right)^2, \quad \cot^2 \phi_4 = \left(\frac{\rho_2}{\rho_3}\right)^2 + \left\{\rho_1\left(\frac{\rho_2}{\rho_3}\right)'\right\}^2, \text{ and so on.}^2 \tag{6}$$

It is to be understood, however, that in calculating ϕ_6, for example, for a curve given by $x_1 = f_1(s)$, $x_2 = f_2(s)$, $\ldots . x_6 = f_6(s)$, ϕ_3 is to be determined from $x_1 = f_1(s)$, $x_2 = f_2(s)$, $x_3 = f_3(s)$.

[1] Since an osculating right cone of the rth order $(3 \leq r \leq n)$ is determined by r consecutive osculating $(r-1)$-spaces, $2r-1$ consecutive points on the curve must lie in an r-space, and so we should regard the curve as of $(r-1)$-tuple curvature.

[2] It will be seen that the expression for $\cot^2 \phi_r$ will contain $\cot^2 \phi_3$, $\cot^2 \phi_4$, $\ldots$ $\cot^2 \phi_{r-2}$; and if $\cot \phi_{r-1}$ is a function of $\rho_{r-2}, \rho_{r-3}, \ldots \rho_1$, $\cot \phi_{r-3}, \ldots \cot \phi_3$, then $\cot \phi_r$ will contain the same function of $\rho_{r-1}, \rho_{r-2}, \ldots \rho_2$, $\cot \phi_{r-2}, \ldots \cot \phi_4$.

(ii) Let $d\eta_r$ be the angle between the axes of two consecutive osculating cones of the rth order for a curve of $(r-1)$-tuple curvature.

The direction-cosines p_{ri} of the axis of the osculating cone at the point are given by $\sin\phi_r(l_{ri}+l_{(r-2)i}b_3+l_{(r-3)i}b_4+\ldots+l_{1i}b_r)$,

where $b_3=\dfrac{\rho_{r-2}}{\rho_{r-1}}$, $b_4=-\rho_{r-3}\left(\dfrac{\rho_{r-2}}{\rho_{r-1}}\right)'$, ...

On differentiating p_{ri}, simplifying by §1(1) and using (6) we have

$$\Sigma p'^{\,2}_{ri}=\frac{\cot^2\phi_r\,.\,\operatorname{cosec}^2\phi_r}{b_r^{\,2}}\phi'^{\,2}_r+\cos^2\phi_r\operatorname{cosec}^2\phi_r\,.\,\phi'^{\,2}_r-2\cos^2\phi_r\operatorname{cosec}^2\phi_r\,.\,\phi'^{\,2}_r$$

$$=\left(\frac{\operatorname{cosec}^2\phi_r}{b_r^{\,2}}-1\right)\cot^2\phi_r\,.\,\phi'^{\,2}_r.$$

Hence, $d\eta_r^{\,2}=\left(\dfrac{\operatorname{cosec}^2\phi_r}{b_r^{\,2}}-1\right)\cot^2\phi_r\,d\phi_r^{\,2}$.

Thus, $d\eta_3^{\,2}=d\phi_3^{\,2}$; $d\eta_4^{\,2}=\left[\dfrac{\operatorname{cosec}^2\phi_4}{\left\{\rho_1\left(\dfrac{\rho_2}{\rho_3}\right)'\right\}^2}-1\right]\cot^2\phi_4\,d\phi_4^{\,2}$; and so on.

If the two r-spaces containing the two consecutive osculating cones lie in an $(r+1)$-space, *i.e.* if the curve be of r tuple curvature, we shall have, since $l'_{ri}=\dfrac{l_{(r+1)i}}{\rho_r}-\dfrac{l_{(r-1)i}}{\rho_{r-1}}$,

$$d\eta_r^{\,2}=\sin^2\phi_r\,d\psi_r^{\,2}+\left(\frac{\operatorname{cosec}^2\phi_r}{b_r^{\,2}}-1\right)\cot^2\phi_r\,d\phi_r^{\,2},\text{ for } r=3,4,..n-1.$$

Corollary. Let the curvatures of a curve be in constant ratios,

$$\rho_1:\rho_2:\rho_3:\ldots:\rho_{n-1}=c_1:c_2:c_3:\ldots:c_{n-1}.$$

It follows at once that ϕ_n is constant and consequently $d\eta_n=0$. Hence, along the curve the axis of the osculating cone has a constant direction and the envelope of this direction is a cylinder of the nth order[1].

By the formulae

$$\frac{dl_{1i}}{ds}+\frac{l_{2i}}{\rho_1},\quad \frac{dl_{2i}}{ds}=\frac{c_1}{\rho_1}\left(\frac{l_{3i}}{c_2}-\frac{l_{1i}}{c_1}\right),\ \ldots,$$

$$\frac{dl_{(n-1)i}}{ds}=\frac{c_1}{\rho_1}\left(\frac{l_{ni}}{c_{n-1}}-\frac{l_{(n-2)i}}{c_{n-2}}\right),\quad \frac{dl_{ni}}{ds}=-\frac{c_1}{\rho_1}\frac{l_{(n-1)i}}{c_{n-1}}.$$

If ρ_1 be an arbitrary function of s, then, for the range of variation of this function, we have a family of curves intrinsically distinct from one another.

[1] Veronese, *loc. cit.*, sec. 180.

(i) If n be odd, it may be seen that

$$l_{1i}+\frac{c_2}{c_1}l_{3i}+\frac{c_2\,c_4}{c_1\,c_3}l_{5i}+\ldots.+\frac{c_2\,c_4\ldots.c_{n-1}}{c_1\,c_3\ldots.c_{n-2}}l_{ni}=k_i\,,$$

where k's are constants.

Therefore $\Sigma k_i\,l_{1i}=1$, $\Sigma k_i\,l_{2i}=0$, $\Sigma k_i\,l_{3i}=\frac{c_2}{c_1}$, $\Sigma k_i\,l_{4i}=0$, $\Sigma\,k_i\,l_{5i}=\frac{c_2\,c_4}{c_1\,c_3}$,..

Thus the principal lines at any point on the curve make constant angles with a fixed direction.

If, for example, $\frac{c_2}{c_1}=\tan\theta_1\cos\theta_2$, $\frac{c_4}{c_3}=\tan\theta_2\cos\theta_3$, ...,

$\frac{c_{n-3}}{c_{n-4}}=\tan\theta_{\frac{n-3}{2}}\cos\theta_{\frac{n-1}{2}}$, $\frac{c_{n-1}}{c_{n-2}}=\tan\theta_{\frac{n-1}{2}}$, where θ_1, θ_2, ... are constants, then the direction-cosines of this fixed direction with respect to the principal lines are $\cos\theta_1$, 0, $\sin\theta_1\cos\theta_2$, 0, ...,

$$\sin\theta_1\sin\theta_2\ldots\sin\theta_{\frac{n-3}{2}}\cos\theta_{\frac{n-1}{2}},\ 0,\ \sin\theta_1\sin\theta_1\ldots.\sin\theta_{\frac{n-1}{2}};$$

in other words r, θ_1, θ_2, .. $\theta_{\frac{n-1}{2}}$ are the polar coordinates of a point on the axis with respect to the tangent, the 2nd normal, the 4th normal, ... the $(n-1)$th normal.

(ii) If n be even, we shall have

$$-\left(\int\frac{l_{1i}}{\rho_1}\,ds-k_i\right)=l_{2i}+\frac{c_3}{c_2}l_{4i}+\frac{c_3\,c_5}{c_2\,c_4}l_{6i}+\ldots+\frac{c_3\,c_5\ldots c_{n-1}}{c_2\,c_4\ldots c_{n-2}}l_{ni}.$$

Let $\rho_1=\frac{k}{f(s)}$, where $f(s)$ is any arbitrary function of s, and k constant.

Then, $\int l_{1i}f(s)\,ds=\xi_i$, where ξ_i are the coordinates of a point on a curve whose principal lines at a point are respectively parallel to those of the given curve at the corresponding point, and

$$\sqrt{\Sigma\,d\xi_i{}^2}=f(s)\,ds.$$

Squaring and adding the above relations it is seen that the radius of the spheric of closest contact along the curve is constant.

In particular, if ρ_1 is constant $=r\cos\theta_1$, and $\frac{c_3}{c_2}=\tan\theta_1\cos\theta_2$,

$\frac{c_5}{c_4}=\tan\theta_2\cos\theta_3$, .., $\frac{c_{n-1}}{c_{n-2}}=\tan\theta_{\frac{n-2}{2}}$, where r, θ_1, θ_2 .. are constant, *the radius is* r.

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