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JACOBI-LIKE FORMS, PSEUDODIFFERENTIAL OPERATORS, AND GROUP COHOMOLOGY

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Abstract

Pseudodifferential operators are formal Laurent series in the formal inverse ∂^{-1} of the derivative operator ∂ whose coefficients are holomorphic functions on the Poincaré upper half-plane. Given a discrete subgroup Γ of $SL(2, \mathbb{R})$, automorphic pseudodifferential operators for Γ are pseudodifferential operators that are Γ-invariant, and they are closely linked to Jacobi-like forms and modular forms for Γ. We construct linear maps from the space of automorphic pseudodifferential operators and from the space of Jacobi-like forms for Γ to the cohomology space of the group Γ, and prove that these maps are compatible with the respective Hecke operator actions.

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1. Introduction

Modular forms are holomorphic functions on the Poincaré upper half-plane $\mathcal{H}$ satisfying a certain transformation formula with respect to the linear fractional action of a discrete subgroup Γ of $SL(2, \mathbb{R})$, and they play a major role in number theory and are also related to various other areas of mathematics. In particular, it is well known that the space of modular forms for Γ of a given weight corresponds to some cohomology group of the discrete group Γ (see [1, 2, 8]).

Pseudodifferential operators are formal Laurent series in the formal inverse ∂^{-1} of the derivative operator ∂ whose coefficients are complex-valued functions, and they have been studied extensively over the years in connection with a variety of topics in pure and applied mathematics. For example, they play a critical role in the theory of nonlinear integrable partial differential equations, also known as soliton equations (see, for example, [5]). If the coefficients of a pseudodifferential operator Ψ belong to the space R of holomorphic functions on $\mathcal{H}$, then the usual linear fractional action of Γ on $\mathcal{H}$ induces an operation of the same group on Ψ. Pseudodifferential operators that are invariant under such an operation are automorphic pseudodifferential operators for Γ, and they are closely linked to modular forms. Indeed, given an automorphic pseudodifferential operator Ψ for Γ, a certain linear combination of derivatives of coefficients of Ψ determines a modular form for Γ, and conversely, each coefficient of Ψ can be expressed as a linear combination of derivatives of modular forms for Γ of various weights. These relations can be used to establish a one-to-one correspondence between automorphic pseudodifferential operators and certain sequences of modular forms. One of the applications of this correspondence is the construction of a lifting map from modular forms to automorphic pseudodifferential operators.

Jacobi-like forms for Γ are formal power series with coefficients in R satisfying a certain functional equation, and they are in one-to-one correspondence with automorphic pseudodifferential operators for Γ. Jacobi-like forms generalize the usual Jacobi forms developed by Eichler and Zagier [7] in some sense, and they are also related to vertex operator algebras and the conformal field theory as is suggested in [6, 20]. Various aspects of automorphic pseudodifferential operators and Jacobi-like forms were studied systematically by Cohen *et al.* in [4] (see also [20]). Some of their results can be extended to the case of several variables, so that pseudodifferential operators and Jacobi-like forms of several variables correspond to certain sequences of Hilbert modular forms (see [12]).

Hecke operators are certain averaging operators acting on the space of automorphic forms (see [1, 16, 18]), and they are an important component of the theory of automorphic forms. For example, they are used to obtain Euler products associated with modular forms which lead to some multiplicative properties of Fourier coefficients of those automorphic forms. In light of the fact that modular forms for Γ are closely linked to the cohomology of Γ, it would be natural to study the Hecke operators on the cohomology of discrete groups associated with modular forms or other automorphic forms as was done in a number of papers (see, for example, [9–11, 19]). Hecke operators on the cohomology of more general groups were also investigated by Rhie and Whaples in [17], and they can also be introduced on the spaces of automorphic pseudodifferential operators and Jacobi-like forms (see [3, 13, 14]).

In this paper we construct linear maps from the space of automorphic pseudodifferential operators and from the space Jacobi-like forms for Γ to the cohomology space of the group Γ and prove that these maps are compatible with the respective Hecke operator actions.

2. Hecke operators on group cohomology

In this section we review Hecke operators acting on group cohomology in terms of homogeneous cochains introduced by Rhie and Whaples [17]. We also describe these operators in terms of nonhomogeneous cochains and apply this description to the case of the cohomology of a discrete subgroup of $SL(2, \mathbb{R})$ to obtain the usual Hecke operators on such cohomology (see, for example, [8, 18]).

Let G be a group, and let M be a left G-module. Given a nonnegative integer q, the group $\mathfrak{C}^q(G, M)$ of homogeneous q-cochains is an abelian group generated by the maps $\phi : G^{q+1} \to M$ satisfying

$$\phi(\sigma\sigma_0, \dots, \sigma\sigma_q) = \sigma\phi(\sigma_0, \dots, \sigma_q)$$

for all $\sigma, \sigma_0, \dots, \sigma_q \in G$, and the associated coboundary map $\delta_q : C^q(G, M) \to C^{q+1}(G, M)$ is defined by

$$(\delta_q\phi)(\sigma_0, \dots, \sigma_{q+1}) = \sum_{i=0}^{q+1}(-1)^i\phi(\sigma_0, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_{q+1}) \tag{2.1}$$

for all $\phi \in C^q(G, M)$ and $(\sigma_0, \dots, \sigma_{q+1}) \in G^{n+2}$. Then $\delta_q \circ \delta_{q-1} = 0$ for each $q \geq 0$ with $\delta_{-1} = 0$, and the qth cohomology group of G with coefficients in M is given by

$$H^q(G, M) = \mathfrak{Z}^q(G, M)/\mathfrak{B}^q(G, M), \tag{2.2}$$

where $\mathfrak{Z}^q(G, M)$ and $\mathfrak{B}^q(G, M)$ denote the kernel of δ_q and the image of δ_{q-1}, respectively.

Given subgroups Γ and Γ' of G, we write $\Gamma \sim \Gamma'$ if they are commensurable, that is, if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. Then $\sim$ is an equivalence relation, and the commensurator $\widetilde{\Gamma}$ of Γ is the subgroup of G containing Γ given by

$$\widetilde{\Gamma} = \{\alpha \in G \mid \alpha^{-1}\Gamma\alpha \sim \Gamma\}.$$

If $\alpha \in \widetilde{\Gamma}$, then the corresponding double coset $\Gamma\alpha\Gamma$ can be written as a disjoint union of right cosets of Γ in G of the form

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\alpha_i \tag{2.3}$$

for some $\alpha_1, \dots, \alpha_d \in \widetilde{\Gamma}$. If $\gamma \in \Gamma$, then the same double coset can be written as

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\alpha_i\gamma,$$

which follows from the fact that $\Gamma\alpha\Gamma\gamma = \Gamma\alpha\Gamma$. Thus for $1 \leq i \leq d$, we see that

$$\alpha_i\gamma = \xi_i(\gamma) \cdot \alpha_{i(\gamma)} \tag{2.4}$$

for some element $\xi_i(\gamma) \in \Gamma$, where $\{\alpha_{1(\gamma)}, \dots, \alpha_{d(\gamma)}\}$ is a permutation of $\{\alpha_1, \dots, \alpha_d\}$.

Given an element $\phi \in \mathfrak{C}^q(\Gamma, M)$ with $q \geq 0$ and a double coset $\Gamma\alpha\Gamma$ with $\alpha \in \widetilde{\Gamma}$ that has a decomposition as in (2.3), we consider the associated map $\mathfrak{T}(\alpha)\phi : \Gamma^{q+1} \to M$ given by

$$(\mathfrak{T}(\alpha)\phi)(\gamma_0, \dots, \gamma_q) = \sum_{i=1}^{d} \alpha_i^{-1}\phi(\xi_i(\gamma_0), \dots, \xi_i(\gamma_q)), \tag{2.5}$$

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where the maps $\xi_i : \Gamma \to \Gamma$ are determined by (2.4). Then it is known that $\mathfrak{T}(\alpha)\phi$ is an element of $\mathfrak{C}^q(\Gamma, M)$ and is independent of the choice of representatives of the coset decomposition of $\Gamma\alpha\Gamma$ modulo Γ (see [17]). Thus each double coset $\Gamma\alpha\Gamma$ with $\alpha \in \widetilde{\Gamma}$ determines the $\mathbb{C}$-linear map

$$\mathfrak{T}(\alpha) : C^q(\Gamma, M) \to C^q(\Gamma, M) \tag{2.6}$$

for each $q \geq 0$. It can be shown that

$$\mathfrak{T}(\alpha) \circ \delta_{q-1} = \delta_q \circ \mathfrak{T}(\alpha)$$

for $q \geq 1$. Hence it follows that the map $\mathfrak{T}(\alpha)$ in (2.6) induces the homomorphism

$$\mathfrak{T}(\alpha) : H^q(\Gamma, M) \to H^q(\Gamma, M),$$

which is the Hecke operator on $H^q(\Gamma, M)$ corresponding to α introduced by Rhie and Whaples [17].

The cohomology of the group Γ can also be defined by using nonhomogeneous cochains. Indeed, the group $C^q(\Gamma, M)$ of nonhomogeneous q-cochains consists of the maps $\psi : \Gamma^q \to M$, and the associated coboundary map $\partial_q : C^q(\Gamma, M) \to C^{q+1}(\Gamma, M)$ is given by

$$\begin{aligned}(\partial_q \psi)(\gamma_1, \dots, \gamma_{q+1}) = \gamma_1 \psi(\gamma_2, \dots, \gamma_{q+1}) \\ + \sum_{i=1}^{q} (-1)^i \psi(\gamma_1, \dots, \gamma_{i-1}, \gamma_i \gamma_{i+1}, \dots, \gamma_{q+1}) \\ + (-1)^{q+1} \psi(\gamma_1, \dots, \gamma_q)\end{aligned} \tag{2.7}$$

for all $\psi \in C^q(\Gamma, M)$ and $(\gamma_1, \dots, \gamma_{q+1}) \in \Gamma^{q+1}$. Then it can be shown that the cohomology for the cochain complex $\{C^q(\Gamma, M), \partial_q\}_{q \geq 0}$ is canonically isomorphic to the cohomology of the cochain complex $\{\mathfrak{C}^q(\Gamma, M), \delta_q\}_{q \geq 0}$ defined by using homogeneous cochains. Thus we may write the qth cohomology group in (2.2) in the form

$$H^q(G, M) = Z^q(G, M)/B^q(G, M),$$

where $Z^q(G, M)$ and $B^q(G, M)$ are the kernel of ∂_q and the image of ∂_{q-1}, respectively. In terms of the nonhomogeneous q-cochains, the Hecke operator in (2.5) can now be written as

$$\begin{aligned}&(T(\alpha)\psi)(\gamma_1, \dots, \gamma_q) \\ &= \sum_{i=1}^{d} \alpha_i^{-1} \psi(\xi_i(\gamma_1), \xi_{i(\gamma_1)}(\gamma_2), \xi_{i(\gamma_1\gamma_2)}(\gamma_3), \dots, \xi_{i(\gamma_1 \cdots \gamma_{q-1})}(\gamma_q))\end{aligned} \tag{2.8}$$

for all $\psi \in C^q(\Gamma, M)$ and $\gamma_1, \dots, \gamma_q \in \Gamma$, which determines another version of the Hecke operator

$$T(\alpha) : H^q(\Gamma, M) \to H^q(\Gamma, M),$$

on $H^q(\Gamma, M)$ corresponding to α.

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We now consider the case where $G = GL(2, \mathbb{C})$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ be the standard basis for the complex vector space $\mathbb{C}^{n+1}$, and for $\binom{z_1}{z_2} \in \mathbb{C}^2$ set

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^n = {}^t(z_1^n, z_1^{n-1} z_2, \dots, z_1 z_2^{n-1}, z_2^n) = \sum_{k=0}^{n} z_1^{n-k} z_2^k \, \mathbf{e}_{k+1} \in \mathbb{C}^{n+1},$$

where ${}^t(\cdot)$ denotes the transpose of the matrix $(\cdot)$. Let

$$\rho_n : GL(2, \mathbb{C}) \to GL(n+1, \mathbb{C})$$

be the nth symmetric tensor representation of $GL(2, \mathbb{C})$, which is given by

$$\rho_n(\gamma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^n = \left(\gamma \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^n$$

for all $\gamma \in \Gamma$ and $\binom{z_1}{z_2} \in \mathbb{C}^2$. We also define the map $\mathbf{v}_n : \mathcal{H} \to \mathbb{C}^{n+1}$ by

$$\mathbf{v}_n(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}^n = \sum_{k=0}^{n} z^{n-k} \, \mathbf{e}_{k+1} \tag{2.9}$$

for all $z \in \mathbb{C}$. Then for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$ we see that

$$\begin{aligned} \rho_n(\gamma) \mathbf{v}_n(z) = \rho_n(\gamma) \begin{pmatrix} z \\ 1 \end{pmatrix}^n &= \begin{pmatrix} az+b \\ cz+d \end{pmatrix}^n \\ &= \sum_{k=0}^{n} (az+b)^{n-k} (cz+d)^k \, \mathbf{e}_{k+1} \\ &= (cz+d)^{-n} \sum_{k=0}^{n} (\gamma z)^{n-k} \, \mathbf{e}_{k+1} = (cz+d)^{-n} \mathbf{v}_n(\gamma z), \end{aligned} \tag{2.10}$$

where $\gamma z = (az+b)(cz+d)^{-1}$. We denote by $\mathfrak{S}^n(\mathbb{C}^2)$ the complex vector space $\mathbb{C}^{n+1}$ equipped with the structure of a $GL(2, \mathbb{C})$-module given by

$$(\gamma, v) \mapsto (\det \gamma)^{n/2} \rho_n(\gamma) v$$

for $\gamma \in GL(2, \mathbb{C})$ and $v \in \mathbb{C}^{n+1}$.

Let Γ be a discrete subgroup of $SL(2, \mathbb{R}) \subset GL(2, \mathbb{C})$. Then by (2.7) its first cohomology group with coefficients in $\mathfrak{S}^n(\mathbb{C}^2)$ can be described as follows. The set $Z^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$ of nonhomogeneous 1-cocycles consists of all maps $u : \Gamma \to \mathbb{C}^{n+1}$ satisfying

$$u(\gamma\gamma') = u(\gamma) + \rho_n(\gamma) u(\gamma') \tag{2.11}$$

for all $\gamma, \gamma' \in \Gamma$. Given an element $v_0 \in \mathbb{C}^{n+1}$, the set $B^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$ of coboundaries consists of the maps $v : \Gamma \to \mathbb{C}^{n+1}$ such that

$$v(\gamma) = (\rho_n(\gamma) - 1_{n+1}) v_0$$

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for all $\gamma \in \Gamma$, where 1_{n+1} is the identity map on $\mathbb{C}^{n+1}$. Then the first cohomology group of Γ with coefficients in $\mathfrak{S}^n(\mathbb{C}^2)$ is given by

$$H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2)) = \frac{Z^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))}{B^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))}. \tag{2.12}$$

To consider Hecke operators acting on this cohomology group, we choose an element $\alpha \in \widetilde{\Gamma} \subset GL(2, \mathbb{R})$ such that the corresponding double coset has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{s} \Gamma\alpha_i$$

with $\alpha_1, \ldots, \alpha_s \in \widetilde{\Gamma}$. Then by (2.8) the Hecke operator $T_n(\alpha)$ on $H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$ can be written as

$$(T_n(\alpha)(\phi))(\gamma) = \sum_{i=1}^{s} (\det \alpha_i)^{n/2} \rho_n(\alpha_i)\phi(\xi_i(\gamma)) \tag{2.13}$$

for each 1-cocycle ϕ and $\gamma \in \Gamma$, where ξ_i is as in (2.4).

3. Jacobi-like forms and pseudodifferential operators

In this section we review modular forms, Jacobi-like forms, and pseudodifferential operators as well as relations among these objects studied by Cohen *et al.* in [4, 20]. We also describe Hecke operators acting on the spaces consisting of those objects (see [3, 13, 14]).

Let R be the space of holomorphic functions on the Poincaré upper half-plane $\mathcal{H}$, and let $R[[X]]$ be the complex algebra of formal power series in X with coefficients in R. Let Γ be a discrete subgroup of $SL(2, \mathbb{R})$, which acts on $\mathcal{H}$ as usual by linear fractional transformations, that is,

$$\gamma z = \frac{az + b}{cz + d}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$.

DEFINITION 3.1. (i) A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is a *modular form of weight k for* Γ if it satisfies

$$f(\gamma z) = (cz + d)^k f(z)$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

(ii) A formal power series $\Phi(z, X) \in R[[X]]$ is a *Jacobi-like form for* Γ if it satisfies

$$\Phi(\gamma z, (cz + d)^{-2} X) = \exp(cX/(cz + d))\Phi(z, X) \tag{3.1}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

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(iii) A *pseudodifferential operator over* R is a formal Laurent series $\Psi(z)$ in the formal inverse ∂^{-1} of $\partial = d/dz$ with coefficients in R, that is, a series of the form

$$\Psi(z) = \sum_{n=-\infty}^{n_0} \xi_n(z)\partial^n$$

for some $n_0 \in \mathbb{Z}$ with $\xi_n \in R$ for each n.

Note that we have slightly modified the usual definition of modular forms by suppressing the cusp condition. We denote by $\mathcal{M}_k(\Gamma)$ the space of modular forms of weight k for Γ, by $\mathcal{J}(\Gamma)$ the space of all Jacobi-like forms for Γ, and by ΨDO the space of all pseudodifferential operators over R.

The space ΨDO has the structure of an algebra over $\mathbb{C}$ whose multiplication is given by the Leibniz rule. Thus

$$\left(\sum_{n=-\infty}^{n_0} \xi_n(z)\partial^n\right)\left(\sum_{m=-\infty}^{m_0} \eta_m(z)\partial^m\right) = \sum_{n=-\infty}^{n_0} \sum_{m=-\infty}^{m_0} \sum_{r=0}^{\infty} \binom{n}{r} \xi_n(z)\eta_m^{(r)}(z)\partial^{n+m-r},$$

where $\eta_m^{(r)}$ denotes the derivative of η_m of order r with respect to z, and

$$\binom{n}{0} = 1, \quad \binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

for $r > 0$. Given an integer v, we denote by $\Psi\mathrm{DO}_v$ the subspace of ΨDO given by

$$\Psi\mathrm{DO}_v = \left\{\sum_{n=0}^{\infty} \xi_n(z)\partial^{v-n} \;\middle|\; \xi_n \in R\right\}, \tag{3.2}$$

and define the symbol map $\Xi_v^\partial : \Psi\mathrm{DO}_v \to R$ by

$$\Xi_v^\partial\left(\sum_{n=0}^{\infty} \xi_n(z)\partial^{v-n}\right) = \xi_0(z). \tag{3.3}$$

Since Ξ_v^∂ is a $\mathbb{C}$-linear map whose kernel is $\Psi\mathrm{DO}_{v-1}$, we obtain a short exact sequence

$$0 \to \Psi\mathrm{DO}_{v-1} \to \Psi\mathrm{DO}_v \xrightarrow{\Xi_v^\partial} R \to 0 \tag{3.4}$$

of complex vector spaces.

We now describe the action of $SL(2,\mathbb{R})$ on pseudodifferential operators. If $\widetilde{\partial}$ denotes the differentiation operator with respect to the transformed coordinate γz of z by an element $\gamma \in SL(2,\mathbb{R})$,

$$\widetilde{\partial} = \left(\frac{d(\gamma z)}{dz}\right)^{-1}\partial = (cz+d)^2\partial. \tag{3.5}$$

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We note that for each $m \in \mathbb{Z}$,

$$((cz+d)^2\partial)^m = \sum_{\ell=0}^{\infty} \ell! \binom{m}{\ell}\binom{m-1}{\ell} c^{\ell}(cz+d)^{2m-\ell}\partial^{m-\ell} \tag{3.6}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$ (see [4, (1.7)]). Then it can be shown that the map $\partial \mapsto \partial \circ \gamma = \widetilde{\partial}$ induces a right action $\Psi(z) \mapsto (\Psi \circ \gamma)(z)$ of $SL(2, \mathbb{R})$ on ΨDO.

DEFINITION 3.2. An element $\Psi(z) \in \Psi\mathrm{DO}$ is an *automorphic pseudodifferential operator for* Γ if it satisfies

$$(\Psi \circ \gamma)(z) = \Psi(z)$$

for all $\gamma \in \Gamma$. We denote by $\Psi\mathrm{DO}^{\Gamma}$ the space of automorphic pseudodifferential operators for Γ.

If $v \in \mathbb{Z}$, we set

$$\Psi\mathrm{DO}_v^{\Gamma} = \Psi\mathrm{DO}_v \cap \Psi\mathrm{DO}^{\Gamma}.$$

Then, using (3.6), we see that the coefficient $\Xi_{-w}^{\partial}(\Psi(z))$ of ∂^{-w} of an element $\Psi(z) \in \Psi\mathrm{DO}_{-w}^{\Gamma}$ with $w \geq 0$ is a modular form belonging to $\mathcal{M}_{2w}(\Gamma)$. Thus the sequence (3.4) induces the short exact sequence

$$0 \to \Psi\mathrm{DO}_{-w-1}^{\Gamma} \to \Psi\mathrm{DO}_{-w}^{\Gamma} \xrightarrow{\Xi_{-w}^{\partial}} \mathcal{M}_{2w}(\Gamma) \to 0, \tag{3.7}$$

which actually splits.

Given a positive integer w, let $R[[X]]_w$ be the subspace of $R[[X]]$ consisting of formal power series of the form $\sum_{k=w}^{\infty} \phi_k(z)X^k$, and set

$$\mathcal{J}(\Gamma)_w = \mathcal{J}(\Gamma) \cap R[[X]]_w.$$

If $(\Xi_w^X \Phi)(z)$ denotes the coefficient of $\Phi(z, X) \in R[[X]]_w$, we obtain the short exact sequence

$$0 \to R[[X]]_{w-1} \to R[[X]]_w \xrightarrow{\Xi_w^X} R \to 0, \tag{3.8}$$

which induces the sequence

$$0 \to \mathcal{J}(\Gamma)_{w-1} \to \mathcal{J}(\Gamma)_w \xrightarrow{\Xi_w^X} R \to 0. \tag{3.9}$$

We now introduce a map

$$\mathfrak{F}_w : R[[X]]_w \to \Psi\mathrm{DO}_{-w} \tag{3.10}$$

defined by

$$\mathfrak{F}_w\left(\sum_{k=w}^{\infty} \phi_k(z)X^k\right) = \sum_{k=w}^{\infty} (-1)^k k!(k-1)!\phi_k(z)\partial^{-k}, \tag{3.11}$$

which is clearly a $\mathbb{C}$-linear isomorphism satisfying

$$\Xi^{\partial}_{-w} \circ \mathfrak{F}_w = \Xi^{X}_{w},$$

where Ξ^{∂}_{-w} and Ξ^{X}_{w} are as in (3.4) and (3.8), respectively.

PROPOSITION 3.3. (i) *A formal power series $\Phi(z, X) \in R[[X]]_w$ with $w \geq 1$ is a Jacobi-like form belonging to $\mathcal{J}(\Gamma)_w$ if and only if $\mathfrak{F}_w(\Phi(z, X))$ is an automorphic pseudodifferential operator belonging to $\Psi\mathrm{DO}^{\Gamma}_{-w}$.*

(ii) *Let $\Phi(z, X) = \sum_{k=w}^{\infty} \phi_k(z) X^k \in R[[X]]_w$ with $w \geq 1$. Then $\Phi(z, X) \in \mathcal{J}(\Gamma)_w$ if and only if there is a sequence $\{h_\ell\}_{\ell=w}^{\infty}$ of modular forms with $h_\ell \in \mathcal{M}_{2\ell}(\Gamma)$ for each $\ell \geq w$ such that*

$$\phi_k = \sum_{r=0}^{k-w} \frac{1}{r!(2k-r-1)!} h^{(r)}_{k-r} \tag{3.12}$$

for all $k \geq w$.

(iii) *The modular forms h_ℓ satisfying (3.12) can be written in the form*

$$h_\ell = (2\ell - 1) \sum_{s=0}^{\ell-w} (-1)^s \frac{(2\ell-2-s)!}{s!} \phi^{(s)}_{\ell-s}$$

for all $\ell \geq w$.

PROOF. Statements (i), (ii) and (iii) can be proved by slightly modifying the proof of [4, Proposition 2]. □

By Proposition 3.3 the map $\mathfrak{F}_w$ in (3.10) induces the $\mathbb{C}$-isomorphism

$$\mathfrak{F}_w : \mathcal{J}(\Gamma)_w \to \Psi\mathrm{DO}^{\Gamma}_{-w} \tag{3.13}$$

satisfying

$$\Xi^{\partial}_{-w} \circ \mathfrak{F}_w = \Xi^{X}_{w},$$

where Ξ^{∂}_{-w} and Ξ^{X}_{w} are as in (3.7) and (3.9), respectively.

Let $GL^+(2, \mathbb{R})$ be the multiplicative group of 2×2 real matrices of positive determinant, which acts on $\mathcal{H}$ by linear fractional transformations. Given $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL^+(2, \mathbb{R})$ and elements $f \in R$ and $\Phi(z, X) \in R[[X]]$, we set

$$(f|_k\alpha)(z) = \det(\alpha)^{k/2}(cz+d)^{-k} f(\alpha z),$$

$$(\Phi \mid^X \alpha)(z, X) = e^{-cX/(cz+d)} \Phi(\alpha z, (\det\alpha)(cz+d)^{-2} X) \tag{3.14}$$

for all $z \in \mathcal{H}$ and $k \in \mathbb{Z}$. Then it can be shown that

$$(f|_k\alpha)|_k\alpha' = f \mid_k (\alpha\alpha'), \quad (\Phi \mid^X \alpha) \mid^X \alpha' = \Phi \mid^X (\alpha\alpha')$$

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for all $\alpha, \alpha' \in GL^+(2, \mathbb{R})$, and by Definition 3.1

$$f \in \mathcal{M}_k(\Gamma), \quad \Phi(z, X) \in \mathcal{J}(\Gamma)$$

if and only if

$$f|_k\gamma = f, \quad (\Phi \mid^X \gamma)(z, X) = \Phi(z, X)$$

for all $\gamma \in \Gamma$. On the other hand, if $\Psi(z) \in \Psi\mathrm{DO}_w$ with $w \in \mathbb{Z}$ is a pseudodifferential operator of the form

$$\Psi(z) = \sum_{k=w}^{\infty} \psi_k(z)\partial^{-k}, \tag{3.15}$$

we define the pseudodifferential operator $(\Psi \mid^\partial \alpha)(z) \in \Psi\mathrm{DO}_w$ by

$$(\Psi \mid^\partial \alpha)(z) = \sum_{k=w}^{\infty} \sum_{\ell=0}^{k-w} \ell! \binom{k}{\ell} \binom{k-1}{\ell} \frac{(\det \alpha)^{k-\ell} c^\ell}{(cz+d)^{2k-\ell}} \psi_{k-\ell}(\alpha z)\partial^{-k} \tag{3.16}$$

for all $z \in \mathcal{H}$.

DEFINITION 3.4. Let α be an element of $\widetilde{\Gamma}$ such that the corresponding double coset has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{s} \Gamma\alpha_i \tag{3.17}$$

with $\alpha_1, \ldots, \alpha_s \in GL^+(2, \mathbb{R})$. The associated *Hecke operators* on $\mathcal{M}_k(\Gamma)$, $\mathcal{J}(\Gamma)$ and ΨDO are the linear endomorphisms

$$T_k^{\mathcal{M}}(\alpha) : \mathcal{M}_k(\Gamma) \to \mathcal{M}_k(\Gamma), \quad T^{\mathcal{J}}(\alpha) : \mathcal{J}(\Gamma) \to \mathcal{J}(\Gamma), \quad T^{\Psi}(\alpha) : \Psi\mathrm{DO} \to \Psi\mathrm{DO},$$

respectively, given by

$$T_k^{\mathcal{M}}(\alpha) f = \sum_{i=1}^{s} (f|_k\alpha_i), \tag{3.18}$$

$$(T^{\mathcal{J}}(\alpha)\Phi)(z, X) = \sum_{i=1}^{s} (\Phi \mid^X \alpha_i)(z, X), \tag{3.19}$$

$$(T^{\Psi}(\alpha)\Psi)(z) = \sum_{i=1}^{s} (\Psi \mid^\partial \alpha_i)(z) \tag{3.20}$$

for all $f \in \mathcal{M}_k(\Gamma)$, $\Phi(z, X) \in \mathcal{J}(\Gamma)$, and $\Psi(z) \in \Psi\mathrm{DO}$.

The power series $T^{\mathcal{J}}(\alpha)\Phi(z, X)$ given by (3.19) is indeed a Jacobi-like form belonging to $\mathcal{J}(\Gamma)$ and is independent of the choice of the coset representatives $\alpha_1, \ldots, \alpha_s$ (see [14, Proposition 3.2]). We also see that

$$T^{\mathcal{J}}(\alpha)(\mathcal{J}(\Gamma)_w) \subset \mathcal{J}(\Gamma)_w, \quad (\Xi_w^X \circ T^{\mathcal{J}}(\alpha))\Phi = T_k^{\mathcal{M}}(\alpha)(\Xi_w^X \Phi)$$

for all $\Phi(z, X) \in \mathcal{J}(\Gamma)_w$, where Ξ_w is as in (3.9).

PROPOSITION 3.5. *Given $w \geq 1$, let $\mathfrak{F}_w$ be the isomorphism in (3.13). Then*

$$\mathfrak{F}_w \circ T^{\mathcal{J}}(\alpha) \circ \mathfrak{F}_w^{-1} = T^{\Psi}(\alpha)$$

for each $\alpha \in \widetilde{\Gamma}$. In particular,

$$(T^{\Psi}(\alpha)\Psi)(z) \in \Psi\mathrm{DO}_w^{\Gamma}$$

for all $\Psi(z) \in \Psi\mathrm{DO}_w^{\Gamma}$.

PROOF. Let $\Psi(z) \in \Psi\mathrm{DO}_w^{\Gamma}$ be as in (3.15). Then by Proposition 3.3 the formal power series

$$\Phi(z, X) = (\mathfrak{F}_w^{-1}\Psi)(z, X) = \sum_{k=w}^{\infty} \frac{(-1)^k}{k!(k-1)!} \psi_k(z) X^k$$

is a Jacobi-like form belonging to $\mathcal{J}(\Gamma)_w$. Given $\alpha \in \widetilde{\Gamma}$, from (3.14) we obtain

$$\begin{aligned}
(\Phi \mid^X \alpha_i)(z, X) &= \exp\left(\frac{-c_i X}{c_i z + d_i}\right) \Phi(\alpha_i z, (\det \alpha_i)(c_i z + d_i)^{-2} X) \\
&= \left(\sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{-c_i}{c_i z + d_i}\right)^r X^r\right) \\
&\quad \times \left(\sum_{k=w}^{\infty} \frac{(-1)^k \psi_k(\alpha_i z)}{k!(k-1)!} (\det \alpha_i)^k (c_i z + d_i)^{-2k} X^k\right) \\
&= \sum_{k=w}^{\infty} \sum_{r=0}^{\infty} \frac{(\det \alpha_i)^k (-c_i)^r}{r!} \frac{(-1)^k \psi_k(\alpha_i z)}{k!(k-1)!(c_i z + d_i)^{2k+r}} X^{k+r} \\
&= \sum_{k=w}^{\infty} \sum_{\ell=0}^{k-w} \frac{(\det \alpha_i)^{k-\ell} (-c_i)^{\ell}}{\ell!} \frac{(-1)^{k-\ell} \psi_{k-\ell}(\alpha_i z)}{(k-\ell)!(k-\ell-1)!(c_i z + d_i)^{2k-\ell}} X^k
\end{aligned}$$

for $1 \leq i \leq s$ with $\alpha_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in GL^+(2, \mathbb{R})$. Thus by using this and (3.19) we see that

$$(T^{\mathcal{J}}(\alpha)\Phi)(z, X) = \sum_{i=1}^{s} (\Phi \mid^X \alpha_i)(z, X) = \sum_{k=w}^{\infty} \sum_{i=1}^{s} \phi_{i,k}(z) X^k,$$

where

$$\phi_{i,k}(z) = \sum_{\ell=0}^{k-w} \frac{(-1)^k (\det \alpha_i)^{k-\ell} c_i^{\ell} \psi_{k-\ell}(\alpha_i z)}{\ell!(k-\ell)!(k-\ell-1)!(c_i z + d_i)^{2k-\ell}} X^k.$$

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On the other hand, from (3.16) and (3.20) we obtain

$$
\begin{aligned}
(T^{\Psi}(\alpha)\Psi)(z) &= \sum_{i=1}^{s}\sum_{k=w}^{\infty}\sum_{\ell=0}^{k-w} \ell! \binom{k}{\ell}\binom{k-1}{\ell} \frac{(\det \alpha_i)^{k-\ell} c_i^{\ell}}{(c_i z + d_i)^{2k-\ell}} \psi_{k-\ell}(\alpha_i z)\partial^{-k} \\
&= \sum_{i=1}^{s}\sum_{k=w}^{\infty} (-1)^k k!(k-1)! \sum_{\ell=0}^{k-w} \frac{(-1)^k (\det \alpha_i)^{k-\ell} c_i^{\ell} \psi_{k-\ell}(\alpha_i z)}{\ell!(k-\ell)!(k-\ell-1)!(c_i z + d_i)^{2k-\ell}} \partial^{-k} \\
&= \sum_{k=w}^{\infty} (-1)^k k!(k-1)! \sum_{i=1}^{s} \phi_{i,k}(z)\partial^{-k} = (\mathfrak{F}_w(T^{\mathcal{J}}(\alpha)\Phi))(z);
\end{aligned}
$$

hence the proposition follows from this and Proposition 3.3. □

By Proposition 3.5 the Hecke operator on ΨDO given by (3.20) induces a linear endomorphism

$$T^{\Psi}(\alpha) : \Psi\mathrm{DO}^{\Gamma} \to \Psi\mathrm{DO}^{\Gamma}$$

on the space $\Psi\mathrm{DO}^{\Gamma}$ of automorphic pseudodifferential operators for Γ. Furthermore, we see that the diagram

$$
\begin{tikzcd}
\mathcal{J}(\Gamma)_w \arrow[r, "T^{\mathcal{J}}(\alpha)"] \arrow[d, "\mathfrak{F}_w"'] & \mathcal{J}(\Gamma)_w \arrow[d, "\mathfrak{F}_w"] \\
\Psi\mathrm{DO}^{\Gamma}_{-w} \arrow[r, "T^{\Psi}(\alpha)"] & \Psi\mathrm{DO}^{\Gamma}_{-w}
\end{tikzcd}
\tag{3.21}
$$

commutes for each $w \geq 1$.

4. Group cohomology

Let $\mathcal{J}(\Gamma) \subset R[[X]]$ be the space of Jacobi-like forms for a discrete subgroup Γ of $SL(2, \mathbb{R})$ as in Section 3, and let $H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$ with $n \geq 1$ be the cohomology space of Γ in (2.12). In this section we construct a linear map from $\mathcal{J}(\Gamma)_1 = \mathcal{J}(\Gamma) \cap R[[X]]_1$ to $H^1(\Gamma, \mathfrak{S}^{2m}(\mathbb{C}^2))$ for each positive integer m and show that it is compatible with respect to the Hecke operator actions.

Let $\Phi(z, X)$ be an element of $\mathcal{J}(\Gamma)_1$ of the form

$$\Phi(z, X) = \sum_{k=1}^{\infty} \phi_k(z) X^k \tag{4.1}$$

for $z \in \mathcal{H}$. Although the coefficients ϕ_k are not modular forms, they are in fact special types of quasimodular forms (see, for example, [15]). We fix a base point $z_0 \in \mathcal{H}$ and, by analogy with periods of modular forms, consider certain integrals over paths in $\mathcal{H}$

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originating at z_0 which may be regarded as periods of coefficients of the Jacobi-like form $\Phi(z, X)$. If m, r and ℓ are integers with $m \geq 1$ and $0 \leq r, \ell \leq m$, we set

$$\varpi_{m,r,\ell}(\gamma) = \int_{z_0}^{\gamma z_0} \phi_{m+1-r}^{(r)}(z) z^\ell \, dz \tag{4.2}$$

for all $\gamma \in \Gamma$. Note that the integral is independent of the choice of the path $z_0 \to \gamma z_0$ because the coefficients $\phi_k(z)$ of $\Phi(z, X)$ are holomorphic. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_{2m+1}\}$ be the standard basis for $\mathbb{C}^{2m+1}$, and set

$$\widehat{\varpi}_{m,r}(\gamma) = \sum_{\ell=0}^{2m} \varpi_{m,r,\ell}(\gamma) \mathbf{e}_{\ell+1} \in \mathbb{C}^{2m+1}. \tag{4.3}$$

We now define the map $\mathcal{L}_m(\Phi) : \Gamma \to \mathbb{C}^{2m+1}$ by

$$\mathcal{L}_m(\Phi)(\gamma) = \sum_{r=0}^{m} (-1)^r \frac{(2m-r)!}{r!} \widehat{\varpi}_{m,r}(\gamma) \tag{4.4}$$

for all $\gamma \in \Gamma$.

PROPOSITION 4.1. *Given a Jacobi-like form $\Phi(z, X) \in \mathcal{J}(\Gamma)_1$, the associated map $\mathcal{L}_m(\Phi) : \Gamma \to \mathbb{C}^{2m+1}$ given by (4.4) is a cocycle belonging to $Z^1(\Gamma, \mathfrak{S}^{2m}(\mathbb{C}^2))$ for each positive integer m, where the Γ-module $\mathfrak{S}^{2m}(\mathbb{C}^2)$ is as in Section 2.*

PROOF. Given $\Phi(z, X) \in \mathcal{J}(\Gamma)_1$ and a positive integer m, since a cocycle belonging to $Z^1(\Gamma, \mathfrak{S}^{2m}(\mathbb{C}^2))$ must satisfy (2.11), we need to show that

$$\mathcal{L}_m(\Phi)(\gamma\gamma') = \mathcal{L}_m(\Phi)(\gamma) + \rho_{2m}(\gamma)\mathcal{L}_m(\Phi)(\gamma') \tag{4.5}$$

for all $\gamma, \gamma' \in \Gamma$. Assuming that $\Phi(z, X)$ is as in (4.1), by using (4.2), (4.3) and (4.4), we see that

$$\begin{aligned} \mathcal{L}_m(\Phi)(\gamma) &= \sum_{r=0}^{m} \sum_{\ell=0}^{2m} (-1)^r \frac{(2m-r)!}{r!} \varpi_{m,r,\ell}(\gamma) \mathbf{e}_{\ell+1} \\ &= \int_{z_0}^{\gamma z_0} \sum_{r=0}^{m} \sum_{\ell=0}^{2m} (-1)^r \frac{(2m-r)!}{r!} \phi_{m+1-r}^{(r)}(z) z^\ell \mathbf{e}_{\ell+1} \, dz \\ &= \int_{z_0}^{\gamma z_0} \sum_{r=0}^{m} (-1)^r \frac{(2m-r)!}{r!} \phi_{m+1-r}^{(r)}(z) \mathbf{v}_{2m}(z) \, dz, \end{aligned}$$

where $\mathbf{v}_{2m}(z)$ is as in (2.9). If we set

$$f_m = \sum_{r=0}^{m} (-1)^r \frac{(2m-r)!}{r!} \phi_{m+1-r}^{(r)}, \tag{4.6}$$

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then by Proposition 3.3 the function f_m is a modular form belonging to $\mathcal{M}_{2m+2}(\Gamma)$, and

$$\mathcal{L}_m(\Phi)(\gamma) = \int_{z_0}^{\gamma z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz. \tag{4.7}$$

Thus for $\gamma, \gamma' \in \Gamma$ we see that

$$\begin{aligned} \mathcal{L}_m(\Phi)(\gamma\gamma') &= \int_{z_0}^{\gamma\gamma' z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz \\ &= \int_{z_0}^{\gamma z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz + \int_{\gamma z_0}^{\gamma\gamma' z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz \\ &= \mathcal{L}_m(\Phi)(\gamma) + \int_{z_0}^{\gamma' z_0} f_m(\gamma z)\mathbf{v}_{2m}(\gamma z)\,d(\gamma z). \end{aligned} \tag{4.8}$$

However, using (2.10),

$$\mathbf{v}_{2m}(\gamma z) = (cz+d)^{-2m}\rho_{2m}(\gamma)\mathbf{v}_{2m}(z).$$

From this, the relations

$$f_m(\gamma z) = (cz+d)^{2m+2} f(z), \quad d(\gamma z) = (cz+d)^{-2}\,dz,$$

and (4.7) we obtain

$$\begin{aligned} \int_{z_0}^{\gamma' z_0} f_m(\gamma z)\mathbf{v}_{2m}(\gamma z)\,d(\gamma z) &= \rho_{2m}(\gamma) \int_{z_0}^{\gamma' z_0} f_m(z)\mathbf{v}_{2m}(z)\,d(z) \\ &= \rho_{2m}(\gamma)\mathcal{L}_m(\Phi)(\gamma'); \end{aligned}$$

hence (4.5) follows from this and (4.8). □

By Proposition 4.1 for each $m \geq 1$ there is a linear map

$$\mathcal{L}_m : \mathcal{J}(\Gamma)_1 \to H^1(\Gamma, \mathfrak{S}^{2m}(\mathbb{C}^2)) \tag{4.9}$$

sending a Jacobi-like form $\Phi(z, X) \in \mathcal{J}(\Gamma)_1$ to the cohomology class of $\mathcal{L}_m(\Phi(z, X))$ in $H^1(\Gamma, \mathfrak{S}^{2m}(\mathbb{C}^2))$.

THEOREM 4.2. *Given a positive integer m, the linear map $\mathcal{L}_m$ in (4.9) satisfies*

$$\mathcal{L}_m \circ T^{\mathcal{J}}(\alpha) = T_{2m}(\alpha) \circ \mathcal{L}_m$$

for each $\alpha \in \widetilde{\Gamma}$, where the Hecke operators

$$T_{2m}(\alpha) : H^1(\Gamma, \mathfrak{S}^{2m}(\mathbb{C}^2)) \to H^1(\Gamma, \mathfrak{S}^{2m}(\mathbb{C}^2)), \quad T^{\mathcal{J}}(\alpha) : \mathcal{J}(\Gamma)_1 \to \mathcal{J}(\Gamma)_1$$

are as in (2.13) and (3.19), respectively.

PROOF. Let $\alpha \in \widetilde{\Gamma}$, and assume that the associated double coset of Γ has a decomposition of the form

$$\Gamma \alpha \Gamma = \coprod_{i=1}^{s} \Gamma \alpha_i$$

with $\alpha_1, \ldots, \alpha_s \in GL^+(2, \mathbb{R})$ as in (3.17). Let $\Phi(z, X) \in \mathcal{J}(\Gamma)$ be as in (4.1), and for each $m \geq 1$ let f_m be as in (4.6). We write $T^{\mathcal{J}}(\alpha)\Phi$ in the form

$$T^{\mathcal{J}}(\alpha)\Phi(z, X) = \sum_{k=1}^{\infty} \widetilde{\phi}_k(z) X^k,$$

and set

$$\widetilde{f}_m = \sum_{r=0}^{m} (-1)^r \frac{(2m-r)!}{r!} \widetilde{\phi}_{m+1-r}^{(r)}$$

for $m \geq 1$. Then, by Proposition 3.3, $\widetilde{f}_m$ is a modular form belonging to $\mathcal{M}_{2m+2}(\Gamma)$, and it can be shown easily that

$$\widetilde{f}_m = T_{2m+2}^{\mathcal{M}}(\alpha) f_m,$$

where $T_{2m+2}^{\mathcal{M}}(\alpha)$ is as in (3.18). From this and (4.7) we obtain

$$\begin{aligned}
\mathcal{L}_m(T^{\mathcal{J}}(\alpha)\Phi)(\gamma) &= \int_{z_0}^{\gamma z_0} (T_{2m+2}^{\mathcal{M}}(\alpha) f_m)(z) \mathbf{v}_{2m}(z)\, dz \\
&= \sum_{i=1}^{s} (\det \alpha_i)^{m+1} (c_i z + d_i)^{-2m-2} \int_{z_0}^{\gamma z_0} f_m(\alpha_i z) \mathbf{v}_{2m}(z)\, dz \\
&= \sum_{i=1}^{s} (\det \alpha_i)^{m+1} \rho_{2m}(\alpha_i)^{-1} \int_{z_0}^{\gamma z_0} f_m(\alpha_i z) \mathbf{v}_{2m}(\alpha_i z)\, d(\alpha_i z) \\
&= \sum_{i=1}^{s} (\det \alpha_i)^{m+1} \rho_{2m}(\alpha_i)^{-1} \int_{\alpha_i z_0}^{\alpha_i \gamma z_0} f_m(z) \mathbf{v}_{2m}(z)\, dz
\end{aligned}$$

for all $\gamma \in \Gamma$, where (c_i, d_i) is the second row of the matrix $\alpha_i \in GL^+(2, \mathbb{R})$ for $1 \leq i \leq s$. Using (2.4), we may write

$$\begin{aligned}
\int_{\alpha_i z_0}^{\alpha_i \gamma z_0} &= \int_{z_0}^{\xi_i(\gamma)\alpha_{i(\gamma)} z_0} - \int_{z_0}^{\alpha_i z_0} \\
&= \bigg(\int_{z_0}^{\xi_i(\gamma)\alpha_{i(\gamma)} z_0} - \rho_{2m}(\xi_i(\gamma)) \int_{z_0}^{\alpha_{i(\gamma)} z_0} \bigg) + \bigg(\rho_{2m}(\xi_i(\gamma)) \int_{z_0}^{\alpha_{i(\gamma)} z_0} - \int_{z_0}^{\alpha_i z_0} \bigg).
\end{aligned}$$

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However, we see that

$$\begin{aligned}\bigg(\int_{z_0}^{\xi_i(\gamma)\alpha_{i(\gamma)}z_0} &- \rho_{2m}(\xi_i(\gamma))\int_{z_0}^{\alpha_{i(\gamma)}z_0}\bigg) f_m(z)\mathbf{v}_{2m}(z)\,dz \\ &= \int_{z_0}^{\xi_i(\gamma)\alpha_{i(\gamma)}z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz - \int_{\xi_i(\gamma)z_0}^{\xi_i(\gamma)\alpha_{i(\gamma)}z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz \\ &= \int_{z_0}^{\xi_i(\gamma)z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz \\ &= \mathcal{L}_m(\Phi)(\xi_i(\gamma)),\end{aligned}$$

where we have used (4.7), and

$$\begin{aligned}\bigg(\rho_{2m}(\xi_i(\gamma))\int_{z_0}^{\alpha_{i(\gamma)}z_0} &- \int_{z_0}^{\alpha_i z_0}\bigg) f_m(z)\mathbf{v}_{2m}(z)\,dz \\ &= \rho_{2m}(\alpha_i)\rho_{2m}(\gamma)\rho_{2m}(\alpha_{i(\gamma)})^{-1}\int_{z_0}^{\alpha_{i(\gamma)}z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz \\ &\quad - \int_{z_0}^{\alpha_i z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz.\end{aligned}$$

Using the above relations, (2.13), and the fact that $\det \alpha_{i(\gamma)} = \det \alpha_i$, we obtain

$$\begin{aligned}&\mathcal{L}_m(T^{\mathcal{J}}(\alpha)\Phi)(\gamma) \\ &\quad= \sum_{i=1}^{s}(\det \alpha_i)^{m+1}\rho_{2m}(\alpha_i)^{-1}\mathcal{L}_m(\Phi)(\xi_i(\gamma)) \\ &\qquad + \rho_{2m}(\gamma)\sum_{i=1}^{s}(\det \alpha_{i(\gamma)})^{m+1}\rho_{2m}(\alpha_{i(\gamma)})^{-1}\int_{z_0}^{\alpha_{i(\gamma)}z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz \\ &\qquad + \sum_{i=1}^{s}(\det \alpha_i)^{m+1}\rho_{2m}(\alpha_i)^{-1}\int_{z_0}^{\alpha_{i(\gamma)}z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz \\ &\quad= (T_{2m}(\alpha)\mathcal{L}_m(\Phi))(\gamma) + (\rho_{2m}(\gamma) - 1_{2m+1})u,\end{aligned}$$

where 1_{2m+1} is the identity map on $\mathbb{C}^{2m+1}$, and

$$u = \sum_{i=1}^{s}(\det \alpha_i)^{m+1}\rho_{2m}(\alpha_i)^{-1}\int_{z_0}^{\alpha_i z_0} f_m(z)\mathbf{v}_{2m}(z)\,dz.$$

Hence the theorem follows. □

COROLLARY 4.3. *Let $\mathfrak{F}_{-1}^{-1} : \Psi\mathrm{DO}_{-1} \to \mathcal{J}(\Gamma)_1$ be the inverse of the isomorphism in (3.13) for $w = 1$, and set*

$$\mathcal{L}_m^{\partial} = \mathcal{L}_m \circ \mathfrak{F}_{-1}^{-1} : \Psi\mathrm{DO}_{-1} \to H^1(\Gamma, \mathfrak{S}^{2m}(\mathbb{C}^2)).$$

Then

$$\mathcal{L}_m^{\partial} \circ T^{\Psi}(\alpha) = T_{2m}(\alpha) \circ \mathcal{L}_m^{\partial}$$

for each $m \geq 1$.

PROOF. This follows immediately from Theorem 4.2 and the commutativity of the diagram (3.21). □

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