JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS Vol. 57, No. 7, Nov. 2022, pp. 2693–2723

doi:10.1017/S0022109022000473

The Shadow Costs of Illiquidity

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Abstract

We solve a flexible model that captures transactions costs and infrequencies of trading opportunities for illiquid assets to better understand the shadow costs of illiquidity for different origins of asset illiquidity and heterogeneous investor types. We show that illiquidity that results in suboptimal asset allocation carries low shadow costs, whereas these costs are high when illiquidity restricts consumption. As a result, the shadow costs are high for short-term investors, investors who face substantial liquidity shocks, and investors who desire to allocate a large fraction of their wealth to illiquid assets.

I. Introduction

Illiquid assets increasingly have a role in investors' portfolios. For instance, they account for 55% of the total portfolios of U.S. endowment funds in 2015 (Dimmock, Wang, and Yang (2019)). Moreover, the seven largest pension funds in the world have increased their average allocations of illiquid assets from 4% in 1997 to 25% in 2017 (Watson (2018)). One potential reason for investing in illiquid assets is to capture liquidity premiums (OECD (2014), Watson (2019)). In other words, investing in illiquid assets might compensate the investor for bearing liquidity risk. Yet, there is no consensus in the empirical literature on the question which asset classes have first-order liquidity premiums. In this study, we ask a related question, namely, how costly is illiquidity from the perspective of the investor for different origins of asset illiquidity? We answer this question by computing shadow costs of illiquidity. We define the shadow costs as the decrease in the expected return on the illiquid asset that a price-taking investor is willing to pay to convert the illiquid asset into a liquid one.

We thank an anonymous referee, Hendrik Bessembinder (the editor), Jules van Binsbergen, Marie Brière, Dirk Broeders, Joost Driessen, Esther Eiling, Frank de Jong, Ralph Koijen, Olivia Mitchell, Theo Nijman, Antoon Pelsser, Patrick Tuijp, seminar participants at Tilburg University, the Wharton School, Maastricht University, Ortec Finance Canada, Ortec Finance Rotterdam, Achmea Investment Management, Robeco, and conference participants of the 2018 Market Microstructure & High Frequency Data Stevanovich Center University of Chicago, 2019 SoFiE, 2018 Netspar Pension Day, and the 2019 International Pension Workshop for very useful comments. Jansen gratefully acknowledges financial support from Instituut Gak. Views expressed are those of the author and do not necessarily reflect official positions of De Nederlandsche Bank.

Studies have either modeled illiquidity as proportional transactions costs (e.g., Constantinides (1986) and Vayanos (1998)) or as the inability to trade illiquid assets for random time periods (e.g., Ang, Papanikolaou, and Westerfield (2014)). We combine these two dimensions of illiquidity here for two reasons. First, several asset classes exhibit both aspects of illiquidity simultaneously. For instance, investors in corporate bonds face transaction costs, yet sometimes specific bonds may not trade for several weeks. Similarly, selling real estate may take several years, and it obviously carries transaction costs as well. Second, combining both sources of illiquidity allows us to capture the heterogeneity across asset classes by adjusting the prominence of both effects.

We solve for shadow costs of illiquidity in a partial equilibrium power utility framework. The investor decides to optimally allocate his wealth to three assets: a risk-less asset, a liquid risky assets, and an illiquid risky asset. The investor optimizes utility over a stream of consumption and is exposed to exogenous liquidity shocks. The investor cannot borrow against the illiquid asset, which implies that consumption and liquidity shocks must be financed out of liquid wealth by selling risk-less and/or liquid risky assets. We then analyze the shadow costs of illiquidity for two different interpretations of the liquidity shock. The first interpretation models the liquidity shock as a sudden random decline in liquid wealth, whereas the second one models the liquidity shock as a forced increase in temporary consumption.

Our model's flexibility allows us to quantify the magnitude of the shadow costs for different origins of asset illiquidity and heterogeneous investor types. This flexibility is important as the costs of illiquidity highly depend on the characteristics of illiquid assets such as the frequency of trade and potential income returns on illiquid assets, as well as the characteristics of the investor types that hold these assets.

We show that the cost of illiquidity involves two aspects: suboptimal asset allocation and suboptimal consumption. If the cost of liquidating the illiquid asset is too high and the investor prefers not to trade the illiquid asset, or the illiquid asset cannot be traded at all, then illiquidity leads to suboptimal portfolio allocations. Yet, deviating from the optimal asset allocation generally only induces small utility costs (Constantinides (1986)). At the same time, illiquidity may lead to suboptimal consumption levels due to insufficient holdings of liquid assets, compared to the case where the illiquid asset can always be traded without costs. That is, if liquid wealth is too low to finance (optimal) consumption and the investor is unable to sell the illiquid asset or only at high costs, they may face a negative consumption shock. Shocks to consumption carry a high utility cost which, in turn, generates substantial shadow costs of illiquidity. Hence, we find that the shadow costs are large for short-term investors, investors who face substantial liquidity shocks, and investors who desire to allocate over 60% of their wealth to illiquid assets if the illiquid asset would also be liquid.

We perform back-of-the-envelope calculations to shed light on the shadow costs of illiquidity for several asset classes. Even though private equity is the most illiquid asset that we analyze, we find annual shadow costs in the range of 0–55 basis points only. Private equity investors have to lock up their money for long periods of time, and mainly for that reason, only long-term investors are present in this market. For these investors, illiquidity is unlikely to substantially harm consumption patterns. For direct real estate, we find annual shadow costs in the range of

0–71 basis points. Direct real estate can often not be traded for a substantial amount of time, the timing of the trading opportunities is uncertain, and the transaction costs are high. Yet, the threat of illiquidity is dampened because of the liquid return component (rents) of real estate investments and the typical long investment horizons for investors in this market. For corporate bonds, the annual shadow cost is in the range of 26–85 basis points. The transaction costs are small, but uncertainty in trading opportunities and its high price of risk amplify shadow costs. For illiquid stocks, we find annual shadow costs in the range of 0–108 basis points. Stocks trade very often and therefore the source of illiquidity is transaction costs. Transaction costs generate small shadow costs for long-term investors, but substantial ones for short-term investors.

A potential shortcoming of our approach is that we are not able to model investors' preferences perfectly. However, we make two assumptions that are likely to overestimate rather than to underestimate shadow costs of illiquidity. First, we assume that the investors cannot borrow against the illiquid assets. This caveat may be a realistic assumption for some asset classes, but not for others. For instance, real estate investors are typically able to borrow a substantial amount using the property as collateral. However, taking this borrowing into account decreases shadow costs because the investors can partially undo the illiquidity of the asset. Second, we allow for liquidity shocks as large as 50% of the investors' total wealth. Even though larger wealth shocks are in practice possible, our model shows that investors substantially reduce their risky asset allocation if faced with such shocks also in the fully liquid case. As a result, large liquidity shocks do not necessarily amplify shadow costs of illiquidity.

Our study contributes to the theoretical literature on liquidity premiums. The early theoretical literature did not find evidence for the existence of sizeable liquidity premiums. Constantinides (1986) and Vayanos (1998) show that transaction costs only have a second-order effect on prices (i.e., a 1% higher transaction cost increases the liquidity premiums by only a few basis points per year). After their work, a great deal of literature was developed that studies illiquidity or transactions costs by using assumptions more in line with real-world investment problems. This work finds that illiquidity can have first-order effects on prices. For instance, theoretical work from Huang (2003) and Garleanu (2009) shows that first-order effects on prices may arise when investors face borrowing constraints. Jang, Koo, Liu, and Loewenstein (2007) add return predictability to the investor's problem in a market with transactions costs and find a slight increase in liquidity premiums. Lynch and Tan (2011) solve a model that comprises labor income, wealth shocks, return predictability, and transaction costs and are able to generate liquidity premiums for stocks in the same order of magnitude as the early empirical literature. We contribute to this literature by i) combining two aspects of illiquidity: transactions costs and nontrading periods and ii) setting up a flexible model that allows to study these dimensions of illiquidity for investors that differ in investment horizons and liquidity needs.

Our study also contributes to the literature on optimal portfolio choice in the presence of jump risk (e.g., Liu, Longstaff, and Pan (2003), Das and Uppal (2004), Jin and Zhang (2012), and Liu and Loewenstein (2013)). For instance, Liu et al. (2003) study optimal asset allocations in the presence of jump risk in prices and volatility. They show that the asset allocation implications are equivalent to a setting where part of the portfolio is treated as being illiquid as in Longstaff (2001). The liquidity shock we model is comparable to a type of jump risk and has similar implications when looking at *total* wealth only: jump risks or liquidity shocks imply lower wealth which leads to lower optimal consumption levels. However, in our setting the liquidity shock can lead to *suboptimal* consumption levels as compared to the case with only liquid assets. As we explicitly model liquid and illiquid assets, our constraint that consumption and liquidity shocks can only be financed out of liquid wealth leads to the shadow cost of illiquidity.

Our study also relates to the empirical literature on liquidity premiums. The early empirical literature found significant effects of illiquidity on stock prices. For instance, Amihud and Mendelson (1986) and Brennan and Subrahmanyam (1996) show that a 1% higher transaction cost means a 1.5% to 2% higher expected return for stocks. Yet, some recent studies have challenged the empirical evidence for liquidity premiums in stocks. For instance, Ben-Rephael, Kadan, and Wohl (2015) show that liquidity premiums have become insignificant in recent decades for public US equities, except for very small stocks. First-order liquidity premiums also exist for corporate bonds (e.g., Chen, Lesmond, and Wei (2007), Bao, Pan, and Wang (2011), and Bongaerts, De Jong, and Driessen (2017)). In particular, Bongaerts et al. (2017) find an average (level) liquidity premium equal to 0.54% for corporate bonds that carry 0.52% transaction costs. Yet, Palhares and Richardson (2019) find only limited evidence for liquidity premiums for corporate bonds after using illiquidity-factor portfolios (i.e., a strategy that goes long in illiquid bonds and short in the liquid ones).[1]

Similarly, for private equity, there is no clear consensus regarding the existence of a liquidity premium, although the evidence is more indirect. Franzoni, Nowak, and Phalippou (2012) report no out-performance of private equity relative to public equity, while Harris, Jenkinson, and Kaplan (2014) find a substantial out-performance of 3% annually.[2] Finally, opposing indirect evidence also exists for real estate investments. Qian and Liu (2012) find a somewhat higher expected return for direct compared to indirect real estate, while Ang, Nabar, and Wald (2013) find comparable performance for direct and indirect real estate investments. Although our model implied shadow costs are not directly comparable to the empirically estimated liquidity premiums that are the result of general equilibrium outcomes, our model gives perspective on the order of magnitude of shadow costs that investors require for the illiquid asset to become liquid in these four asset classes.

The remainder of the study is organized as follows: Section II shows the theoretical framework of the model and describes the corresponding optimal strategies and the partial equilibrium implications for the shadow costs. We compute the model implied shadow costs for a range of different parameter values and provide simple calculations for shadow costs in several asset classes in Section III. Section IV concludes. The code for this project is included as Supplementary Material.

[1]Note that we refer here to the *level* of the liquidity premium. There are also studies on the liquidity *risk* premium (e.g., Pastor and Stambaugh (2003)).

[2]These findings do not necessarily imply a contradiction, because the reported out-performance in Franzoni et al. (2012) is corrected for the exposure to a liquidity *risk* factor, whereas the reported out-performance in Harris et al. (2014) is not.

II. Shadow Costs of Illiquidity: Theory

In this section, we model illiquidity as the inability to trade an asset frequently and by the cost that occurs when trading. This is formalized in Section II.A. In Section II.B, we describe the optimization problem of the investor and its solution is presented in Section II.C. Section II.D describes the numerical solution technique and Section II.E shows how we derive shadow costs of illiquidity.

A. Financial Market

The financial market consists of 3 assets: a risk-free asset B, a liquid risky asset S, and an illiquid risky asset denoted by X. The risk-free asset has a constant annual rate of return r_f and we denote its return over period h by $r_f^{(h)} = r_f h$. The liquid risky asset earns a nominal return r_t^S over the period $(t-h, t]$, while we denote the nominal return on the illiquid asset over the same period by r_t^X. All returns are continuously compounded. Further, we assume that the price of the illiquid asset is observed, even though it cannot be traded every period.

The prices of risk of the liquid and illiquid assets are denoted by λ_S and λ_X, respectively. Their volatilities are similarly denoted by σ_S and σ_X; their correlation by ρ_{SX}. The returns r_t^S and r_t^X are jointly normally distributed:

$$(1)\qquad \begin{bmatrix} r_t^S \\ r_t^X \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} r_f + \left(\lambda_S \sigma_S - \dfrac{1}{2}\sigma_S^2\right) \\ r_f + \left(\lambda_X \sigma_X - \dfrac{1}{2}\sigma_X^2\right) \end{bmatrix} h, \begin{bmatrix} \sigma_S^2 & \rho_{SX}\sigma_S\sigma_X \\ \rho_{SX}\sigma_S\sigma_X & \sigma_X^2 \end{bmatrix} h \right).$$

The differences between the liquid and the illiquid risky asset are the trading opportunities and transaction costs. While the investor can always trade the liquid risky asset S at no cost, the illiquid asset X can only be traded at infrequent points in time and at a cost. We denote the trading indicator for time t with $\mathbf{1}_t^T$ and interpret $\mathbf{1}_t^T = 1$ as a trading opportunity that arises for illiquid asset X, while $\mathbf{1}_t^T = 0$ indicates that the illiquid asset cannot be traded during that period. We assume throughout that the trading indicators are IID Bernoulli random variables. We determine the trading probability by assuming that trading opportunities arrive according to a Poisson process with intensity η. The probability that the investor is able to trade in a given period then equals $p = \mathbb{P}\left\{\mathbf{1}_t^T = 1\right\} = 1 - \exp(-\eta h)$. If a trading opportunity occurs ($\mathbf{1}_t^T = 1$) and the investor decides to trade, then the proportional transaction costs ϕ must be paid, $0 \le \phi \le 1$. We allow for two types of return components on the illiquid asset: income return and capital gains. That is, we separate r_t^X into a liquid part d (income return) and an illiquid part $r_t^X - d$ (capital gains).

B. The Investors' Consumption and Investment Problem

Preferences are represented by a standard constant relative risk aversion (CRRA) expected utility function. The investor has an investment horizon equal to T. We assume that the illiquid asset can be traded (and thus liquidated) against transaction costs ϕ at the final date T. Notice that under the assumption that the illiquid asset might not be liquidated at T, the shadow costs are obviously amplified. We do not consider this because, in that case, the investor is better off postponing the liquidation of the illiquid asset until a trading opportunity arises.

We introduce the following notation to distinguish liquid and illiquid wealth. We denote liquid wealth as available at time t by W_t. This wealth consists of investments in both the risk-free asset B and the liquid risky asset S. The value of the investment in the illiquid asset at time t is denoted by X_t. Therefore, total wealth equals $W_t + X_t$. We denote the fraction of liquid wealth W_t that is invested in the liquid risky asset S by θ_t, and $1-\theta_t$ is invested in the risk-free asset B. Consumption at time t is denoted by C_t and must be financed from liquid wealth W_t.

Illiquid wealth X_t can only be converted into liquid wealth (and, if desired, immediately consumed) if a trading opportunity arises (i.e., if $\mathbf{1}_t^T = 1$). We denote the transfer from liquid to illiquid wealth by ΔX_t. Thus, $\Delta X_t > 0$ means that at time t, an additional amount ΔX_t of the illiquid asset is bought, and thus liquid wealth W_t decreases by $\Delta X_t + \phi \varDelta X_t$. If no trading opportunity arises, then $\mathbf{1}_t^T = 0$; and we automatically have $\Delta X_t = 0$.

Furthermore, we assume that the investor may face a liquidity shock L_t that is assumed to be a nonrandom fraction l of total wealth $W_t + X_t$ that can occur at most once during the interval $(t-h, t]$, where $0 \leq l < 1$ (the size of the liquidity shock is strictly smaller than total wealth). We denote the liquidity shock indicator for time t with $\mathbf{1}_t^L$ and interpret $\mathbf{1}_t^L = 1$ as the occurrence of a liquidity shock, while $\mathbf{1}_t^L = 0$ indicates that no liquidity shock arises during that period. We assume throughout that the liquidity shock indicators are IID Bernoulli random variables. We determine the liquidity shock probability by assuming that the liquidity shocks arrive according to a Poisson process with intensity υ. The probability that the investor faces a liquidity shock in a given period then equals $q = \mathbb{P}\{\mathbf{1}_t^L = 1\} = 1 - \exp(-\upsilon h)$. If a liquidity shock arises, the investor has to pay the liquidity shock out of liquid wealth, even though the size of the liquidity shock depends on its total wealth. This means that the investor cannot avoid the liquidity shock by investing a large fraction of his wealth in the illiquid asset.

We provide two examples to motivate why we model the liquidity shock as a fraction of total wealth, but assume that the liquidity shock has to be financed from the liquid part of the portfolio. The first example concerns a wealthy private investor that faces a sudden raise in wealth taxes.[3] A wealth tax is generally imposed on the total value of all assets and, hence, on total wealth. Although the (additional) tax amount is a fraction of total wealth, these taxes have to be financed from the liquid part of the investor's portfolio. Put differently, the investor has to pay the taxes with cash, and cannot use illiquid assets to pay the tax invoice. The investor must use his liquid wealth by either selling the risk-less asset B or the liquid risky asset S. Only if a trading opportunity in the illiquid assets arises at exactly the same time this wealth tax is due, the investor is also able to sell the illiquid asset and use the proceeds to pay the tax invoice. The lower the trading probability of the illiquid asset, the lower the likelihood that the investor is able to use illiquid wealth to pay the tax invoice.

[3]For instance, in 2001, the wealth tax in the Netherlands went up from 0.7 to 1.2 percentage points: https://www.cpb.nl/sites/default/files/publicaties/download/cpb-discussion-paper-273-saving-behavior-and-risk-taking.pdf.

The second example concerns an institutional investor facing margin calls, resulting from the use of derivatives.[4] For instance, pension funds can (partly) hedge their interest rate risk that results from the long-term nature of the pension liabilities. Typically, pension funds define their interest rate hedge as a fraction of the total liabilities which is thus related to total assets. Suppose a pension fund desires to hedge 50% of its interest rate risk, then the total notional to achieve this hedge is higher when the fund has more (total) liabilities. Because pension funds are typically net receiver swap holders, an increase in interest rates may result in margin calls. That is, the pension fund has to post cash (or highly liquid assets) on a margin account. The larger the notional amount of the receiver swap, the larger the amount of cash that is needed to finance the margin. Thus, margin calls occur as a fraction of the pension fund's total assets, but they can be financed out of liquid wealth only. Again, the investor would have to either sell the risk-less asset B or the liquid risky asset S. Only if a trading opportunity in the illiquid asset occurs at the time of the margin call, the pension fund is able to sell the illiquid asset as well and use the proceeds to finance the margin. The exact same mechanism holds for other types of derivatives (e.g., currency and credit derivatives).

Furthermore, we consider two different interpretations of the liquidity shock. First, the interpretation of the liquidity shock as a sudden random decrease in liquid wealth (i.e., the investor does not receive utility from the liquidity shock). For retail investors, one example is again a raise in wealth taxes or another example could be extreme weather events. For institutional investors, a possible random shock to liquid wealth is a margin call on derivative positions as just described. Second, the liquidity shock can be interpreted as a forced increase in temporary consumption and hence in utility. Examples include some health care costs and unforeseen expenditures for retail investors. For long-term investors such as pension funds, mortality shocks can be interpreted as temporary increases in utility, assuming that utility is measured by lifetime consumption. In these examples, the investor faces a temporary increase in consumption of which it benefits during the period these costs materialize. Our baseline model assumes the first interpretation, but we also study the implications of the alternative setting in Section III.

We now turn to the optimization problem of the investor. The investor optimizes its utility of a stream of consumption levels C_t over a horizon $t = 0, h, 2h, \ldots, T$. Thus, the criterion function is

$$\mathbb{E}_0\left[\sum_{t\in\{0,h,2h,\ldots,T\}} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}\right], \tag{2}$$

[4]The importance of margin requirements has substantially increased over the past decade. In response to the aftermath of the financial crisis, the European Union agreed on a new legislative framework to mitigate systemic risk for over-the-counter derivatives in 2012, the European Market Infrastructure Regulation (EMIR). The EMIR legislation is relevant in the context of illiquid assets, as EMIR requires central clearing of derivative contracts. Central clearing implies that each counterparty of an OTC derivative contract should post an initial margin at the clearing house on entry of the contract, and a variation margin when the value of the derivative contract changes (https://www.bis.org/bcbs/publ/d317.pdf).

where β denotes the time-preference discount factor and $\gamma > 1$ is the risk-aversion parameter.

The investor faces two budget constraints: one for liquid wealth W_t and one for illiquid wealth X_t. Formally, we have

$$\text{(3)} \quad W_t = (W_{t-h} - \Delta X_{t-h} - \phi|\Delta X_{t-h}| - C_{t-h} - L_{t-h}) \left(\exp\left(r_f^{(h)}\right) + \theta_{t-h}\left[\exp\left(r_t^S\right) - \exp\left(r_f^{(h)}\right)\right]\right) + (X_{t-h} + \Delta X_{t-h})\exp(d),$$

$$\text{(4)} \quad X_t = (X_{t-h} + \Delta X_{t-h})\left(\exp\left(r_t^X\right) - \exp(d)\right).$$

We assume that the investor cannot borrow against the illiquid asset. The effect of illiquidity would be strongly reduced if this borrowing was possible, as the investor could always undo the illiquidity by borrowing against the illiquid asset if needed. Thus, we impose

$$\text{(5)} \quad C_t \leq W_t - L_t, t = 0, h, 2h, \ldots, T.$$

The borrowing constraint implies that the investor can only finance consumption out of *liquid* wealth, after liquidity shocks. This means that consumption can only be generated from selling the risk-less bond B and/or selling the liquid risky stock S. Only if a trading opportunity arises for the illiquid asset when the investor desires to consume, the investor can also use the illiquid asset to finance consumption at exactly the same moment in time. Hence, illiquidity impacts the investor because he may not be able to attain the desired consumption level due to insufficient holdings of liquid assets. Furthermore, the borrowing constraint implies that the investor's liquid wealth will always be larger than the liquidity shock L_t (i.e., $W_t > L_t$ for all t, as zero consumption leads to negative infinite utility in (2)). In other words, if the investor locks up a substantial amount of its wealth in the illiquid asset and a liquidity shock occurs, the investor faces the risk of not having sufficient liquid wealth left to consume, a scenario the investor avoids at all times.

The optimal consumption problem can now be stated as follows:

Problem II.1. The investor maximizes

$$\text{(6)} \quad \max_{\{\theta_t, \Delta X_t, C_t\}_{t=0}^T} \mathbb{E}_0 \left[\sum_{t \in \{0, h, 2h, \ldots, T\}} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}\right]$$

subject to the budget constraints (3) and (4) and the borrowing constraint (5). Moreover, when $1_t^T = 0$, we must have $\Delta X_t = 0$.

The decision variables C_t, θ_t, and ΔX_t are nonanticipative. Formally, $\{C_t, \theta_t, \Delta X_t\}$ is adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=1}^T$, where $\mathcal{F}_t$ is the natural filtration generated by $\left\{r_t^S, r_t^X, \mathbf{1}_t^T, \mathbf{1}_t^L\right\}$.

In the setting where we assume that the liquidity shock results in a utility gain, the optimization problem of the investor becomes as follows.

Problem II.2. The investor maximizes

$$\max_{\{C_t,\theta_t,\Delta X_t\}_{t=0}^{T}} \mathbb{E}_0\left[\sum_{t\in\{0,h,2h,\ldots,T\}} \beta^t \frac{(C_t+L_t)^{1-\gamma}}{1-\gamma}\right] \tag{7}$$

subject to the budget constraints (3) and (4) and the borrowing constraint (5). Moreover, when $1_t^T=0$, we must have $\Delta X_t=0$.

To summarize, illiquidity limits the investor's consumption and investment decisions in three ways compared to the case where the illiquid asset is fully liquid, that is, the two risky asset Merton case (Merton (1969)): the inability to trade the illiquid assets for uncertain periods of time; transaction costs of the illiquid asset when a trading opportunity arises; and the investor cannot borrow against the illiquid assets. All three assumptions are important characteristics of (most) illiquid assets.

C. Optimal Strategies

The optimization in Problem II.1 cannot be solved analytically, so we resort to numerical methods. For these problems, the numerical complexity is well-known to strongly increase with the number of endogenous state variables. In the formulation of Problem II.1, there are two: W_t and X_t. Yet, in line with Ang et al. (2014), a simple transformation leads to a partly analytical result due to the homogeneity of the CRRA utility function we consider, see Theorem II.3 below.

More precisely, we consider as endogenous state variables the total wealth W_t+X_t and the fraction of total wealth invested in the illiquid asset, that is,

$$\xi_t=\frac{X_t}{W_t+X_t}. \tag{8}$$

With this reparametrization, we define the value function using the Bellman principle as

$$V_t(W_t+X_t,\xi_t)=\max_{C_t,\theta_t,\xi_t}\beta^t\frac{C_t^{1-\gamma}}{1-\gamma}+\mathbb{E}_t V_{t+h}(W_{t+h}+X_{t+h},\xi_{t+h}), \tag{9}$$

with the boundary condition at time T given by

$$V_T(W_T+X_T,\xi_T)=\beta^T\frac{(W_T+(1-\phi)X_T)^{1-\gamma}}{1-\gamma}. \tag{10}$$

The boundary condition means that we assume that all assets can be traded (and thus liquidated) at time T against transaction costs ϕ. Moreover, the optimal investment decision concerning illiquid wealth (i.e., ΔX_t^*) is determined by the choice ξ_t^*

$$\Delta X_t^*=\begin{cases}(\xi_t^*-\xi_t)(W_t+X_t) & \text{if } \mathbf{1}_t^T=1\\ 0 & \text{if } \mathbf{1}_t^T=0.\end{cases} \tag{11}$$

https://doi.org/10.1017/S0022109022000473 Published online by Cambridge University Press

With the above-introduced change of variables, that is, the pair (W_t, X_t) is replaced by the pair $(W_t + X_t, \xi_t)$, the solution to the investor's problem satisfies the following theorem. A proof is provided in Appendix A.

Theorem II.3. There are time-dependent (deterministic) functions α_t, θ_t, and H_t such that the optimal solution $\{C_t^*, \theta_t^*, \xi_t^*\}$ to Problem II.1 can be written as:

$$V_t(W_t + X_t, \xi_t) = \beta^t \frac{(W_t + X_t)^{1-\gamma}}{1-\gamma} H_t(\xi_t), \tag{12}$$

$$C_t^* = \alpha_t(\xi_t)(W_t + X_t), \tag{13}$$

$$\theta_t^* = \theta_t(\xi_t), \tag{14}$$

$$\xi_t^* = \underset{\xi_t}{\arg\min}\, H_t(\xi_t). \tag{15}$$

The function $H_t(\xi_t)$ can be viewed as a penalty function, which is minimized at the optimal fraction of total wealth invested in the illiquid asset ξ_t^*. If the investor is able to trade the illiquid asset at time t, then they will rebalance their portfolio toward the optimal ratio of illiquid wealth to total wealth ξ_t^*, if the decrease in the penalty function is sufficient to outweigh the transaction cost ϕ. Thus, in line with Constantinides (1986), there is a no-trading region where the investor will not rebalance their portfolio.

Theorem II.3 furthermore indicates that the optimal consumption choice and the optimal investment strategy in the liquid risky asset depend on the fraction of total wealth invested in the illiquid asset ξ_t. As we show in Section III, if illiquid wealth is substantial relative to liquid wealth, for instance after a liquidity shock L_t occurs; then the investor might have to cut their consumption relative to the case where the illiquid asset can always be traded. Moreover, to compensate for the increased risk exposure that results from the high fraction invested in illiquid wealth, the investor reduces their allocation to the liquid risky asset.

D. Solving the Model

The model is solved by means of backward induction, where we start solving the problem at the final date $t = T$ and solve the model backward for each period until arriving at time $t = 0$. The advantage of Theorem II.3 is that the dependence of the value function on total wealth $W_t + X_t$ is known analytically. The fact that the value function is proportional to $(W_t + X_t)^{1-\gamma}$ simplifies the numerical optimization to a 1-dimensional grid search over ξ_t only. Details on how we solve the model are provided in Appendix B.

E. Willingness to Pay for Liquidity

To understand why illiquidity is costly in some cases but not in others, we analyze the willingness to pay for liquidity. We define the investor's willingness to pay δ_t as the decrease in the expected return on the illiquid asset over period

$(t-h,t]$ that they are willing to pay to convert the illiquid asset into a liquid one. In other words, δ_t can be interpreted as a shadow cost and is the compensation the investor demands for holding the illiquid asset. To formalize the willingness to pay, denote the value function for Problem II.1 by assuming that the asset X is actually also liquid by $V_t^{LIQ}(W_t+X_t)$. In other words, we solve Problem II.1 subject to the budget constraints (3) and (4), where $\eta \to \infty$, $\phi = 0$. This value function factorizes as:

$$V_t^{LIQ}(W_t+X_t) = \beta^t \frac{(W_t+X_t)^{1-\gamma}}{1-\gamma} H_t^{LIQ}, \tag{16}$$

for a deterministic constant H_t^{LIQ}, where H_t^{LIQ} no longer depends on ξ_t as the illiquid asset is tradeable as well.

The value function V_t^{LIQ} depends on the expected return $\left(r_f+\lambda_X\sigma_X-0.5\sigma_X^2\right)h$ of asset X. Subtracting δ_t from this expected return leads to a (lower) value function that we denote by $V_t^{LIQ}(W_t+X_t|\delta_t)$. We then define the willingness to pay as the value of δ_t that solves

$$V_t^{LIQ}(W_t+X_t|\delta_t) = V_t(W_t+X_t,\xi_t). \tag{17}$$

Given (12) and (16), we can find δ_t by solving

$$H_t^{LIQ}(\delta_t) = H_t(\xi_t), \tag{18}$$

where $H_t^{LIQ}(\delta_t)$ denotes the penalty function when the illiquid asset is actually liquid at a risk premium reduced by δ_t. This willingness to pay or shadow cost depends on the actual allocation to the illiquid asset: ξ_t. To determine the shadow cost, we assume that an investor with horizon T chooses the optimal allocation to the illiquid asset when entering the investment, so the actual allocation at time $t=0$ equals $\xi_{t=0}^T = \xi_{t=0}^{T,*}$.

III. Shadow Costs of Illiquidity: Comparative Statics

We now turn to the qualitative and quantitative implications of the model. First, we show how illiquidity affects the asset allocation and consumption patterns of the investor. We then compute the shadow costs of illiquidity depending on different parameter configurations. We end the section by computing model implied shadow costs for different asset classes. For ease of interpretation, all the shadow costs are annualized in the subsequent sections.

A. Parameter Values Baseline Model

With respect to the investor's preferences, we assume the investor faces a liquidity shock with intensity $\upsilon = 10\%$, which implies the liquidity shock occurs on average once in 10 years and the probability of a shock each month equals $q = 0.83\%$. The magnitude of the liquidity shock is equal to $l = 30\%$ of total wealth (i.e., liquid and illiquid wealth combined). The investor has a risk-aversion

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parameter equal to $\gamma = 5$ and the time-preference discount factor equals $\beta = 0.91$.[5] We assume that the investor can consume and trade each month and hence $h = 1/12$.

With respect to the financial market, we assume the liquid asset has a price of risk $\lambda_S = 38\%$ and volatility $\sigma_S = 18.5\%$, and the risk-free rate is $r_f = 2\%$. These parameter values result in an optimal risky asset allocation of approximately 40% and a risk-free bond allocation of 60%. The parameter values of the illiquid asset are set equal to the parameter values of the liquid risky asset: $\lambda_X = 38\%$ and $\sigma_X = 18.5\%$. In this way, we isolate the effect of illiquidity instead of relying on a higher Sharpe ratio for the illiquid asset. In line with this reasoning, we also assume no correlation between the liquid and illiquid risky assets in the baseline model; $\rho_{SX} = 0$. We also assume no income return for the illiquid assets (i.e., $d_t = 0$). For the illiquidity parameters, we assume that the investor can trade the illiquid asset on average once in 2 years, or in other words trading opportunities occur with intensity $\eta = 50\%$, which implies a trading probability each month equal to $p = 4.08\%$. If the investor decides to trade the illiquid asset when a trading opportunity arises, then the proportional transactions costs equal $\phi = 1\%$. At the final date, the investor has to pay transaction costs $\phi = 1\%$ in all states of the world. Table 1 summarizes the parameter values for the baseline model.

B. Optimal Consumption and Asset Allocation Baseline Model

We now describe the optimal consumption pattern and asset allocation decisions for the baseline model described in Section II. Both consumption and the allocation to the liquid risky assets are functions of the investment horizon and the fraction invested in the illiquid asset, as derived in Theorem II.3.

Figure 1 shows that the optimal consumption decreases as a function of time and the fraction of wealth invested in the illiquid asset. To smooth consumption over time, the shorter the investment horizon, the larger the fraction the investor optimally desires to consume out of his total wealth. Moreover, the larger the fraction of illiquid wealth relative to total wealth, the less room for the investor to consume, because of the constraint that consumption and liquidity shocks can only be financed out of liquid wealth.

The allocation to the liquid risky asset (after consumption) is stable over time, because investment opportunities are constant.[6] However, Figure 2 shows that the liquid risky asset allocation is decreasing as a function of the allocation to illiquid wealth. If the fraction allocated to illiquid wealth is high, the investor reduces its total exposure to market risk by investing less in the liquid risky asset. When the fraction invested in illiquid wealth gets closer to 1, zero liquid wealth might be left after consumption and potential liquidity shocks, and hence the allocation to the liquid risky asset turns zero as well.

The optimal allocation to the illiquid asset (after consumption) is in Figure 3 and is found by minimizing the penalty function $H_t(\xi_t)$, as shown in Theorem II.3. We compare this allocation to the case where the illiquid asset is fully liquid, that is, the two risky asset Merton case (Merton (1969)). Generally, the shorter the

[5] A lower value for the time-preference discount factor has a negligible effect on shadow costs.

[6] Because we simulate the returns each month, the optimal allocation to the liquid risky asset varies slightly from month to month.

TABLE 1

Parameter Values

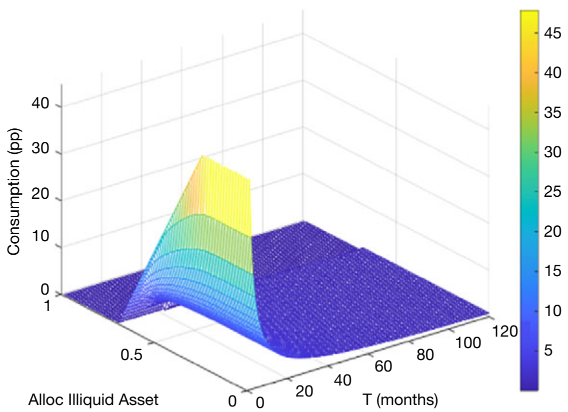
Table 1 summarizes the parameter values of our baseline model.

Parameters	Symbol	Value
Trading frequency risky assets	h	1/12
Liquidity shock	l	30%
Intensity liquidity shock	υ	10%
Risk aversion parameter	γ	5
Time-preference discount factor	β	0.91
Risk-free rate	r_f	2%
Price of risk liquid risky asset	λ_S	38%
Volatility liquid risky asset	σ_S	18.5%
Price of risk illiquid asset	λ_X	38%
Volatility illiquid asset	σ_X	18.5%
Correlation coefficient	ρ	0
Income return	d	0
Intensity trading opportunity illiquid asset	η	50%
Transaction costs illiquid asset	ϕ	1%

FIGURE 1

Optimal Consumption

Figure 1 shows the optimal consumption as a function of the investment horizon T and fraction invested in the illiquid asset that uses the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$ with intensity $\upsilon = 10\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the income return $d = 0$, the trading intensity of the illiquid asset $\eta = 50\%$, and the transactions costs $\phi = 1\%$.



investment horizon, the lower the optimal allocation to the illiquid asset relative to the Merton case. The kink in the illiquid asset allocation for horizons between 4 to 12 months results from the way we model the final date. At horizons of 4 to 6 months, the investor knows for sure he is able to liquidate illiquid wealth at the final date, with a very small probability of facing liquidity shocks. Even though he has to pay transaction costs to liquidate illiquid wealth at the final date, the period is sufficiently long to earn investment returns that outweigh these transaction costs in expectation. On the other hand, at horizons between 7 and 12 months, the risk of facing liquidity shocks before the final date increases, resulting in a lower allocation to the illiquid asset.

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FIGURE 2

Optimal Liquid Risky Asset Allocation

Figure 2 shows the optimal liquid risky asset allocation as a function of the investment horizon T and fraction invested in the illiquid asset that uses the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$ with intensity $\upsilon = 10\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the income return $d = 0$, the trading intensity of the illiquid asset $\eta = 50\%$, and the transactions costs $\phi = 1\%$.

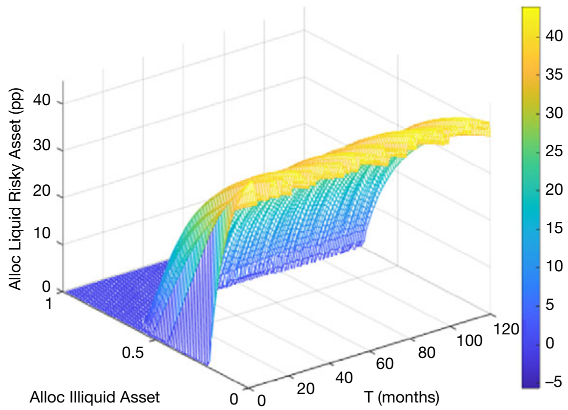
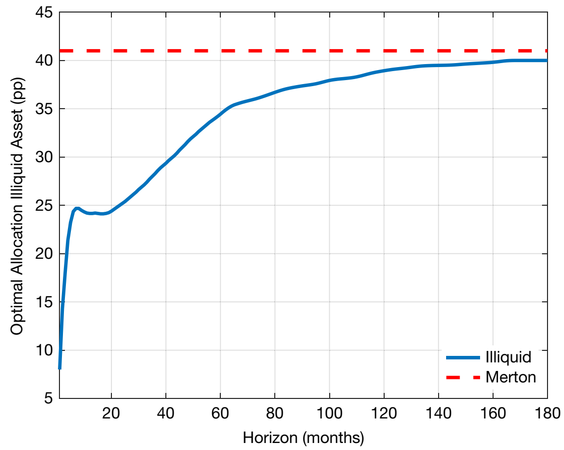


FIGURE 3

Optimal Illiquid Asset Allocation

Figure 3 shows the optimal illiquid asset allocation as a function of the investment horizon T that uses the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$ with intensity $\upsilon = 10\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the income return $d = 0$, the trading intensity of the illiquid asset $\eta = 50\%$, and the transactions costs $\phi = 1\%$.

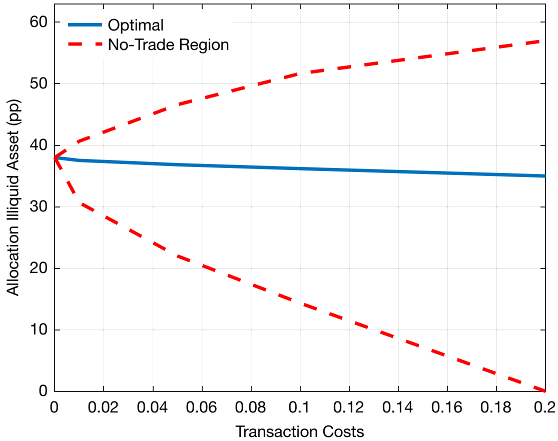


The no-trading region induced by transaction costs is in Figure 4. The solid line represents the optimal allocation to the illiquid assets given transaction costs ϕ, and the dashed lines represent the boundaries of the no-trading region. As long as the illiquid asset allocation is within the no-trading region (i.e., the area within the two dashed lines), the investor does not trade if a trading opportunity arises, while

FIGURE 4

No-Trading Region

Figure 4 shows the no-trading region of the investor with an investment horizon $T = 10$ (in years) that uses the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$ with intensity $\upsilon = 10\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the income return $d = 0$, the trading intensity of the illiquid asset $\eta = 50\%$, and the transactions costs $\phi = 1\%$.



the investor rebalances back to the optimal allocation when outside of the no-trading region. Compared to Constantinides (1986), the upper bound of the no-trading region is lower in our model; an over-investment in the illiquid asset relative to the optimal amount may prevent the investor from smoothing consumption due to the borrowing constraints and/or potential liquidity shocks. In order to avoid these states of the world, they re-balance back to the optimal illiquid asset allocation more quickly as opposed to under-investment in the illiquid asset.

In the next subsections, we convert the suboptimal consumption and asset allocation patterns resulting from illiquidity to implications for shadow costs of holding illiquid assets.

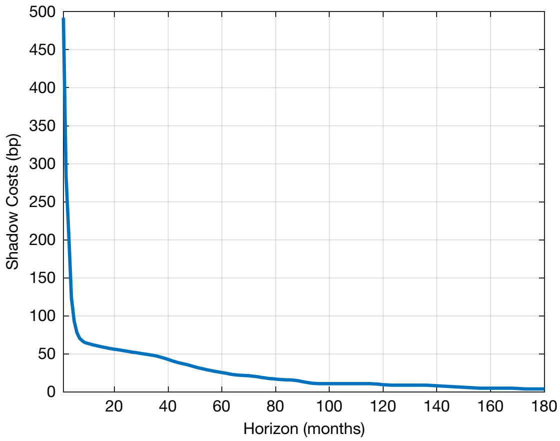
C. Shadow Costs and the Investment Horizon

Perhaps the most straightforward, but nevertheless important result, is that the willingness to pay depends on the investment horizon. Figure 5 shows that the shorter the investor's investment horizon amplifies the shadow cost. Illiquidity is typically a bigger threat for short-term investors as they want to consume or payout a large part of their total wealth compared to long-term investors. Under the baseline parameters, the shadow cost demanded by an investor with a horizon equal to a month equals 490 basis points, while this cost converges to only a few basis points as the horizon T becomes large. The steep decline in shadow costs at the 1 to 3 months horizons results from transaction costs: At the 1 and 2 months investment horizons, the expected excess return on the illiquid asset is below the 1% transaction costs, making the illiquid asset unattractive. On the other hand, for an investor with a horizon equal to 3 months, the expected excess return exceeds the transaction costs.

FIGURE 5

The Shadow Costs As a Function of the Investment Horizon

Figure 5 shows the shadow costs as a function of the investment horizon T (in months) that assumes the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$ with intensity $\upsilon = 10\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the income return $d = 0$, the trading intensity of the illiquid asset $\eta = 50\%$, and the transactions costs $\phi = 1\%$.



D. Shadow Costs and Trading Opportunities

Higher trading opportunities decrease the willingness to pay for illiquidity. Figure 6 shows that for the short-term investor ($T = 1$ year) the shadow cost equals 60 basis points if the investor is unable to trade the illiquid asset before the final date and decreases to 31 basis points when the probability to trade each month is high. For long-term investors ($T = 10$ years), the shadow cost equals 20 basis points if the investor is unable to trade the illiquid asset before the final date and converges to zero if the probability to trade each month is high. The relation between the trading probability and the shadow cost is approximately linear for the short-term investor but decreases exponentially for the long-term investor. The short-term investor knows for sure that they are able to trade in 12 months from now on, so the trading probability only affects trading opportunities in the upcoming 11 months. However, the inability to trade lengthens with a lower trading probability for the long-term investor. As a result, the probability of scenarios where illiquid wealth grows too fast relative to liquid wealth increases more rapidly at low trading probabilities for the long-term investor compared to the short-term investor.

E. Shadow Costs and Transaction Costs

Figure 7 shows that rising transaction costs increase the willingness to pay for both short-term ($T = 1$ year) and long-term ($T = 10$ year) investors. Nevertheless, the increase is more substantial for the short-term investor, such that a 1 percentage point increase in transaction costs increases the shadow cost demanded by approximately 10 basis points. Short-term investors always liquidate their illiquid wealth 12 months from now, and they can only do so at cost ϕ. Long-term investors only

FIGURE 6

The Shadow Costs As a Function of the Trading Probability

Figure 6 shows the shadow costs as a function of the monthly trading probability p $(= 1 - \exp(-\eta h))$ for the investor at horizons $T = 1$ and $T = 10$ (in years) that assumes the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$ with intensity $\upsilon = 10\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the income return $d = 0$, and the transactions costs $\phi = 1\%$.

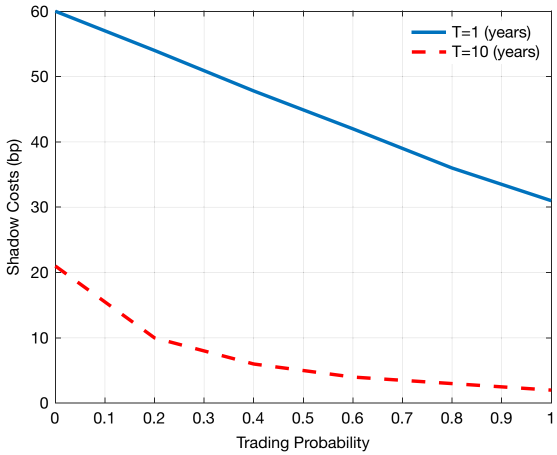
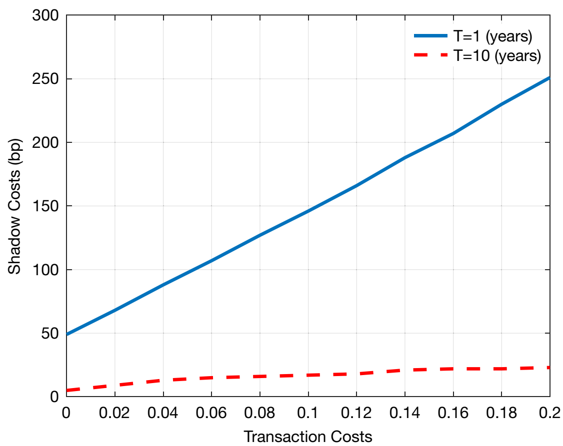


FIGURE 7

The Shadow Costs As a Function of the Transaction Costs

Figure 7 shows the shadow costs as a function of the transaction costs ϕ for the investor at horizons $T = 1$ and $T = 10$ (in years) that assumes the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$ with intensity $\upsilon = 10\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the income return $d = 0$, and the trading intensity of the illiquid asset $\eta = 50\%$.



liquidate their illiquid wealth at proportional cost ϕ if the realized illiquid asset allocation deviates too much from the optimal level. As a result, the shadow cost increases by only 1–2 basis points when transaction costs increase by 1 percentage point. This increase confirms earlier results from Constantinides (1986),

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where the investor has an infinite horizon and transactions costs endogenously decrease their trading frequency in the illiquid asset. The larger the transaction costs, the larger the investor's no-trading region. As the investor's value function is fairly insensitive to small deviations from the optimal (nontransaction) portfolio allocation, the transaction costs lead to second-order effects on shadow costs.

F. Shadow Costs and Liquidity Shocks

Figure 8 shows that the shadow cost is hump-shaped in the level of the liquidity shock. This hump-shaped relation means that the shadow cost is amplified when the level of the liquidity shock increases up to a shock of $l = 50\%$ for the short-term investors and $l = 60\%$ for the long-term investors, but decreases again for larger shocks. Because a liquidity shock can only be financed out of liquid wealth, a shock increases the probability that liquid wealth becomes insufficient to fulfill consumption needs. In order to prevent these states of the world, the investor reduces their optimal allocation to the illiquid asset substantially as compared to the liquid case and shadow costs increase. However, when the liquidity shock gets too severe, the allocation to the illiquid asset if it were fully liquid also gets closer to zero. Such large liquidity shocks make risky assets unattractive also in the fully liquid case, and as a result, shadow costs drop.

For the baseline parameter values, Figure 9 shows that the shadow costs are slightly lower when we model the liquidity shock as a temporary increase in consumption (Problem II.2). To remain able to smooth consumption, the optimal consumption level decreases if the liquidity shock results in a temporary increase in consumption (Panel A of Table 2). Hence, the budget constraint for liquid wealth, equation (3), becomes less stringent and shadow costs drop.

This effect becomes more apparent if we increase the occurrence of the liquidity shock compared to the baseline where the liquidity shock occurs only once in 10 years. A high probability of facing liquidity shocks reduces shadow costs in general, because consumption decreases more substantially also if the illiquid asset is fully liquid and the budget constraint for liquid wealth becomes less binding. As a result, both the dashed red and the dotted-dashed purple line that represent a monthly liquidity shock probability of $q = 90\%$ are below the lines of the cases where the liquidity shock occurs with probability $q = 0.83\%$. Yet, if the liquidity shock occurs frequently, the optimal consumption level decreases more substantially to ensure consumption smoothing over time in case the liquidity shocks increase temporary consumption, reducing the shadow costs (Table 2, Panel B).[7] If the liquidity shock occurs with probability $q = 90\%$ each month, the shadow costs equal to 38 basis points for the interpretation of the liquidity shock as a sudden decrease in liquid wealth for the investor with a 1-year horizon, but reduce to 16 basis points when the liquidity shock increases temporary consumption.

[7]For a high probability of facing liquidity shocks, consumption turns on average negative if the liquidity shock increases temporary consumption, however, $L_t + C_t$ remains positive and the objective function in Problem II.2 is identified.

FIGURE 8

The Shadow Costs As a Function of the Level Liquidity Shock

Figure 8 shows the shadow costs as a function of the level liquidity shock l for the investor at horizon $T=1$ and $T=10$ (in years) that assumes the following parameter values: the risk-aversion parameter $\gamma=5$, the time-preference discount factor $\beta=0.91$, intensity of the liquidity shock $\upsilon=10\%$, the return on the risk-free rate $r_f=2\%$, the average return on the liquid and illiquid risky asset $\mu_S=\mu_X=9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S=\sigma_X=18.5\%$, the correlation coefficient $\rho_{SX}=0$, the income return $d=0$, the trading intensity of the illiquid asset $\eta=50\%$, and the transactions costs $\phi=1\%$.

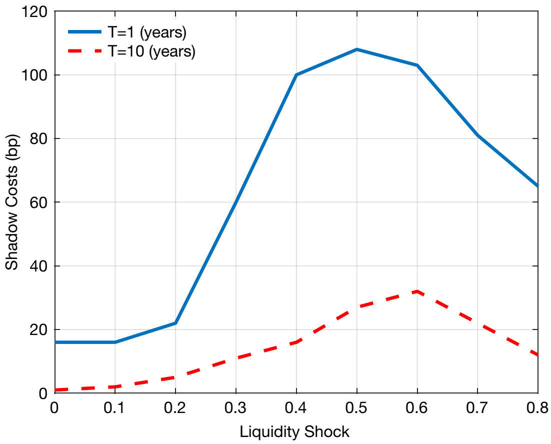


FIGURE 9

The Shadow Costs and Two Interpretations for the Liquidity Shock

Figure 9 shows the shadow costs as a function of the investment horizon T ($\geq$2), comparing the liquidity shock modeled as a temporary increase in consumption to the baseline specification. We show the results for monthly liquidity shock probabilities $q=0.83\%$ and $q=90\%$. We use the following parameter values: the risk-aversion parameter $\gamma=5$, the time-preference discount factor $\beta=0.91$, a liquidity shock $l=30\%$, the return on the risk-free rate $r_f=2\%$, the average return on the liquid and illiquid risky asset $\mu_S=\mu_X=9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S=\sigma_X=18.5\%$, the correlation coefficient $\rho_{SX}=0$, the income return $d=0$, the trading intensity of the illiquid asset $\eta=50\%$, and the transactions costs $\phi=1\%$.

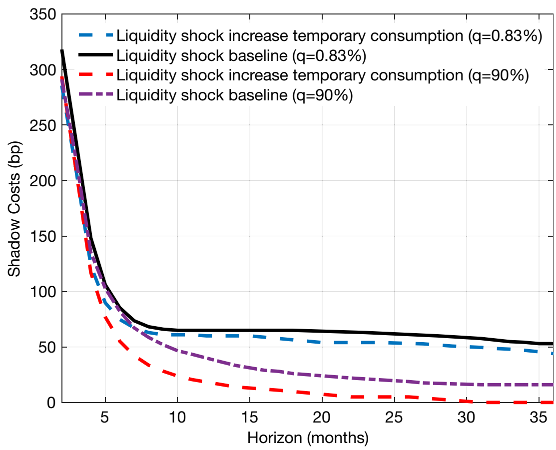


TABLE 2

Optimal Strategies Two Interpretations for the Liquidity Shock

Table 2 shows the optimal allocation to the illiquid asset and the corresponding optimal consumption level for the baseline specification and when the liquidity shock is modeled as a temporary increase in consumption, for an investor with horizon T (in months). We show the results for monthly liquidity shock probabilities of $q = 0.83\%$ and $q = 90\%$. We use the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the income return $d = 0$, the trading intensity of the illiquid asset $\eta = 50\%$, and the transactions costs $\phi = 1\%$. The optimal consumption level is determined when the investor can trade the illiquid asset and as a result, it does not depend on the endogenous state variable ξ_t. All values are in percentage points.

	Baseline		Increase Consumption L_t		
T	ξ_t^*	α_t^*	ξ_t^*	α_t^*	Diff. α_t^*
Panel A. Monthly Liquidity Shock Probability $q = 0.83\%$					
1	5.75	47.74	5.80	47.69	−0.05
2	17.22	32.32	18.18	32.21	−0.12
3	21.15	24.34	23.76	24.24	−0.09
4	22.36	19.51	28.53	19.38	−0.12
5	27.48	16.29	27.44	16.17	−0.12
6	26.73	13.96	26.69	13.83	−0.14
7	26.20	12.22	26.16	12.09	−0.13
8	25.81	10.87	25.77	10.74	−0.13
9	25.49	9.78	25.46	9.66	−0.12
10	25.25	8.90	25.21	8.75	−0.15
11	25.04	8.16	25.00	8.02	−0.14
12	24.87	7.53	24.84	7.41	−0.12
Panel B. Monthly Liquidity Shock Probability $q = 90\%$					
1	6.19	35.42	5.13	21.99	−13.43
2	19.90	19.61	16.92	5.45	−14.15
3	23.97	12.39	21.40	−2.78	−15.17
4	20.74	8.38	23.22	−7.68	−16.06
5	22.31	5.89	24.33	−10.99	−16.88
6	21.93	4.25	25.59	−13.32	−17.57
7	24.77	3.11	26.09	−14.99	−18.10
8	24.57	2.31	26.61	−16.48	−18.79
9	24.42	1.73	27.22	−17.54	−19.27
10	24.32	1.30	27.02	−18.41	−19.71
11	25.25	0.98	27.69	−19.19	−20.17
12	25.19	0.75	28.38	−19.81	−20.56

G. Shadow Costs and Income Return

Figure 10 shows that the shadow costs are decreasing in the level of income return, although the magnitude of the effects is relatively small. For the short-term investor ($T = 1$ year), returns on the illiquid asset that consist of capital gains only result in shadow costs equal to 61 basis points, and this cost decreases to 41 basis points if the return on the illiquid asset is fully paid as income. For the long-term investor, the shadow cost of illiquidity without income return equals 12 basis points, and this decreases to 5 basis points if the return consists fully of income. For the levels of income return we consider, its effect on reducing the shadow costs is limited, because the income return is a fairly small fraction of the total return on liquid and illiquid assets combined.

H. Shadow Costs and Price of Risk

Figure 11 shows that a higher price of risk for the illiquid asset strongly amplifies shadow costs. A higher price of risk increases the illiquid asset's attractiveness so that the optimal fraction of wealth allocated to it increases in case the illiquid asset becomes fully liquid. The higher the fraction of total wealth that the

FIGURE 10

The Shadow Costs As a Function of Income Return

Figure 10 shows the shadow costs as a function of the income return d for the investor at horizon $T = 1$ and $T = 10$ (in years) that assumes the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$ with intensity $\upsilon = 10\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the trading intensity of the illiquid asset $\eta = 50\%$, and the transactions costs $\phi = 1\%$.

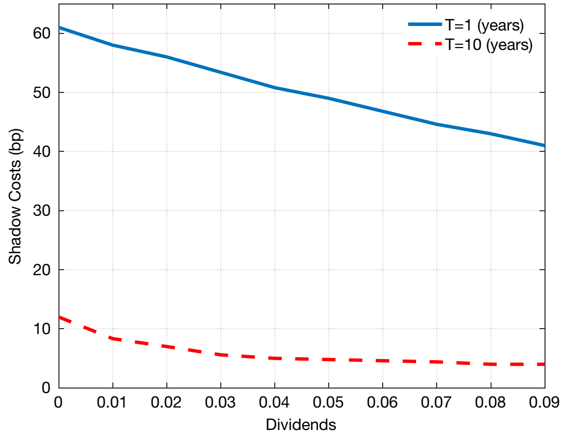
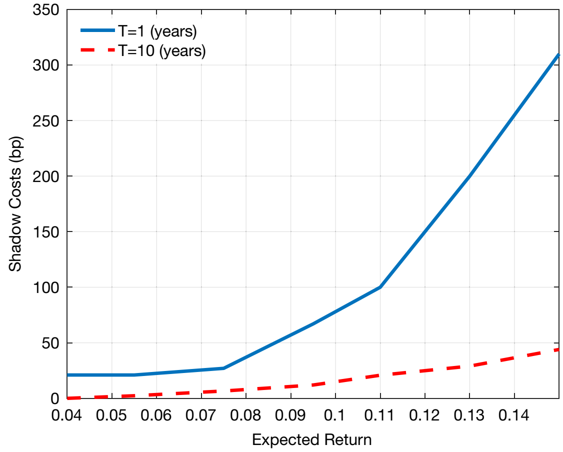


FIGURE 11

The Shadow Costs As a Function of the Price of Risk

Figure 11 shows the shadow costs as a function of the expected return of the illiquid asset for the investor at horizon $T = 1$ and $T = 10$ (in years) that assumes the following parameter values: the risk-aversion parameter $\gamma = 5$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 30\%$ with intensity $\upsilon = 10\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid risky asset $\mu_S = 9\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the income return $d = 0$, the trading intensity of the illiquid asset $\eta = 50\%$, and the transactions costs $\phi = 1\%$.



investor optimally wants to invest in the illiquid asset, the stronger the threat of illiquidity. Because the investor cannot borrow against the illiquid asset and to remain able to smooth consumption, the gap between the optimal allocation to the illiquid asset compared to when it is fully liquid widens as the illiquid asset becomes

https://doi.org/10.1017/S0022109022000473 Published online by Cambridge University Press

more attractive. These findings are consistent with Kahl, Liu, and Longstaff (2003) and Longstaff (2009) who show that the welfare effects of illiquidity are much larger when more wealth is tied up in the illiquid asset. For ease of interpretation, we show the shadow costs as a function of the expected return as the driver of the price of risk. However, we obtain similar results if we adjust the standard deviation. A lower standard deviation increases the optimal allocation to the illiquid asset if its also liquid and hence shadow costs are amplified. Similarly, a low correlation between the illiquid and the liquid risky asset increases the optimal allocation to the illiquid asset, leading to higher shadow costs of illiquidity.

I. Shadow Costs in Four Asset Classes

The previous subsections show that the shadow costs are high for short-term investors, investors who face substantial liquidity shocks, and investors who desire to allocate a large fraction of their wealth to illiquid assets. Next, to provide perspective on the quantitative and qualitative implications of the model, we compute rough estimates of the shadow costs of illiquidity in different asset classes. The asset classes we consider are private equity, real estate, corporate bonds, and stocks. To quantify shadow costs of illiquidity in each asset class, we use parameter values that we feel are representative for the asset class in consideration. We provide these as stylized examples to illustrate the effect of illiquidity in different markets.

Throughout, we assume that the liquid risky asset S represents a liquid stock index. We use the annualized mean and the standard deviation of the S&P500 Index to model the diversified liquid stock index. Calibrated over the last 25 years, the average return is $\mu_S = 11.3\%$ and the standard deviation $\sigma_S = 17.8\%$. Moreover, we use as the risk-free rate the annualized 1-year Treasury yield over the last 25 years that gives us $r_f = 2.8\%$.

The preferences of investors in each market are less well-known, as researchers only have a very rough idea about investors' investment horizons and their liquidity needs that are usually represented as holdings periods (e.g., Atkins and Dyl (1997)) or investors' funding constraints (e.g., Chen, Huang, Sun, Yao, and Yu (2020)). These measures are generally incomplete as these proxies do not measure other liquidity risks such as margin calls on derivative positions or rare disasters that investors potentially face. For this reason, we provide qualitative indicators of investors' preferences for each asset class.

Despite this drawback, we argue that our findings can be interpreted as upper bounds on the shadow costs of illiquidity in each of the asset classes. We make two assumptions that are more likely to overestimate rather than to underestimate the shadow costs. First, we assume that the investors cannot borrow against the illiquid assets. This constraint may be a realistic assumption for some asset classes but not for others. For instance, real estate investors are typically able to borrow a substantial amount using the property as collateral. However, taking this borrowing into account decreases shadow costs because the investors can partially undo the illiquidity of the asset. Second, we allow for liquidity shocks as large as 50% of the investors' total wealth. Even though larger wealth shocks are in practice possible, our model shows that investors substantially reduce their risky asset allocation if faced with such shocks also in the fully liquid case. As a result, large liquidity shocks do not necessarily have a positive effect on shadow costs (Figure 8).

1. Private Equity

To assess the shadow costs for private equity, we use the mean and standard deviation of the S&P500 Index to model the liquid counterpart of the illiquid private equity investment in our model, as the S&P500 Index is generally taken as the benchmark for private equity (see, e.g., Franzoni et al. (2012) and Harris et al. (2014)). This benchmark means that $\mu_X = 11.3\%$ and $\sigma_X = 17.8\%$. We do not take a stance on the correlation between private equity and the S&P500 Index. The performance of private equity varies substantially across investments, as for instance noted by Phalippou and Gottschalg (2009) (and as a result the correlation coefficient varies across the specifications as well). We therefore analyze the results for the correlation coefficients of $\rho_{\mathrm{SX}} = 0.25$ (baseline) and $\rho_{\mathrm{SX}} = 0.60$.

Private equity contracts generally run for 10 years, and trading is unusual before a contract expires (Metrick and Yasuda (2010)). Therefore, we set $\eta = p = 0$ over the first 10 years of the investment horizon.[8] We furthermore assume that the transaction cost at exiting the contract is $\phi = 1\%$. A study by Dechert and Preqin (2011) shows that transaction fees at completion of a private equity contract vary between 0.84% and 1.25% depending on the size of the investment.

Turning to investors' preferences in the private equity market, we posit that they likely have a low demand for liquidity, as the lock-up period of private equity is long and known beforehand. Indeed, Harris et al. (2014) report that for the Burgiss database that covers \$1 trillion of committed capital to private equity over 20% is held by endowment funds and 60% by pension funds. Hence, we analyze the shadow costs for horizons equal to $T = 10$ (baseline) and $T = 15$. Finally, we assess the shadow cost for long-term investors who face different liquidity shocks. Across these specifications, Panel A of Table 3 shows that the shadow costs for private equity vary between 0 and 55 basis points.

2. Real Estate

To model a direct investment in real estate, we use the first two moments of the S&P US Real Estate Investment Trust (REIT) Index to model the liquid counterpart of a direct real estate investment: $\mu_X = 12.22\%$ and $\sigma_X = 18.31\%$. The correlation coefficient between the liquid and illiquid asset is calibrated as the correlation between the S&P500 Index and S&P US REIT Index, which equals $\rho_{\mathrm{SX}} = 0.4$.

The return on real estate includes both the income return and capital gains. The income return refers to the rent on properties, so the real estate returns are partially liquid. Following Hardin III, Liano, and Huang (2002) we assume that the income return or rent payments explain the majority of the total investment returns for REITs. In the model, we therefore set the income return equal to $d = \mu_X - r_f$. The volatility is instead largely defined by the volatility in the capital

[8]Another distinct feature of private equity is that they usually involve capital commitment agreements. The investor agrees to provide a preset amount of capital over the first 3–5 years of the project. Yet, the capital commitment is preset, so we treat it as an upfront investment in our model.

TABLE 3
Shadow Costs in Four Asset Classes

Table 3 shows the shadow costs in basis points for investors who face liquidity shock l in four asset classes: private equity, real estate, corporate bonds, and stocks. In all asset classes we assume the following parameter values: risk-aversion parameter $\gamma=5$, time-preference discount factor $\beta=0.91$, return on the risk-free rate $r_f=2.8\%$, average and standard deviation of the return on the liquid risky asset $\mu_S=11.3\%$ and $\sigma_S=17.8\%$. For each asset class we then use different parameter values to characterize that asset class. Private equity: horizon $T=10$ year, average and standard deviation of the return on the illiquid asset $\mu_X=11.3\%$ and $\sigma_X=17.8\%$, the correlation coefficient $\rho_{SX}=0.25$, the income return $d=0$, the trading intensity of the illiquid asset $\eta=0$ ($p=0$), and the transaction costs equal to $\phi=1\%$. Real estate: horizon $T=10$ year, average and standard deviation of the return on the illiquid asset $\mu_X=12.2\%$ and $\sigma_X=18.3\%$, the correlation coefficient $\rho_{SX}=0.4$, the income return $d=9.4\%$, trading intensity $\eta=0.2$ ($p=1.65\%$), and the transaction costs $\phi=6\%$. Corporate bonds: horizon $T=10$ year, average and standard deviation of the return on the illiquid asset $\mu_X=7.0\%$ and $\sigma_X=6.6\%$, the correlation coefficient $\rho_{SX}=0.35$, the income return $d=4.2\%$, the trading intensity of the illiquid asset $\eta=28$ ($p=90\%$), and transaction costs $\phi=0.46\%$. Stocks: horizon $T=1$ year, average and standard deviation of the return on the illiquid asset $\mu_X=11.3\%$ and $\sigma_X=17.8\%$, the correlation coefficient $\rho_{SX}=0.8$, the income return $d=0$, the trading intensity of the illiquid asset $\eta=\infty$ ($p=1$), and transaction costs $\phi=4.0\%$.

Panel A. Private Equity

	Baseline	$p=0.6$	$T=15$	$p=0.6$, $T=15$
Liquidity shock				
0.0	4	0	0	0
0.3	23	12	8	4
0.5	55	29	24	9

Panel B. Real Estate

	Baseline	$\eta=10\%$	$T=5$	$\eta=10\%$, $T=5$
Liquidity shock				
0.0	0	1	4	6
0.3	16	16	28	33
0.5	36	39	51	71

Panel C. Corporate Bonds

	Baseline	$\phi=0.58\%$	$T=1$	$\phi=0.58\%$, $T=1$
Liquidity shock				
0.0	80	85	74	79
0.3	64	65	50	59
0.5	35	38	26	32

Panel D. Stocks

	Baseline	$\phi=8\%$	$T=10$	$\phi=8\%$, $T=10$
Liquidity shock				
0.0	54	108	0	0
0.3	50	100	6	11
0.5	46	83	12	17

gains, and we therefore assume that the volatility relates to volatility in capital gains only.

To describe the illiquidity parameters for real estate, we set transaction costs equal to $\phi=6\%$, in line with estimates by Ommeren (2008). The typical time between transactions for residential housing is 4–5 years and 8–11 years for institutional real estate (see, e.g., Hansen (1998) and Miller, Peng, and Sklarz (2011)). We thus assume trading intensities η in the range of 10% and 20% (baseline), which equals a monthly trading probability of $p=0.83\%$ or $p=1.65\%$. We are not aware of public holdings data on real estate. However, pension funds are large investors in real estate markets worldwide (Watson (2018)). We therefore conjecture that the majority of investors are either medium term ($T=5$) or long term ($T=10$, baseline). Across these specifications, we find that shadow costs are in the range of 0–71 basis points (Table 3, Panel B).

3. Corporate Bonds

We use the first two moments of the Bloomberg Barclays U.S. Corporate Bond Index to model the liquid counterpart of corporate bonds: $\mu_X = 7.0\%$ and $\sigma_X = 6.6\%$. We calibrate the correlation coefficient as the correlation between the Bloomberg Barclays US Corporate Bond Index and the S&P500 Index, which equals $\rho_{SX} = 0.35$. Finally, we assume fixed coupon payments, and as for real estate, we assume that the income return equals $d = \mu_X - r_f$.

Turning to the illiquidity parameters, we set transaction costs equal to $\phi = 0.46\%$ (baseline) and $\phi = 0.58\%$ (Bongaerts et al. (2017)). We translate the periods between trades found in Bao et al. (2011) and Bongaerts et al. (2017) to monthly trading probabilities that equal $p = 90\%$, which means trades occur on average slightly less than once a month.[9]

The Financial Accounts of the United States (Fed (2020)) reports the holdings of several asset classes within the United States by investor type. We define long-term investors as the insurance companies, pension funds, and the government. The short-term investors are the banks, broker-dealers, households, and mutual funds. The Fed (2020) reports that for 2019, 38% of corporate bonds were held by long-term investors and 34% by short-term investors. The remaining 28% of the holdings are unspecified. We therefore analyze the shadow costs for investors with horizons equal to $T = 1$ year and $T = 10$ years (baseline). Across these specifications, we find shadow costs in the range of 26–85 basis points for corporate bonds (Panel C of Table 3).

4. Stocks

For our model, we again use the first two moments of the S&P500 Index to model the liquid counterpart of the illiquid stock, so $\mu_X = 11.3\%$ and $\sigma_X = 17.8\%$. Equity asset classes, such as small-growth stocks, are highly correlated with the US stock market, so we set $\rho_{SX} = 0.8$. Given the improved liquidity of U.S. equities, we assume that the monthly trading probability equals $p = 100\%$. The transaction costs for stocks range from 0.25% for the most liquid stocks to 8% for the least liquid stocks (Beber, Driessen, Neuberger, and Tuijp (2021)). We therefore assess the shadow cost for transaction costs equal to 4% (baseline) and 8%.

The Fed (2020) reports that for 2019, 13% of U.S. corporate equity is held by long-term investors and 70% by short-term investors. The remaining 17% of the holdings are unspecified. We therefore again assess shadow costs for investment horizons $T = 1$ year (baseline) and $T = 10$ years. Across these specifications, Panel D of Table 3 shows shadow costs in the range of 0–108 basis points for illiquid stocks.

IV. Conclusion

In this study, we solve a flexible model that captures transactions costs and infrequencies of trading opportunities for illiquid assets to achieve better

[9]For instance, Bao et al. (2011) show an annualized turnover of corporate bonds that varies between 25% and 35% and Bongaerts et al. (2017) show that 15% to 25% of corporate bonds are not traded in a given week.

understanding of the shadow costs of illiquidity. The cost of illiquidity may be twofold: suboptimal asset allocation and suboptimal consumption smoothing. We show that only the illiquidity that results in suboptimal consumption smoothing is able to generate substantial shadow costs, while the illiquidity that leads to suboptimal asset allocations does not. Hence, we find that the shadow costs are larger for short-term investors, investors who face substantial liquidity shocks, and investors who desire to allocate a large fraction of their wealth to illiquid assets if the same illiquid asset would otherwise be liquid. Looking at separate asset classes, back-of-the-envelope calculations suggest low average shadow costs for private equity and direct real estate, but these costs can become substantial for illiquid stocks and corporate bonds.

Appendix A. Proof of Optimal Consumption and Investment Strategies

Proof of Theorem II.3. Instead of the endogenous variables (W_t, X_t), we use the pair (W_t+X_t, ξ_t) as endogenous state variables. That is, we write

$$C_t = \alpha_t(W_t+X_t,\xi_t)(W_t+X_t), \tag{A-1}$$

$$\theta_t = \theta_t(W_t+X_t,\xi_t). \tag{A-2}$$

Now, rewrite the evolution of total wealth W_t+X_t using budget constraints (3) and (4) as:

$$\begin{aligned} W_t+X_t = {} & (W_{t-h}+X_{t-h}) \\ & \times\left[(1-\xi_{t-h}-\alpha_{t-h}-\phi|\Delta\xi_{t-h}|-l_{t-h})\right. \\ & \times\left(\exp\left(r_f^{(h)}\right)+\theta_{t-h}\left(\exp\left(r_t^S\right)-\exp\left(r_f^{(h)}\right)\right)\right) \\ & \left.+\xi_{t-h}\exp(d)+\xi_{t-h}\left(\exp\left(r_t^X\right)-\exp(d)\right)\right], \end{aligned} \tag{A-3}$$

$$\xi_t = \frac{\xi_{t-h}\left(\exp\left(r_t^X\right)-\exp(d)\right)}{\begin{aligned}&(1-\xi_{t-h}-\alpha_{t-h}-\phi|\Delta\xi_{t-h}|-l_{t-h})\times\left(\exp\left(r_f^{(h)}\right)+\theta_{t-h}\left(\exp\left(r_t^S\right)-\exp\left(r_f^{(h)}\right)\right)\right)\\&+\xi_{t-h}\exp(d)+\xi_{t-h}\left(\exp\left(r_t^X\right)-\exp(d)\right)\end{aligned}}, \tag{A-4}$$

where $\Delta\xi_{t-h}=\xi_{t-h}^{*}-\xi_{t-h}$ if $\mathbf{1}_{t-h}^{T}=1$ and $\Delta\xi_{t-h}=0$ if $\mathbf{1}_{t-h}^{T}=0$.

The proof is by backward induction. At the final horizon $t=T$, the claim is obviously correct with $\alpha_T\equiv 1$ and $H_T(\xi_T)\equiv(1-\phi\xi_T)^{1-\gamma}$. At time T, θ_T is irrelevant. Now, for the induction argument, assume that (12)–(15) holds at time t. Then, we need to show that (12)–(15) also holds at time $t-h$. From the value function (9), evaluated at time $t-h$ and substituting (12), we find:

$$
\begin{aligned}
\text{(A-5)}\quad & V_{t-h}(W_{t-h}+X_{t-h},\xi_{t-h}) \\
&= \max_{\theta_t,\xi_t,C_t} \beta^{t-h}\frac{C_{t-h}^{1-\gamma}}{1-\gamma}+\mathbb{E}_{t-h}V_t(W_t+X_t,\xi_t) \\
&= \max_{\theta_t,\xi_t,C_t} \beta^{t-h}\frac{(\alpha_{t-h}(W_{t-h}+X_{t-h}))^{1-\gamma}}{1-\gamma} \\
&+\mathbb{E}_{t-h}\left[\beta^t\frac{(W_t+X_t)^{1-\gamma}}{1-\gamma}H_t(\xi_t)\right] \\
&= \max_{\theta_t,\xi_t,C_t} \beta^{t-h}\frac{(W_{t-h}+X_{t-h})^{1-\gamma}}{1-\gamma} \\
&\times\left(\alpha_{t-h}^{1-\gamma}+\beta\mathbb{E}_{t-h}[\{(1-\xi_{t-h}-\alpha_{t-h}-\phi|\Delta\xi_{t-h}|-l_{t-h})\right. \\
&\times\left(\exp\left(r_f^{(h)}\right)+\theta_{t-h}\left(\exp\left(r_t^S\right)-\exp\left(r_f^{(h)}\right)\right)\right)+\xi_{t-h}\exp(d) \\
&\left.+\xi_{t-h}\left(\exp\left(r_t^X\right)-\exp(d)\right)\}^{1-\gamma}H_t(\xi_t)]\right) \\
&= \max_{\theta_t,\xi_t,C_t} \beta^{t-h}\frac{(W_{t-h}+X_{t-h})^{1-\gamma}}{1-\gamma}H_t(\xi_t).
\end{aligned}
$$

At time $t-h$, the penalty function $H_{t-h}(\xi_{t-h})$ equals:

$$
\begin{aligned}
\text{(A-6)}\quad H_{t-h}(\xi_{t-h}) = \alpha_{t-h}^{1-\gamma}+\beta\mathbb{E}_{t-h}\Big[&\{(1-\xi_{t-h}-\alpha_{t-h}-\phi|\Delta\xi_{t-h}|-l_{t-h}) \\
&\times\left(\exp\left(r_f^{(h)}\right)+\theta_{t-h}\left(\exp\left(r_t^S\right)-\exp\left(r_f^{(h)}\right)\right)\right) \\
&+\xi_{t-h}\exp(d)+\xi_{t-h}\left(\exp\left(r_t^X\right)-\exp(d)\right)\}^{1-\gamma}H_t(\xi_t)\Big].
\end{aligned}
$$

Therefore, the function $H_{t-h}(\xi_{t-h})$ is a function of $t-h$ and ξ_{t-h} only and hence (12) holds for all t. We continue with proving (13) and (14) at time $t-h$. The first-order conditions of the decision variables α_{t-h} and θ_{t-h} equal

$$
\text{(A-7)}\quad \alpha_{t-h}^{UC*} = \arg\max_{\alpha_{t-h}} \frac{(\alpha_{t-h}(W_{t-h}+X_{t-h}))^{1-\gamma}}{1-\gamma}+\mathbb{E}_{t-h}V_t(W_t+X_t,\xi_t),
$$

$$
\text{(A-8)}\quad \theta_{t-h}^{*} = \arg\max_{\theta_{t-h}} \frac{(\alpha_{t-h}(W_{t-h}+X_{t-h}))^{1-\gamma}}{1-\gamma}+\mathbb{E}_{t-h}V_t(W_t+X_t,\xi_t),
$$

where α_{t-h}^{UC*} is the solution if the investor were unconstrained (i.e., when constraint (5) does not bind). Because we assume that the investor cannot borrow against the illiquid asset the constrained solution becomes

$$
\text{(A-9)}\quad \alpha_{t-h}^{C*} = \begin{cases} \alpha_{t-h}^{UC*} & \text{if } \alpha_{t-h}^{UC*} \le 1-\xi_{t-h}-l \\ 1-\xi_{t-h}-l & \text{if } \alpha_{t-h}^{UC*} > 1-\xi_{t-h}-l. \end{cases}
$$

We can now rewrite (A-7) and (A-8) as

$$\text{(A-10)}\quad \frac{\partial V_{t-h}}{\partial \alpha_{t-h}} = \beta^{t-h}(\alpha_{t-h}(W_{t-h}+X_{t-h}))^{-\gamma} - \mathbb{E}_{t-h}\left[\frac{\partial V_t}{\partial W_t + X_t}\left(\exp\left(r_f^{(h)}\right) + \theta_{t-h}\left(\exp\left(r_t^S\right) - \exp\left(r_f^{(h)}\right)\right)\right)\right] + \mathbb{E}_{t-h}\left[\frac{\partial V_t}{\partial \xi_t}\xi_t \frac{1}{W_t + X_t}\left(\exp\left(r_f^{(h)}\right) + \theta_{t-h}\left(\exp\left(r_t^S\right) - \exp\left(r_f^{(h)}\right)\right)\right)\right] = 0,$$

$$\text{(A-11)}\quad \frac{\partial V_{t-h}}{\partial \theta_{t-h}} = \mathbb{E}_{t-h}\left[\frac{\partial V_t}{\partial W_t + X_t}\left(\exp\left(r_t^S\right) - \exp\left(r_f^{(h)}\right)\right)\right] + \mathbb{E}_{t-h}\left[\frac{\partial V_t}{\partial \xi_t}\xi_t \frac{1}{W_t + X_t}\left(\exp\left(r_t^S\right) - \exp\left(r_f^{(h)}\right)\right)\right] = 0.$$

To see that both α^*_{t-h} and θ^*_{t-h} depend only on ξ_{t-h}, we solve (A-10) and (A-11) and substitute (A-3) into (A-10) and (A-11), we get

$$\text{(A-12)}\quad \frac{\partial V_{t-h}}{\partial \alpha_{t-h}} = \alpha_{t-h}^{-\gamma} + \beta\mathbb{E}_{t-h}[\{(1-\xi_{t-h}-\alpha_{t-h}-\phi|\Delta\xi_{t-h}|-l_{t-h}) \left(\exp\left(r_f^{(h)}\right) + \theta_{t-h}\left(\exp\left(r_t^S\right) - \exp\left(r_f^{(h)}\right)\right)\right) + \xi_{t-h}\exp(d) + \xi_{t-h}\left(\exp\left(r_t^X\right) - \exp(d)\right)\}^{-\gamma} \times \left(\frac{H_t{}'(\xi_t)}{1-\gamma}\xi_t - H_t(\xi_t)\right)\exp\left(r_f^{(h)}\right)] = 0,$$

$$\text{(A-13)}\quad \frac{\partial V_{t-h}}{\partial \theta_{t-h}} = \mathbb{E}_{t-h}\left[\left\{(1-\xi_{t-h}-\alpha_{t-h}-\phi|\Delta\xi_{t-h}|-l_{t-h})\left(\exp\left(r_f^{(h)}\right) + \theta_{t-h}\left(\exp\left(r_t^S\right) - \exp\left(r_f^{(h)}\right)\right)\right) + \xi_{t-h}\exp(d) + \xi_{t-h}\left(\exp\left(r_t^X\right) - \exp(d)\right)\right\}^{-\gamma} \times \left(H_t(\xi_t) - \frac{H_{t'}(\xi_t)}{1-\gamma}\xi_t\right)\left(\exp\left(r_t^S\right) - \exp\left(r_f^{(h)}\right)\right)\right] = 0.$$

The first-order conditions (A-12) and (A-13) depend only on time $t-h$ and the fraction invested in the illiquid asset ξ_{t-h}. In this way, the optimal consumption and the fraction invested in the liquid risky assets can indeed be written as in (13) and (14), so (13) and (14) from (12)–(15) holds for all t. We finish the proof by showing that (15) also holds at time $t-h$. When a trading opportunity arises at $t-h$, the investor chooses ξ_{t-h} such that the value function at $t-h$ is optimized:

$$\text{(A-14)}\quad \xi^*_{t-h} = \underset{\xi_{t-h}}{\arg\max}\, V_{t-h}(W_{t-h}+X_{t-h}, \xi_{t-h}) = \underset{\xi_{t-h}}{\arg\max}\, \beta^{t-h}\frac{(W_{t-h}+X_{t-h})^{1-\gamma}}{1-\gamma}H_{t-h}(\xi_{t-h}) = \underset{\xi_{t-h}}{\arg\max}\, H_{t-h}(\xi_{t-h}).$$

□

Appendix B. Numerical Implementation

Appendix B provides an outline of the numerical method to solve the baseline model. First, we describe the sequence of making decisions. Second, we explain the numerical solution technique to solve the decision variables.

Figure B1 depicts the sequence of making decisions. The endogenous variables, liquid wealth W_t and illiquid wealth X_t, are defined as total wealth before consumption, liquidity shocks, and returns earned in period $(t, t+h]$. Based on the actual fraction allocated to the illiquid asset ξ_t, the investor chooses the optimal fraction of total wealth to be consumed in period $(t, t+h]$, $\alpha_t^*(\xi_t)$, and the optimal allocation toward the liquid risky asset, $\theta_t^*(\xi_t)$. If a trading opportunity arises at time t, the investor chooses simultaneously ξ_t^*, $\alpha_t^*(\xi_t^*)$ and $\theta_t^*(\xi_t^*)$. Further, by the assumption $X_0 \geq 0$ and the inability to borrow against the illiquid asset, the possible values for ξ_t are restricted to the interval $[0,1]$.

The model is solved by means of backward induction, where we start solving the problem at the final date $t = T$ and solve the model backward for each period until arriving at time $t = 0$. At the final horizon $t = T$, the investor can always liquidate illiquid wealth and we have $\alpha_T \equiv 1$ and $H_T(\xi_t) \equiv (1 - \phi\xi_T)^{1-\gamma}$. To solve for ξ_{T-h}^*, $\alpha_t(\xi_{T-h})$, and $\theta_t(\xi_{T-h})$, we construct a grid for $\xi_{T-h} \in [0,1]$. We simulate $M = 10{,}000$ trajectories for the exogenous state variables, the returns on the liquid and illiquid risky asset in period $(T-h, T]$, r_T^S and r_T^X, from a multinormal distribution with mean and variance–covariance matrix as described in Section II. We also simulate $M = 10{,}000$ trajectories for the liquidity shock indicator $\mathbf{1}_{T-h}^L$ from a Bernoulli distribution as described in Section II.

For each grid point, by using nonlinear least squares, we solve the first-order conditions with respect to consumption (A-12) and the allocation toward the liquid risky asset (A-13) by using $H_T(\xi_T) \equiv (1 - \phi\xi_T)^{1-\gamma}$, r_T^S, r_T^X, and $\mathbf{1}_{T-h}^L$ to find $\alpha_{T-h}^*(\xi_{T-h})$ and $\theta_{T-h}^*(\xi_{T-h})$. Then we are able to compute $H_{T-h}(\xi_{T-h})$ and solve for $\xi_{T-h}^* = \arg\min_{\xi_{T-h}} H_{T-h}(\xi_{T-h})$ with the corresponding consumption level $\alpha_{T-h}^*(\xi_{T-h}^*)$ and the allocation to the liquid risky asset $\theta_{T-h}^*(\xi_{T-h}^*)$. This gives us the optimal solution at time $T-h$.

We then solve for the optimal solution at time $T-2h$ in the same way, except that we also simulate $M = 10{,}000$ trajectories for the trading indicator $\mathbf{1}_{T-h}^T$ from a Bernoulli distribution as described in Section II. In the scenarios the investor is able to trade ($\mathbf{1}_{T-h}^T = 1$), we use $H_{T-h}(\xi_{T-h}^*)$ and in the scenarios the investor is unable to trade ($\mathbf{1}_{T-h}^T = 0$), we use $H_{T-h}(\xi_{T-h})$. Together with r_{T-h}^S, r_{T-h}^X, and $\mathbf{1}_{T-2h}^L$ we find $\alpha_{T-2h}^*(\xi_{T-2h})$ and $\theta_{T-2h}^*(\xi_{T-2h})$. We again compute $H_{T-2h}(\xi_{T-2h})$ and solve for $\xi_{T-2h}^* = \arg\min_{\xi_{T-2h}} H_{T-2h}(\xi_{T-2h})$ with the corresponding consumption level $\alpha_{T-2h}^*(\xi_{T-2h}^*)$ and the allocation to the liquid risky asset $\theta_{T-2h}^*(\xi_{T-2h}^*)$. We can continue this approach until we arrive at $t = 0$.

FIGURE B1

Timeline

Figure B1 depicts the investor's sequence of decision making.

https://doi.org/10.1017/S0022109022000473 Published online by Cambridge University Press

Supplementary Material

Supplementary Material for this article is available at https://doi.org/10.1017/S0022109022000473.

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