Proceedings of the Edinburgh Mathematical Society (1996) 39, 325–336 ©

BIORTHOGONALITY IN THE BANACH SPACES $\ell^p(n)$*

by ANTHONY J. FELTON and H. P. ROGOSINSKI

(Received 13th July 1994)

We consider the finite-dimensional Banach spaces $\ell^p(n)$, where $p>1$. On these spaces there is a unique homogeneous semi-inner-product $[.,.]$ consistent with the norm. If $p\neq 2$ this semi-inner product is not symmetric. We define a pair of vectors $\mathbf{x}$ and $\mathbf{y}$ to be *biorthogonal* if $[\mathbf{x},\mathbf{y}]=[\mathbf{y},\mathbf{x}]=0$. For a given non-zero $\mathbf{x}$, let $\tau(\mathbf{x})$ be the number of elements in a maximal linearly independent set of vectors biorthogonal to $\mathbf{x}$. If $p=2$ it is well-known that this number is $n-1$. The aim of this paper is to find $\tau(\mathbf{x})$ when $p\neq 2$. Our investigation shows that the situation differs from the Euclidean case in that the value of $\tau(\mathbf{x})$ can be either $n-1$ or $n-2$. The 'exceptional' vectors $\mathbf{x}$ for which $\tau(\mathbf{x})=n-2$ are characterised.

1991 *Mathematics subject classification*: 46C50.

0. Introduction

The following definition is due to Lumer [2]. Let V be a real vector space. A *semi-inner-product* (s.i.p.) on $V\times V$ is a map $[.,.]:V\times V\to\mathbb{R}$ satisfying the following properties: for all $\mathbf{x},\mathbf{y},\mathbf{z}\in V$,

(a) $[\mathbf{x}+\mathbf{y},\mathbf{z}]=[\mathbf{x},\mathbf{z}]+[\mathbf{y},\mathbf{z}]$,

(b) $[\lambda\mathbf{x},\mathbf{y}]=\lambda[\mathbf{x},\mathbf{y}]\ \forall\lambda\in\mathbb{R}$,

(c) $[\mathbf{x},\mathbf{x}]>0$ if $\mathbf{x}\neq\mathbf{0}$,

(d) $|[\mathbf{x},\mathbf{y}]|^2\leqq[\mathbf{x},\mathbf{x}][\mathbf{y},\mathbf{y}]$.

We note in general $[\mathbf{x},\mathbf{y}]\neq[\mathbf{y},\mathbf{x}]$. A semi-inner-product is called *homogeneous* [1] if

$$[\mathbf{x},\lambda\mathbf{y}]=\lambda[\mathbf{x},\mathbf{y}]\ \forall\lambda\in\mathbb{R},\quad\text{and for all }\mathbf{x},\mathbf{y}\in V.$$

It is readily verified that $\|\mathbf{x}\|=[\mathbf{x},\mathbf{x}]^{1/2}$ defines a norm on V. We note the well-known result that in a smooth normed linear space X there exists a *unique* semi-inner-product on X which is consistent with the norm on X. In fact $[\mathbf{x},\mathbf{y}]=(W\mathbf{y})(\mathbf{x})$ where $W\mathbf{y}$ is the unique linear functional such that $\|W\mathbf{y}\|=\|\mathbf{y}\|$ and $(W\mathbf{y})(\mathbf{x})=\|\mathbf{y}\|^2$. [2]

Definition 0.1. Let X be a smooth normed linear space, with norm $\|.\|$ and

*This research was funded by the University of Wales.

https://doi.org/10.1017/S001309150002304X Published online by Cambridge University Press

associated semi-inner-product $[.,.]$. Let $\mathbf{x},\mathbf{y}\in X$. We say that $\mathbf{x}$ is *biorthogonal* to $\mathbf{y}$ if, and only if,

$$[\mathbf{y},\mathbf{x}]=[\mathbf{x},\mathbf{y}]=0.$$

In this case, we write $\mathbf{x}\pm\mathbf{y}$.

In the following we consider real finite-dimensional normed linear spaces $\ell^p(n)$, where $1<p<\infty$, $p\neq 2$. It is well-known that these spaces are smooth, and that the unique consistent s.i.p. on such spaces is given by

$$[\mathbf{x},\mathbf{y}]=\frac{1}{\|\mathbf{y}\|^{p-2}}\sum_{i=1}^{n}x_i|y_i|^{p-1}\operatorname{sgn}y_i=\frac{1}{\|\mathbf{y}\|^{p-2}}\sum_{i=1}^{n}x_iy_i|y_i|^{p-2}$$

for $\mathbf{x},\mathbf{y}\in\ell^p(n)$ and $\mathbf{y}\neq\mathbf{0}$ (see e.g. [1]).

Throughout this paper p will denote a real number such that $p>1$ and $p\neq 2$, Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\ell^p(n)$. We define $|\mathbf{x}|=(|x_1|,|x_2|,\ldots,|x_n|)$. Further we define $\mathbf{x}_\pi=(x_{\pi(1)},x_{\pi(2)},\ldots,x_{\pi(n)})$, where $\pi\in S_n$, and S_n is the group of permutations of $\{1,2,\ldots,n\}$.

Definition 0.2. Let $\mathbf{x}\in\ell^p(n)$. Define $\tau(\mathbf{x})$ to be the number of elements in a maximally linearly independent set of vectors biorthogonal to $\mathbf{x}$.

Our purpose is to find $\tau(\mathbf{x})$ for each $\mathbf{x}\in\ell^p(n)$.

1. Basic properties of $\tau(\mathbf{x})$

Proposition 1.1. *Let* $\mathbf{x}\in\ell^p(n)$. *For fixed* $k\in\mathbb{N}$, *let* $\hat{\mathbf{x}}\in\ell^p(n+k)$ *be defined by* $\hat{\mathbf{x}}=(x,\overbrace{0,0,\ldots,0}^{k})$. *Then*

(i) $\tau(\lambda\mathbf{x})=\tau(\mathbf{x})\ \forall\lambda\neq 0$,

(ii) $\tau(\mathbf{x})=\tau(|\mathbf{x}|)$,

(iii) $\tau(\mathbf{x})=\tau(\mathbf{x}_\pi)$ *where* $\pi\in S_n$,

(iv) $\tau(\hat{\mathbf{x}})=\tau(\mathbf{x})+k$,

(v) $\tau(\mathbf{x})=n$ *if, and only if,* $\mathbf{x}=\mathbf{0}$,

(vi) $n-2\leqq\tau(\mathbf{x})\leqq n$.

Proof. (i) Since

$$[\lambda\mathbf{x},\mathbf{y}]=\lambda[\mathbf{x},\mathbf{y}],\ [\mathbf{y},\lambda\mathbf{x}]=\lambda[\mathbf{y},\mathbf{x}],$$

$\mathbf{x}\pm\mathbf{y}$ if, and only if, $\lambda\mathbf{x}\pm\mathbf{y}$, when $\lambda\neq 0$.

(ii) Define $\phi_{\mathbf{x}}:\ell^p(n)\to\ell^p(n)$ by

$$\phi_{\mathbf{x}}((y_1,y_2,\ldots,y_n))=((\operatorname{sgn}x_1)y_1,(\operatorname{sgn}x_2)y_2,\ldots,(\operatorname{sgn}x_n)y_n).$$

Then it is clear that the map $\phi_{\mathbf{x}}$ is linear and onto, and so preserves the linear dependence and linear independence of sets of vectors. Moreover the map $\phi_{\mathbf{x}}$ preserves the semi-inner-product on $\ell^p(n)$, and so

$$\mathbf{y}\pm\mathbf{x}\quad\text{if, and only if,}\quad \phi_{\mathbf{x}}(\mathbf{y})\pm\phi_{\mathbf{x}}(\mathbf{x}).$$

Since $\phi_{\mathbf{x}}(\mathbf{x})=|\mathbf{x}|$, it follows that $\tau(\mathbf{x})=\tau(|\mathbf{x}|)$.

(iii) The proof of this follows from the identity

$$\sum_{i=1}^{n}x_iy_i|y_i|^{p-2}=\sum_{i=1}^{n}x_{\pi(i)}y_{\pi(i)}|y_{\pi(i)}|^{p-2},$$

where $\pi\in S_n$, as well as from the fact that the map $\mathbf{x}\to\mathbf{x}_\pi$ preserves linear independence.

(iv) Note that $(y_1,\ldots,y_{n+k})$ is biorthogonal to $\hat{\mathbf{x}}$ in $\ell^p(n+k)$ if, and only if, $(y_1,\ldots,y_n)$ is biorthogonal to $\mathbf{x}$ in $\ell^p(n)$. If $\mathbf{e}_1,\ldots,\mathbf{e}_{n+k}$ are the standard basis vectors in $\ell^p(n+k)$, and if $\mathbf{b}_1,\ldots,\mathbf{b}_{\tau(\mathbf{x})}$ is a set of $\tau(\mathbf{x})$ linearly independent vectors biorthogonal to $\mathbf{x}$, it follows that $\{\hat{\mathbf{b}}_1,\hat{\mathbf{b}}_2,\ldots,\hat{\mathbf{b}}_{\tau(x)},\mathbf{e}_{n+1},\ldots,\mathbf{e}_{n+k}\}$ is a set of $\tau(\mathbf{x})+k$ linearly independent vectors biorthogonal to $\hat{\mathbf{x}}$ in $\ell^p(n+k)$. Moreover *every* vector $(y,\ldots,y_{n+k})$ biorthogonal to $\hat{\mathbf{x}}$ is in the linear span of these $\tau(\mathbf{x})+k$ vectors. Indeed, since $(y_1,\ldots,y_n)$ is biorthogonal to $\mathbf{x}$ there exist scalars $\lambda_1,\ldots,\lambda_{\tau(\mathbf{x})}$ so that

$$(y_1,\ldots,y_n)=\lambda_1\mathbf{b}_1+\lambda_2\mathbf{b}_2+\ldots+\lambda_{\tau(\mathbf{x})}\mathbf{b}_{\tau(\mathbf{x})},$$

and so

$$(y_1,\ldots,y_{n+k})=\lambda_1\hat{\mathbf{b}}_1+\lambda_2\hat{\mathbf{b}}_2+\ldots+\lambda_{\tau(\mathbf{x})}\hat{\mathbf{b}}_{\tau(\mathbf{x})}+y_{n+1}\mathbf{e}_{n+1}+\ldots+y_{n+k}\mathbf{e}_{n+k}.$$

This completes the proof that $\tau(\hat{\mathbf{x}})=\tau(\mathbf{x})+k$.

(v) If $\tau(\mathbf{x})=n$ then there exists a basis in $\ell^p(n)$ in which each vector is biorthogonal to $\mathbf{x}$. Since the semi-inner-product in $\ell^p(n)$ is left-linear, it follows that *all* vectors in $\ell^p(n)$ are left-orthogonal to $\mathbf{x}$. In particular $[\mathbf{x},\mathbf{x}]=0$, and so $\mathbf{x}=\mathbf{0}$.

(vi) We need only show that $\tau(\mathbf{x})\geqq n-2$ whenever $\mathbf{x}\in\ell^p(n)$. This is obvious when $n=2$. We proceed by induction. Fix k, with $k\geqq 2$, and assume that $\tau(\mathbf{x})\geqq k-2$ whenever $\mathbf{x}\in\ell^p(k)$. Let $\mathbf{x}=(x_1,x_2,\ldots,x_{k+1})\in\ell^p(k+1)$. We shall show in Section 3 (Proposition 3.3(i)) that there exists a *non-zero* vector (b_{k-1},b_k,b_{k+1}) biorthogonal to (x_{k-1},x_k,x_{k+1}). Let $\mathbf{b}_1$ be the vector in $\ell^p(k+1)$ given by

$$\mathbf{b}_1=(\overbrace{0,\ldots,0}^{k-2},b_{k-1},b_k,b_{k+1}).$$

Then $\mathbf{b}_1$ is biorthogonal to $\mathbf{x}$. We shall assume that $b_{k+1}\neq 0$ (If $b_{k+1}=0$ then either b_{k-1}

or b_k must be non-zero, and it is clear how to modify the argument which follows). By the inductive assumption there exists a set of $k-2$ linearly independent vectors biorthogonal to $(x_1, x_2, \ldots, x_k) \in \ell^p(k)$. Let $\mathbf{b}_2, \ldots, \mathbf{b}_{k-1}$ be the $k-2$ vectors in $\ell^p(k+1)$ arising from this set by the addition of a final coordinate which is equal to 0. Then each of these 'augmented' vectors is biorthogonal to $\mathbf{x}$. Moreover, since the final coordinate of $\mathbf{b}_1$ is *non-zero*, the set of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{k-1}\}$ is linearly independent, and it follows that $\tau(x) \geqq k-1$. This completes the proof. □

2. Biorthogonality in $\ell^p(2)$

Proposition 2.1. *Let* $\mathbf{x} = (a, b) \in \ell^p(2)$. *Then*

(i) $\tau(\mathbf{x}) = 2$ *if, and only if,* $\mathbf{x} = \mathbf{0}$.
(ii) *If either a or b is equal to zero, and if* $\mathbf{x} \neq \mathbf{0}$, *then* $\tau(\mathbf{x}) = 1$.
(iii) *If both a and b are non-zero then* $\tau(\mathbf{x}) = 1$ *if, and only if,* $|a| = |b|$.

Proof. (i) This is covered by Proposition 1.1(v).

(ii) If $a = 0$ then $(1, 0)$ is biorthogonal to $\mathbf{x}$, and if $b = 0$ then $(0, 1)$ is biorthogonal to $\mathbf{x}$. Since $\mathbf{x} \neq \mathbf{0}$, it follows that $\tau(\mathbf{x}) = 1$.

(iii) If $a = b$ then $(1, -1)$ is biorthogonal to $\mathbf{x}$, and if $a = -b$ then $(1, 1)$ is biorthogonal to $\mathbf{x}$. In either case since $\mathbf{x} \neq \mathbf{0}$, $\tau(\mathbf{x}) = 1$. Suppose conversely that $\tau(\mathbf{x}) = 1$. Then there exists a *non-zero* vector (c, d) biorthogonal to (a, b). Since $b \neq 0$ it follows that $c \neq 0$. Since the s.i.p. is homogeneous we can assume w.l.o.g. that $c = 1$. We then have

$$a|a|^{p-2} + db|b|^{p-2} = 0 \quad \text{and} \quad a + bd|d|^{p-2} = 0.$$

The first equation implies that

$$|a|^{p-1} = |d|\,|b|^{p-1}, \tag{1}$$

whilst the second equation implies that

$$|a| = |b|\,|d|^{p-1}. \tag{2}$$

Substituting for $|d|^{p-1}$ from (2) into (1) gives

$$|a|^{(p-1)^2} = \frac{|a|}{|b|}|b|^{(p-1)^2}.$$

Hence $|a|^{p(p-2)} = |b|^{p(p-2)}$, and since $p \neq 2$ it follows that $|a| = |b|$. □

3. Biorthogonality in $\ell^p(3)$

We start with a lemma.

Lemma 3.1. *Let* $p > 1$ *with* $p \neq 2$, *and let* $a \geqq b \geqq 1$. *Let f be defined on* $(-\infty, \infty)$ *by*

https://doi.org/10.1017/S001309150002304X Published online by Cambridge University Press

$$f(t)=a+bt|t|^{p-2}-[a^{p-1}+b^{p-1}t]|a^{p-1}+b^{p-1}t|^{p-2}.$$

Then

(i) *if* $b>1$, f *has at least one zero, and* f *has more than one zero if, and only if,* $a^p \leqq b^p+1$;

(ii) *if* $b=1$, f *has a zero if, and only if,* $a^p \leqq 2$.

(iii) $(1,x,y)$ *is biorthogonal to* $(a,b,1)$ *in* $\ell^p(3)$ *if, and only if,* $f(x)=0$ *and* $y=-[a^{p-1}+b^{p-1}x]$.

Proof. We shall assume throughout that $p>2$. The case where $1<p<2$ is treated similarly.

(i) Let $b>1$. Note that

$$\lim_{|t|\to\infty}\frac{f(t)}{t|t|^{p-2}}=b[1-b^{p(p-2)}]<0.$$

It follows that

$$\lim_{t\to-\infty} f(t)=\infty \quad \text{and} \quad \lim_{t\to\infty} f(t)=-\infty.$$

Since f is continuous, the intermediate-value theorem shows that f has a zero on $(-\infty,\infty)$. By elementary calculus

$$f'(t)=(p-1)[b|t|^{p-2}-b^{p-1}|a^{p-1}+b^{p-1}t|^{p-2}].$$

Let

$$t_0=\frac{-a^{p-1}b}{b^p-1} \quad \text{and} \quad t_1=\frac{-a^{p-1}b}{b^p+1}.$$

Then it is easily verified that

$$f'(t_0)=f'(t_1)=0, f'(t)>0 \quad \text{if} \quad t_0<t<t_1, \text{and } f'(t)<0 \quad \text{otherwise}.$$

Hence f attains a minimum value when $t=t_0$ and a maximum value when $t=t_1$. Since $a^p>b^p-1$ and $p>2$,

$$f(t_0)=\frac{a}{(b^p-1)^{p-2}}[(b^p-1)^{p-2}-a^{p(p-2)}]<0.$$

It is now clear that $f(t_1)\geqq 0$ is a necessary and sufficient condition for f to have more than one zero. Since

$$f(t_1)=\frac{a}{(b^p+1)^{p-2}}[(b^p+1)^{p-2}-a^{p(p-2)}],$$

this condition is equivalent to $a^p \leqq b^p + 1$.

(ii) Let $b=1$. Then, putting $c=a^{p-1}$,

$$f(t)=a+t|t|^{p-2}-(c+t)|c+t|^{p-2}.$$

We have

$$f'(t)=(p-1)[|t|^{p-2}-|c+t|^{p-2}],$$

and so $f'(t)>0$ if $|t|>|c+t|$ and $f'(t)<0$ if $|t|<|c+t|$. Hence $f'(t)>0$ if $t<-\frac{1}{2}c$ and $f'(t)<0$ if $t>-\frac{1}{2}c$, and consequently f attains its *maximum value* when $t=-\frac{1}{2}c$. This maximum value is given by

$$f\left(-\frac{1}{2}c\right)=a-\frac{c^{p-1}}{2^{p-2}}=\frac{a}{2^{p-2}}(2^{p-2}-a^{p(p-2)}).$$

If $a^p>2$ then this maximum value is negative, and so f has no zeros. Otherwise, $f(-\frac{1}{2}c)\geqq 0$ whereas $f(0)=a-c^{p-1}=a-a^{(p-1)^2}\leqq 0$, and by the intermediate-value theorem, f has a zero in the closed interval $[-\frac{1}{2}c, 0]$.

(iii) $(1,x,y)$ is biorthogonal to $(a,b,1)$ if, and only if,

$$a+bx|x|^{p-2}+y|y|^{p-2}=0,$$

and

$$a^{p-1}+b^{p-1}x+y=0.$$

This is clearly the case if, and only if,

$$y=-[a^{p-1}+b^{p-1}x],$$

and

$$f(x)=a+bx|x|^{p-2}-[a^{p-1}+b^{p-1}x]|a^{p-1}+b^{p-1}x|^{p-2}=0. \qquad \square$$

Corollary 3.2. *Let* $\mathbf{x}=(a,b,1)\in\ell^p(3)$ *where* $a>b>1$. *Then there exist real numbers* α,α' *with* $\alpha<0$ *and* $\alpha'>0$ *such that* $(1,\alpha,\alpha')$ *is biorthogonal to* $\mathbf{x}$.

Proof. Let $p>2$. The case $1<p<2$ is treated similarly. Let f be the function defined in Lemma 3.1. Since $b>1$, we have seen that $f(t)\to\infty$ as $t\to-\infty$. Since

$$f\left(-\left(\frac{a}{b}\right)^{p-1}\right)=\frac{a}{b^{p(p-2)}}[b^{p(p-2)}-a^{p(p-2)}]<0.$$

it follows that $f(\alpha)=0$ for some α, with $\alpha<-(\frac{a}{b})^{p-1}<0$. If $\alpha'=-[a^{p-1}+b^{p-1}\alpha]$, then $\alpha'>0$. By Lemma 3.1(iii), $(1,\alpha,\alpha')$ is biorthogonal to $\mathbf{x}$. $\square$

Proposition 3.3. *Let $p>1$, $p\neq 2$. Let $\mathbf{x}=(a,b,c)\in\ell^p(3)$. Then*

(i) $\tau(\mathbf{x})\geqq 1$,

(ii) *If a,b,c are non-zero, and if $\{\alpha,\beta,\gamma\}$ is a permutation of $\{a,b,c\}$ with $|\alpha|\geqq|\beta|\geqq|\gamma|$, then $\tau(\mathbf{x})=2$ if, and only if, $|\alpha|^p\leqq|\beta|^p+|\gamma|^p$.*

Proof. In what follows f is the function defined in Lemma 3.1.

(i) Let $\mathbf{x}=(a,b,c)$. We can assume that a,b,c are non-zero, since otherwise one of the vectors $(1,0,0),(0,1,0),(0,0,1)$ will be biorthogonal to $\mathbf{x}$. By Proposition 1.1(i), (ii), and (iii), we can assume w.l.o.g. that $a\geqq b\geqq c=1$. Moreover if $b=c$ then $(0,1,-1)$ is biorthogonal to $\mathbf{x}$, and so we can further assume that $a\geqq b>c=1$. Lemma 3.1(i) then shows that $f(x)=0$ for some real x, and so, by Lemma 3.1(iii), $(1,x,y)$ is biorthogonal to $\mathbf{x}$, where $y=-[a^{p-1}+b^{p-1}x]$. Hence $\tau(\mathbf{x})\geqq 1$.

(ii) Let $\mathbf{x}=(a,b,c)$ where a,b,c are non-zero. Again we may assume that $a\geqq b\geqq c=1$. We consider two cases.

Case 1. Let $b>1$. By Proposition 2.1(iii), $\tau(b,1)=0$, and consequently there is no non-zero vector of the form $(0,x,y)$ biorthogonal to $\mathbf{x}$. Hence $\tau(\mathbf{x})=2$ if, and only if, there are two linearly independent vectors of the form $(1,x,y)$ biorthogonal to $\mathbf{x}$. Lemma 3.1(iii) shows that this happens if, and only if, the function f has more than one zero, and so if, and only if, $a^p\leqq b^p+1=b^p+c^p$ (Lemma 3.1(i)).

Case 2. Let $b=1$. Then $\mathbf{x}=(a,1,1)$. Since $\tau(1,1)=1$, the only vectors of the form $(0,x,y)$ biorthogonal to $\mathbf{x}$ are scalar multiples of $(0,1,-1)$. Hence $\tau(\mathbf{x})=2$ if, and only if, there is some vector $(1,x,y)$ biorthogonal to $\mathbf{x}$. Lemma 3.1(iii) shows that this happens if, and only if, $f(x)=0$ for some x, and so if, and only if, $a^p\leqq 2=b^p+c^p$ (Lemma 3.1(ii)). □

Corollary 3.4. *Let $n\geqq 3$. Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\ell^p(n)$ with $x_1\geqq x_2\geqq\ldots\geqq x_n>0$. Then there exists a vector $\mathbf{y}$ of the form $(y_1,y_2,\ldots,y_{n-1},1)$ which is biorthogonal to $\mathbf{x}$.*

Proof. By putting $y_1=y_2=\ldots=y_{n-3}=0$ we may assume w.l.o.g. that $n=3$. Let $\mathbf{x}=(x_1,x_2,x_3)$ with $x_1\geqq x_2\geqq x_3>0$. If $x_1>x_2$ the result follows from the facts that $\tau(x_1,x_2)=0$ (Proposition 2.1(iii)) and $\tau(x_1,x_2,x_3)\geqq 1$ (Proposition 3.3(i)). If $x_1=x_2$ the result follows from the fact that $\tau(x_1,x_2)=1$ (Proposition 2.1(iii) and $\tau(x_1,x_2,x_3)=2$ (Proposition 3.3(ii)). □

4. The main theorem

The proof of the following proposition makes use of an inequality which we state in the form of a preliminary lemma. We recall that throughout $p>1$ and $p\neq 2$.

Lemma 4.1. *If $b\geqq c\geqq 1$, and $\lambda>0$ then*

$$(b+c\lambda^{p-1})-(b^{p-1}+\lambda c^{p-1})^{p-1}\neq 0. \tag{1}$$

Proof. Suppose that $p>2$. Elementary calculus shows that $(x_1+x_2)^{p-1}>x_1^{p-1}+x_2^{p-1}$ when $x_1,x_2>0$. With $x_1=b^{p-1}$, $x_2=\lambda c^{p-1}$ this inequality reduces to $(b^{p-1}+\lambda c^{p-1})^{p-1}>b^{(p-1)^2}+\lambda^{p-1}c^{(p-1)^2}$. Since $b^{(p-1)^2}\geqq b$ and $c^{(p-1)^2}\geqq c$, we see that the expression on the left-hand side of (1) is negative. A similar argument shows that this expression is positive if $1<p<2$. □

Proposition 4.2. *Let $a\geqq b\geqq c\geqq 1$. Let $\mathbf{x}=(a,b,c,1)\in\ell^p(4)$. Then for each $\lambda>0$ there exist $x_0(\lambda)$ and $z_0(\lambda)$ such that $(1,x_0(\lambda),\lambda x_0(\lambda),z_0(\lambda))$ is biorthogonal to* $\mathbf{x}$. *Moreover if $a>b\geqq c=1$ then $x_0(\lambda)$ is not constant on* $(0,\infty)$.

Proof. Let $\lambda>0$ and let $\mathbf{y}=(1,t,\lambda t,t')\in\ell^p(4)$. Then $\mathbf{x}$ is biorthogonal to $\mathbf{y}$ if, and only if,

$$a+(b+c\lambda^{p-1})t|t|^{p-2}+t'|t'|^{p-2}=0, \tag{1}$$

and

$$a^{p-1}+(b^{p-1}+\lambda c^{p-1})t+t'=0. \tag{2}$$

Substituting (2) into (1) we obtain the equation

$$f_\lambda(t)=a+(b+c\lambda^{p-1})t|t|^{p-2}-(a^{p-1}+(b^{p-1}+\lambda c^{p-1})t)|a^{p-1}+(b^{p-1}+\lambda c^{p-1})t|^{p-2}=0. \tag{3}$$

We have

$$\frac{f_\lambda(t)}{t|t|^{p-2}}\to(b+c\lambda^{p-1})-(b^{p-1}+\lambda c^{p-1})^{p-1}, \quad \text{as } |t|\to\infty.$$

By Lemma 4.1 this limit is non-zero, and it then follows from the intermediate-value theorem that equation (3) has a real root. For $\lambda>0$, let $x_0(\lambda)$ denote the least such root.

If

$$z_0(\lambda)=-[a^{p-1}+(b^{p-1}+\lambda c^{p-1})x_0(\lambda)],$$

it is clear that $(1,x_0(\lambda),\lambda x_0(\lambda),z_0(\lambda))$ is biorthogonal to x.

Let $a>b\geqq c=1$. Suppose, for a contradiction, that for some constant K, $x_0(\lambda)=K$ for all positive λ. Then

$$f_\lambda(K)=a+(b+\lambda^{p-1})K|K|^{p-2}-(a^{p-1}+(b^{p-1}+\lambda)K)|a^{p-1}+(b^{p-1}+\lambda)K|^{p-2}=0\ \forall\lambda>0. \tag{4}$$

Since $a>1$, we see from (4) that $K\neq 0$. Differentiating both sides of (4) with respect to λ we obtain

$$(p-1)\lambda^{p-2}K|K|^{p-2}-|a^{p-1}+(b^{p-1}+\lambda)K|^{p-2}(p-1)K=0\ \forall\lambda>0. \tag{5}$$

Dividing both sides of (5) by $(p-1)K|K|^{p-2}$ and taking $(p-2)^{th}$ roots gives

$$\lambda=\left|\frac{a^{p-1}}{K}+b^{p-1}+\lambda\right| \forall\lambda>0, \tag{6}$$

and (6) implies that

$$K=-\frac{a^{p-1}}{b^{p-1}}.$$

Since

$$f_\lambda\left(\frac{-a^{p-1}}{b^{p-1}}\right)=\frac{a(b^{p(p-2)}-a^{p(p-2)})}{b^{p(p-2)}}\neq 0,$$

we obtain the desired contradiction. □

Proposition 4.3. *Let* $\mathbf{x}=(a,b,c,d)\in\ell^p(4)$, *where* a,b,c,d *are non-zero. Then* $\tau(\mathbf{x})=3$.

Proof. By (i), (ii), (iii) of Proposition 1.1, we can assume w.l.o.g. that $a\geqq b\geqq c\geqq d=1$. We consider three cases.

Case 1. Suppose that at least two of a,b,c are equal.

Suppose first that $a=b$. Then $\tau(a,b,c)=2$ by Proposition 3.3(ii). Hence there are two linearly independent vectors in $\ell^p(4)$ which are biorthogonal to $(a,b,c,1)$ and whose last coordinates are 0. By Corollary 3.4 there is also a vector in $\ell^p(4)$ which is biorthogonal to $(a,b,c,1)$ and whose last coordinate is 1. The three vectors so obtained are linearly independent, and hence $\tau(a,b,c,1)=3$.

Now suppose that $b=c$. Then $\tau(b,c,1)=2$ by Proposition 3.3(ii). Hence there are two linearly independent vectors in $\ell^p(4)$ which are biorthogonal to $(a,b,c,1)$ and whose first coordinates are 0. By Proposition 4.2, there is also a vector in $\ell^p(4)$ which is biorthogonal to $(a,b,c,1)$ whose first coordinate is 1. Again $\tau(a,b,c,1)=3$.

Case 2. Suppose that $a>b>c>1$.

Corollary 3.2 shows that there exist vectors $\mathbf{X}=(1,0,\alpha,\alpha')$, $\mathbf{Y}=(0,1,\beta,\beta')$, $\mathbf{Z}=(1,\gamma,0,\gamma')$ biorthogonal to $\mathbf{x}$, where $\alpha,\beta,\gamma<0$ and $\alpha',\beta',\gamma'>0$. We shall show that the vectors $\mathbf{X},\mathbf{Y},\mathbf{Z}$ are linearly independent, by showing that

$$\begin{vmatrix}1 & 0 & \alpha\\ 0 & 1 & \beta\\ 1 & \gamma & 0\end{vmatrix}\neq 0.$$

In fact suppose, for a contradiction, that this determinant is zero. Then

https://doi.org/10.1017/S001309150002304X Published online by Cambridge University Press

$$\beta = -\frac{\alpha}{\gamma}. \tag{1}$$

Since $[\mathbf{X}, \mathbf{x}] = [\mathbf{Y}, \mathbf{x}] = [\mathbf{Z}, \mathbf{x}] = 0$, we have

$$\alpha' = -(a^{p-1} + c^{p-1}\alpha), \quad \beta' = -(b^{p-1} + c^{p-1}\beta), \quad \gamma' = -(a^{p-1} + b^{p-1}\gamma) \tag{2}$$

and it follows from (1) and (2) that

$$\frac{\gamma' - \alpha'}{\gamma} = \beta'. \tag{3}$$

Since $\gamma < 0$ and $\beta' > 0$, we see that $\alpha' > \gamma'$.

By (1) and (3),

$$-\gamma\mathbf{Y} = (0, -\gamma, -\gamma\beta, -\gamma\beta') = (0, -\gamma, \alpha, \alpha' - \gamma'),$$

and since $[\mathbf{x}, -\gamma\mathbf{Y}] = 0$ we have, noting that $\alpha' > \gamma'$,

$$-b\gamma|\gamma|^{p-2} + c\alpha|\alpha|^{p-2} + (\alpha' - \gamma')^{p-1} = 0 \tag{4}$$

Now $[\mathbf{x}, \mathbf{X}] = [\mathbf{x}, \mathbf{Z}] = 0$, and so we also have

$$[\mathbf{x}, \mathbf{X}] - [\mathbf{x}, \mathbf{Z}] = -b\gamma|\gamma|^{p-2} + c\alpha|\alpha|^{p-2} + \alpha'^{p-1} - \gamma'^{p-1} = 0. \tag{5}$$

Subtracting (5) from (4), we deduce that

$$(\alpha' - \gamma')^{p-1} - \alpha'^{p-1} + \gamma'^{p-1} = 0.$$

If

$$r = \frac{\alpha'}{\gamma'},$$

this gives

$$(r-1)^{p-1} - r^{p-1} + 1 = 0. \tag{6}$$

Elementary calculus shows that, since $p \neq 2$, the expression on the left-hand side of (6) is strictly monotonic in r, and so (6) is satisfied only when $r = 1$. Since

$$r = \frac{\alpha'}{\gamma'} > 1,$$

we have the desired contradiction.

Case 3. Suppose that $a > b > c = 1$.

If $\mathbf{y}_1=(0,0,1,-1)$, then $\mathbf{y}_1$ is biorthogonal to $\mathbf{x}$. Corollary 3.2 shows that there exists a vector $\mathbf{y}_2=(1,\alpha,0,\alpha')$ biorthogonal to $\mathbf{x}$. By Proposition 4.2 for each $\lambda>0$ there exist $x(\lambda)$ and $z(\lambda)$ such that $\mathbf{y}_3(\lambda)$ is biorthogonal to $\mathbf{x}$, where $\mathbf{y}_3(\lambda)=(1,x(\lambda),\lambda x(\lambda),z(\lambda))$. Suppose, for a contradiction, that the three vectors $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3(\lambda)$ are linearly dependent for all $\lambda>0$. Then

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & \alpha & 0 \\ 1 & x(\gamma) & \lambda x(\lambda) \end{vmatrix} \neq 0.$$

This implies that

$$x(\lambda)=\alpha \ \forall\lambda>0,$$

and so contradicts the second part of Proposition 4.2. Hence $\tau(\mathbf{x})=3$. □

Proposition 4.4. *Let $n\geqq 4$ and let $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\ell^p(n)$ where $x_i\neq 0$ $\forall i$. Then*

$$\tau(\mathbf{x})=n-1. \tag{1}$$

Proof. We proceed by induction. Proposition 4.3 shows that (1) holds when $n=4$. Let $k\geqq 4$, and suppose that (1) holds when $n=k$. Let $\mathbf{x}=(x_1,x_1,\ldots,x_{k+1})\in\ell^p(k+1)$. By Proposition 1.1(ii), (iii) we may assume without loss of generality that $x_1\geqq x_2\geqq\ldots\geqq x_{k+1}>0$. Then $(x_1,x_2,\ldots,x_k)\in\ell^p(k)$, and by the inductive hypothesis there exists a linearly independent set of $k-1$ vectors in $\ell^p(k)$ biorthogonal to $(x_1,x_1,\ldots,x_k)$. By adding a final zero coordinate to each of the vectors in this set, we obtain a linearly independent set of $k-1$ vectors $\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_{k-1}$ in $\ell^p(k+1)$ biorthogonal to $\mathbf{x}$. By Corollary 3.4 there exists a vector $\mathbf{y}_k$ with final coordinate equal to 1 which is biorthogonal to $\mathbf{x}$. The set of vectors $\{\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_k\}$ is then a linearly independent set of k vectors in $\ell^p(k+1)$ biorthogonal to $\mathbf{x}$. Since $\mathbf{x}\neq\mathbf{0}$, $\tau(\mathbf{x})=k$. Hence (1) holds for $n=k+1$, and the proof is complete. □

An application of Proposition 2.1(ii), (iii), Proposition 3.3(ii) and Proposition 4.4, together with the properties (ii), (iii), (iv) and (v) of $\tau(\mathbf{x})$ listed in Proposition 1.1 now readily yield our main result.

Theorem 4.5. *Let $n\geqq 2$, and let $\mathbf{x}\in\ell^p(n)$. Let k be the number of non-zero coordinates of* $\mathbf{x}$.

(i) *If $k=0$ then $\tau(\mathbf{x})=n$.*

(ii) *If $k=1$ or $k\geqq 4$ then $\tau(\mathbf{x})=n-1$.*

(iii) *If $k=2$ then $\tau(\mathbf{x})=n-1$ if the two non-zero coordinates have equal modulus, and $\tau(\mathbf{x})=n-2$ otherwise.*

https://doi.org/10.1017/S001309150002304X Published online by Cambridge University Press

(iv) *If* $k=3$, *let* $\{\alpha,\beta,\gamma\}$ *be a permutation of the three non-zero coordinates such that* $|\alpha|\geqq|\beta|\geqq|\gamma|$. *Then* $\tau(\mathbf{x})=n-1$ *if* $|\alpha|^p\leqq|\beta|^p+|\gamma|^p$ *and* $\tau(\mathbf{x})=n-2$ *otherwise.*

Acknowledgement. We would like to thank the referee for his helpful comments concerning the presentation of this paper.

REFERENCES

1. J. R. Giles, Classes of semi-inner-product spaces, *Trans. Amer. Math. Soc.* **129** (1967), 436–446.

2. G. Lumer, Semi-inner-product spaces, *Trans. Amer. Math. Soc.* **100** (1961), 29–43.

Department of Mathematics
University of Wales, Swansea
Singleton Park
Swansea SA2 8PP
Wales