FORCED MOTION OF A CHARGED PARTICLE IN A MAGNETIC FIELD

L. J. GLEESON

(Received 30 August 1965)

1. Introduction and summary

We consider the motion of a particle of mass m and electrical charge e, moving in a constant magnetic field $B\boldsymbol{k}$, where $\boldsymbol{k}$ is a unit vector, and acted upon by a force $m\boldsymbol{f}(t)$. The position vector $\boldsymbol{r}(t)$ of this particle is governed by the differential equation

$$\ddot{\boldsymbol{r}}(t)+\omega\boldsymbol{k}\times\dot{\boldsymbol{r}}(t) = \boldsymbol{f}(t) \tag{1.1}$$

where $\omega = eB/m$.

The subscript $\perp$ will be used to indicate vectors and their components which are perpendicular to $\boldsymbol{k}$.

An analysis of this motion has been given by Alfvén [1], and Alfvén and Fälthammar [2]. A result given on p. 18[1] is that, if projected on to a plane transverse to $\boldsymbol{k}$, "the path is a circle the centre of which drifts with velocity $\boldsymbol{u}_\perp$ given by the differential equation [1]

$$\boldsymbol{u}_\perp = -(m/eB^2)\boldsymbol{B}\times\{\boldsymbol{f}_\perp(t)-(d\boldsymbol{u}_\perp/dt)\}." \tag{1.2}$$

In the relations (26) and (27) on p. 18 of [1] Alfvén gives a series solution for $\boldsymbol{u}_\perp$ in terms of the derivatives of $\boldsymbol{f}_\perp(t)$.

Taking the vector product of each side of the equation (1.2) with $\boldsymbol{k}$ shows that it is an algebraic rearrangement of the perpendicular part of the original equation of motion (1.1) with $\boldsymbol{u}_\perp$ replacing $\dot{\boldsymbol{r}}_\perp$. Thus no integration of equation (1.1) has been achieved.

In this paper we obtain the solution of the differential equation (1.1) using a method which is essentially that given by Westfold [7] for any linear, vector differential equation with a gyroscopic term, $g(D)$, viz.,

$$f(D)\boldsymbol{r}+\omega\boldsymbol{k}\times g(D)\boldsymbol{r} = \boldsymbol{f}(t), \tag{1.3}$$

where D represents the operator d/dt.

Our main result is to show that this solution in the transverse plane may be rearranged systematically by repeated integration by parts, and that each time this is done an expression of the same form is obtained.

[1] A change of units and notation has been used here.

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

These forms may be considered to be a combination of "guiding centre" and "rotation" about the guiding centre. Such descriptions are most helpful when the "rotation" is substantially uniform over many periods of the free motion, and the path of the guiding centre is specified by a relatively simple function.

The "rotation" is further divided into a uniform rotation and a residual term. The residual term can be made to depend on successively higher-order integrals or derivatives of the forcing function $\boldsymbol{f}(t)$. If it is negligible, we have the solution expressed conveniently in terms of a guiding-centre function and uniform rotation.

The form with a higher-order derivative of the forcing function in the integrand of the residual term is useful when the forcing function varies by a small fraction of itself during a period of the free motion. We describe this as varying slowly. An alternative form with a successively higher-order integral of the forcing function in the integrand of the residual term is useful when the forcing function is varying rapidly.

Three results, two from Alfvén [1], [2], and the polarization drift that occurs in plasma physics, Chandrasekhar [4] or Thompson [6], each derived in a different way in these references, are obtained as particular cases of a general result.

A simple example, $\boldsymbol{f}(t) = \boldsymbol{j}K \cos \omega_i t$, is used to illustrate the use of various forms of the solution. This driving force is used in a precision ion selector known as an omegatron described by Hipple et al. [5] and analyzed by Berry [3].

2. The basic solution

In this section we derive the basic solution of the equation of motion (1.1) and in later sections rearrange it into more convenient forms.

Following Westfold [7] we express equation (1.1) in terms of components in the directions of the eigenvectors of the operator $(\boldsymbol{k}\times)$, viz., the solutions of the equation

$$\boldsymbol{k} \times \boldsymbol{e} = \lambda \boldsymbol{e}. \tag{2.1}$$

Referred to a real orthogonal Cartesian base such that $\boldsymbol{k} = (0, 0, 1)$, and with $i = \sqrt{-1}$, the unit eigenvectors are

$$\boldsymbol{e}_1 = (1, i, 0)/\sqrt{2}, \qquad \boldsymbol{e}_2 = (1, -i, 0)/\sqrt{2}, \qquad \boldsymbol{e}_3 = \boldsymbol{k}. \tag{2.2}$$

The corresponding eigenvalues are

$$\lambda_1 = -i, \qquad \lambda_2 = i, \qquad \lambda_3 = 0. \tag{2.3}$$

With these eigenvectors we may express any vector $\boldsymbol{V}$ in the form

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

$$V = V_1 e_1 + V_2 e_2 + V_3 e_3 \tag{2.4}$$

in which the vector components parallel and perpendicular to $\boldsymbol{k}$ are, respectively,

$$V_\parallel = V_3 e_3, \qquad V_\perp = V_1 e_1 + V_2 e_2. \tag{2.5}$$

In particular, denoting the components of $\boldsymbol{r}$ and $\boldsymbol{f}$ by x_j and f_j, $j = 1, 2, 3$, equation (2.1) becomes

$$(\ddot{x}_1 - i\omega\dot{x}_1)e_1 + (\ddot{x}_2 + i\omega\dot{x}_2)e_2 + \ddot{x}_3 e_3 = f_1 e_1 + f_2 e_2 + f_3 e_3. \tag{2.6}$$

The component equations of (2.6) will be solved assuming the initial conditions

$$\dot{\boldsymbol{r}}(0) = \boldsymbol{u}, \qquad \boldsymbol{r}(0) = \boldsymbol{a}. \tag{2.7}$$

The 2-component equation

$$\ddot{x}_2 + i\omega\dot{x}_2 = f_2(t)$$

is an elementary differential equation for which, after successive integration, we have, with initial conditions (2.7),

$$\dot{x}_2 = \left\{u_2 + \int_0^t e^{i\omega\tau} f_2(\tau)d\tau\right\} e^{-i\omega t} \tag{2.8}$$

and

$$x_2 = a_2 + \frac{1}{i\omega}\left\{u_2 + \int_0^t f_2(\tau)d\tau\right\} - \frac{1}{i\omega}\left\{u_2 + \int_0^t e^{i\omega\tau} f_2(\tau)d\tau\right\} e^{-i\omega t}. \tag{2.9}$$

The corresponding results for $\dot{x}_1$, x_1 are obtained by substituting the subscript 1 for 2 and replacing ω by $-\omega$ wherever it occurs.

With these results we may write down the components perpendicular to $\boldsymbol{k}$,

$$\dot{\boldsymbol{r}}_\perp = \dot{x}_1 e_1 + \dot{x}_2 e_2, \qquad \boldsymbol{r}_\perp = x_1 e_1 + x_2 e_2, \tag{2.10}$$

but delay doing this until section 4.

The parallel components

$$\dot{\boldsymbol{r}}_\parallel = \dot{x}_3 e_3, \qquad \boldsymbol{r}_\parallel = x_3 e_3,$$

are obtained directly by integration of the 3-component equation from (2.6). We obtain

$$\begin{aligned} \dot{x}_3 &= u_3 + \int_0^t f_3(\tau)d\tau, \\ x_3 &= a_3 + u_3 t + \int_0^t dt' \int_0^{t'} f_3(\tau)d\tau, \end{aligned} \tag{2.11}$$

the usual results for rectilinear accelerated motion in the direction of $\boldsymbol{k}$. It is of no further interest here.

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

3. Interpreting combinations of complex number components

The functions $\boldsymbol{r}_\perp(t)$ and $\dot{\boldsymbol{r}}_\perp(t)$ are obtained by substituting from the expressions (2.8) and (2.9) for $\dot{x}_2$ and x_2 and the corresponding ones for $\dot{x}_1$, x_1 into equations (2.10). Since the 1-component solutions are obtained from the 2-component solutions by the changes

$$\text{subscript } 2 \to \text{subscript } 1, \qquad \omega \to -\omega,$$

certain characteristic combinations of terms appear in V_1 and V_2, the 1 and 2-component expressions. The combinations that appear are examined below and expressed in a form suitable for physical interpretation. These results are used in section 4 when giving explicit expressions for $\boldsymbol{r}_\perp$ and $\dot{\boldsymbol{r}}_\perp$.

When dealing with a physical problem $\boldsymbol{f}$ will be real and $\boldsymbol{r}(t)$ and $\dot{\boldsymbol{r}}(t)$ will be real vectors even though the eigenvector components may not be.

For our discussion we assume we have 1 and 2-components A_1 and A_2 with the corresponding vector

$$\boldsymbol{A}_\perp = A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 \tag{3.1}$$

and examine components V_1 and V_2 related to these.

Case (*i*) $V_1 = cA_1$, $V_2 = cA_2$. Substitution into

$$\boldsymbol{V}_\perp = V_1\boldsymbol{e}_1 + V_2\boldsymbol{e}_2 \tag{3.2}$$

gives immediately the correspondence

$$(cA_1,\ cA_2) \leftrightarrow c\boldsymbol{A}_\perp. \tag{3.3}$$

Case (*ii*) $V_1 = -iA_1$, $V_2 = iA_2$. Substitution into (3.2) gives

$$\boldsymbol{V}_\perp = -iA_1\boldsymbol{e}_1 + iA_2\boldsymbol{e}_2.$$

But from equations (2.1), (2.2) and (2.3) we have

$$-i\boldsymbol{e}_1 = \boldsymbol{k}\times\boldsymbol{e}_1, \quad i\boldsymbol{e}_2 = \boldsymbol{k}\times\boldsymbol{e}_2. \tag{3.4}$$

Hence

$$\boldsymbol{V}_\perp = \boldsymbol{k}\times(A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2) = \boldsymbol{k}\times\boldsymbol{A}_\perp,$$

and we have the correspondence

$$(-iA_1, iA_2) \leftrightarrow \boldsymbol{k}\times\boldsymbol{A}_\perp. \tag{3.5}$$

Case (*iii*) $V_1 = A_1e^{-i\theta}$, $V_2 = A_2e^{i\theta}$, θ real. Here we have

$$\begin{aligned}\boldsymbol{V}_\perp &= (\cos\theta - i\sin\theta)A_1\boldsymbol{e}_1 + (\cos\theta + i\sin\theta)A_2\boldsymbol{e}_2\\ &= \cos\theta(A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2) + \sin\theta(-iA_1\boldsymbol{e}_1 + iA_2\boldsymbol{e}_2).\end{aligned}$$

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

Thus by equations (3.3) and (3.5)

$$V_\perp = \cos\theta\, A_\perp + \sin\theta(k\times A_\perp) \tag{3.5a}$$

a vector obtained by rotating $A_\perp$ through the angle θ in the positive sense about k.

It is convenient to introduce the following notation, which exhibits the property of rotation noted above,

$$[A_\perp, \theta] \equiv \cos\theta\, A_\perp + \sin\theta(k\times A_\perp). \tag{3.6}$$

We shall always use brackets, [ ], with this meaning in this paper. Then for the correspondence we have

$$(A_1 e^{-i\theta}, A_2 e^{i\theta}) \leftrightarrow [A_\perp, \theta]. \tag{3.7}$$

Cases (i) and (ii) are in fact special cases of case (iii) obtained by putting $\theta = 0$ and $\theta = \pi/2$.

It follows immediately from the definition (3.6) that

$$k\times [A_\perp, \theta] = [k\times A_\perp, \theta] = [A_\perp, \theta+\pi/2]. \tag{3.8}$$

Putting $V_1 = A_1 e^{-i(\theta+\alpha)}$, $V_2 = A_2 e^{i(\theta+\alpha)}$ and using (3.7) we readily establish that

$$[A_\perp, \theta+\alpha] = [[A_\perp, \theta], \alpha] = [[A_\perp, \alpha], \theta]. \tag{3.9}$$

4. The basic solution in vector form

Using the results of section 3 we now express the solutions obtained in section 2 in vector form independent of the base vectors.

Substituting the expressions for $\dot{x}_2$ and $\dot{x}_1$ obtained from (2.8) into (2.10) and using the result (3.7), we obtain

$$\dot{r}_\perp(t) = \left[u_\perp + \int_0^t [f_\perp(\tau), \omega\tau]\,d\tau,\ -\omega t\right]. \tag{4.1}$$

Similarly from (2.10) and (2.9) and using the results (3.5) and (3.7) we obtain

$$\begin{aligned} r_\perp(t) = a_\perp - \frac{1}{\omega} k\times \left\{u_\perp + \int_0^t f_\perp(\tau)\,d\tau\right\} \\ + \left[\frac{1}{\omega} k\times \left(u_\perp + \int_0^t [f_\perp(\tau), \omega\tau]\,d\tau\right), -\omega t\right]. \end{aligned} \tag{4.2}$$

It is convenient to write this expression in the form

$$r_\perp(t) = S(t) + [A(t), -\omega t], \tag{4.3}$$

with

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

$$S(t) = \boldsymbol{a}_\perp - \frac{1}{\omega}\boldsymbol{k}\times\left\{\boldsymbol{u}_\perp + \int_0^t \boldsymbol{f}_\perp(\tau)d\tau\right\}, \tag{4.4}$$

and

$$\boldsymbol{A}(t) = \frac{1}{\omega}\boldsymbol{k}\times\left\{\boldsymbol{u}_\perp + \int_0^t [\boldsymbol{f}_\perp(\tau), \omega\tau]d\tau\right\}. \tag{4.5}$$

If $\boldsymbol{S}(t)$ and $\boldsymbol{A}(t)$ are constant, the transverse motion consists of uniform rotation with radius A and angular velocity $-\omega$ about a fixed centre $\boldsymbol{S}$. This situation occurs when the forcing term $\boldsymbol{f}_\perp(t)$ remains zero.

Suppose the driving-force terminates at time t_1, i.e., if $\boldsymbol{f}(t) = 0$ for $t \geqq t_1$. Then $\boldsymbol{S}(t)$ and $\boldsymbol{A}(t)$ are constant for $t \geqq t_1$ with the values $\boldsymbol{S}(t_1)$ and $\boldsymbol{A}(t_1)$ respectively. The motion is a uniform rotation with radius $A(t_1)$ about the centre $\boldsymbol{S}(t_1)$.

When $\boldsymbol{A}(t)$ alone is sensibly constant the motion consists of a uniform rotation superimposed on a moving centre $\boldsymbol{S}(t)$. The form (4.3) is then a readily visualized specification of the motion, and later we shall see how it may be approximately brought about in some cases.

Beginning with the expressions (4.1) and (4.5) we readily deduce a general relation between the rotation term and the transverse velocity

$$[\boldsymbol{A}(t), -\omega t] = (1/\omega)\boldsymbol{k}\times\dot{\boldsymbol{r}}_\perp \tag{4.6}$$

and with this the expression, (4.3) for $\boldsymbol{r}_\perp$ becomes

$$\boldsymbol{r}_\perp(t) = \boldsymbol{S}(t) + (1/\omega)\boldsymbol{k}\times\dot{\boldsymbol{r}}_\perp. \tag{4.7}$$

The term $(1/\omega)\boldsymbol{k}\times\dot{\boldsymbol{r}}_\perp$ equals the transverse radius vector in the case of uniform circular motion with an angular velocity $-\omega$.

In the form (4.7) a comparison with Alfvén's definitions, equation (14)[2] and equation (9) p. 17[1], shows that $\boldsymbol{S}(t)$ is identical with the function he terms the guiding centre. I prefer to use this term for cases in which $\boldsymbol{A}(t)$ remains substantially constant over several periods as only then can the term $[\boldsymbol{A}(t), -\omega t]$ be properly regarded as a uniform rotation.

Taking the vector product of each side of the equality (4.6) with $\boldsymbol{k}$ and using the result (3.8) we have immediately the alternative form

$$\dot{\boldsymbol{r}}_\perp = \omega[\boldsymbol{A}(t), -\omega t - \pi/2] \tag{4.8}$$

which specifies the transverse velocity in terms of ω and $\boldsymbol{A}(t)$.

5. Some alternative forms of the solution

We have observed that the term $[\boldsymbol{A}(t), -\omega t]$ in the expression for $\boldsymbol{r}_\perp$, (4.3), cannot be regarded as a pure rotation unless $\boldsymbol{f}_\perp(t)$ remains sensibly zero. However, by repeated integration by parts, we are able to find alter-

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

native series forms of the solution (4.3) consisting of a pure rotation $[\boldsymbol{A}_n, -\omega t]$, superimposed on a moving (guiding) centre, $\boldsymbol{S}_n(t)$, and a residual term $[\boldsymbol{R}_n(t), -\omega t]$. In any particular application we would attempt to select a form in which $\boldsymbol{R}_n(t)$ is small and negligible.

Two forms arise corresponding to the alternative ways of expanding the second integral in (4.2). The residual term $\boldsymbol{R}_n(t)$ depends correspondingly on $(D/\omega)^n \boldsymbol{f}_\perp(t)$, $n > 0$, and $(D/\omega)^{-|n|}\boldsymbol{f}_\perp(t)$, $n < 0$. It becomes negligible in the first form or the second form depending on whether $\boldsymbol{f}_\perp(t)$ varies slowly or rapidly during a period of the free motion $2\pi/\omega$.

Case (*a*) The form for $\boldsymbol{f}_\perp(t)$ varying slowly.

The expansions of $\boldsymbol{r}_\perp(t)$ are most readily carried out in terms of the components x_1 and x_2. Partial integration of the second integral in the expression (2.9) for x_2 gives

(5.1)
$$e^{-i\omega t}\int_0^t e^{i\omega t} f_2(\tau)d\tau = \left(\frac{1}{i\omega}\right) f_2(t) - \left(\frac{1}{i\omega}\right) f_2(0)e^{-i\omega t} + e^{-i\omega t}\int_0^t e^{i\omega\tau}\left(\frac{iD}{\omega}\right) f_2(\tau)d\tau.$$

If the typical values of $Df_2(\tau)/\omega$ are small in magnitude compared with those of $f_2(\tau)$ in the range $0 \leqq \tau \leqq t$, the integral on the right side will usually be small compared with that on the left side and may be neglected. This condition implies that $f_2(t)$ remains substantially constant over each period of the free motion $2\pi/\omega$. We describe this as varying slowly.

Repeated integration in this manner gives

(5.2)
$$e^{-i\omega t}\int_0^t e^{i\omega\tau} f_2(\tau)d\tau = \frac{1}{i\omega}\sum_{r=0}^{n-1}\left(\frac{iD}{\omega}\right)^r f_2(t) - e^{-i\omega t}\,\frac{1}{i\omega}\sum_{r=0}^{n-1}\left(\frac{iD}{\omega}\right)^r f_2(0) + e^{-i\omega t}\int_0^t e^{i\omega\tau}\left(\frac{iD}{\omega}\right)^n f_2(\tau)d\tau.$$

Using this in (2.9) we may write for $n > 0$,

(5.3)
$$x_2 = S_{2,n}(t) + \{A_{2,n} + R_{2,n}(t)\}e^{-i\omega t}$$

where

(5.4)
$$S_{2,n}(t) = a_2 + \frac{1}{i\omega}\left\{u_2 + \int_0^t f_2(\tau)d\tau\right\} + \frac{1}{\omega^2}\sum_{r=0}^{n-1}\left(\frac{iD}{\omega}\right)^r f_2(t),$$

(5.5)
$$A_{2,n} = -\left(\frac{u_2}{i\omega}\right) - \frac{1}{\omega^2}\sum_{r=0}^{n-1}\left(\frac{iD}{\omega}\right)^r f_2(0),$$

(5.6)
$$R_{2,n}(t) = -\left(\frac{i}{\omega}\right)\int_0^t e^{i\omega\tau}\left(\frac{iD}{\omega}\right)^n f_2(\tau)d\tau.$$

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

From (5.2) and the previous discussion, $R_{2,n}(t)$ will usually be negligible whenever

$$(5.7) \qquad |(D/\omega)^n f_2(\tau)|_{\text{typical}} \ll |f_2(\tau)|_{\text{typical}}, \quad 0 \leqq \tau \leqq t.$$

Writing down x_1 in terms of $S_{1,n}$, $A_{1,n}$, and $R_{1,n}$ and using the principles of section 3 when combining x_1 and x_2 to form $\boldsymbol{r}_\perp(t)$ we obtain for $n > 0$,

$$(5.8) \qquad \boldsymbol{r}_\perp(t) = \boldsymbol{S}_n(t) + [\boldsymbol{A}_n + \boldsymbol{R}_n(t), -\omega t],$$

where

$$(5.9) \qquad \begin{aligned} \boldsymbol{S}_n(t) &= \boldsymbol{a}_\perp - \frac{1}{\omega}\boldsymbol{k} \times \left\{\boldsymbol{u}_\perp + \int_0^t \boldsymbol{f}_\perp(\tau)d\tau\right\} \\ &\quad + \frac{1}{\omega^2}\left\{1 + \boldsymbol{k} \times \left(\frac{D}{\omega}\right) - \left(\frac{D}{\omega}\right)^2 - \boldsymbol{k} \times \left(\frac{D}{\omega}\right)^3 + \cdots T_{n-1}\left(\frac{D}{\omega}\right)^{n-1}\right\}\boldsymbol{f}_\perp(t), \end{aligned}$$

$$(5.10) \qquad \begin{aligned} \boldsymbol{A}_n = \frac{1}{\omega}\boldsymbol{k} \times \boldsymbol{u}_\perp - \frac{1}{\omega^2}\Bigg\{1 + \boldsymbol{k} \times \left(\frac{D}{\omega}\right) - \left(\frac{D}{\omega}\right)^2 - \boldsymbol{k} \times \left(\frac{D}{\omega}\right)^3 \\ + \cdots T_{n-1}\left(\frac{D}{\omega}\right)^{n-1}\Bigg\}\boldsymbol{f}_\perp(0), \end{aligned}$$

$$(5.11) \qquad \boldsymbol{R}_n(t) = \frac{1}{\omega}\int_0^t \left[\left(\frac{D}{\omega}\right)^n \boldsymbol{f}_\perp(\tau), \omega\tau + (n+1)\pi/2\right] d\tau.$$

The terms in the braces in (5.9) and (5.10) are the operators $T_p(D/\omega)^p$ with

$$T_p\left(\frac{D}{\omega}\right)^p = \begin{cases} (-1)^{(p-1)/2}\boldsymbol{k} \times \left(\frac{D}{\omega}\right)^p, & p \text{ odd}, \\ (-1)^p\left(\frac{D}{\omega}\right)^p, & p \text{ even}. \end{cases}$$

Corresponding to the inequality (5.7) $\boldsymbol{R}_n(t)$ is usually negligible when, typically

$$(5.12) \qquad |(D/\omega)^n \boldsymbol{f}_\perp(\tau)| \ll |\boldsymbol{f}_\perp(\tau)|, \qquad 0 \leqq \tau \leqq t.$$

Case (*b*) The form for $\boldsymbol{f}_\perp(t)$ varying rapidly.

When the second integral on the right of the expression (2.9) is integrated by parts, in the alternative manner to that used to obtain (5.1) we find

$$(5.13) \qquad \begin{aligned} e^{-i\omega t}\int_0^t e^{i\omega\tau} f_2(\tau)d\tau &= -\frac{1}{i\omega}\left(\frac{iD}{\omega}\right)^{-1} f_2(t) + e^{-i\omega t}\frac{1}{i\omega}\left(\frac{iD}{\omega}\right)^{-1} f_2(0) \\ &\quad + e^{-i\omega t}\int_0^t e^{i\omega\tau}\left(\frac{iD}{\omega}\right)^{-1} f_2(\tau)d\tau. \end{aligned}$$

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

Here we define,

$$D^{-1}f(t) \equiv \int_{\alpha_1}^{t} f(\tau_1)d\tau_1, \tag{5.14}$$

and for later

$$D^{-s}f(t) \equiv \int_{\alpha_s}^{t} d\tau_s \int_{\alpha_{s-1}}^{\tau_s} d\tau_{s-1} \cdots \int_{\alpha_1}^{\tau_2} f(\tau_1)d\tau_1, \tag{5.15}$$

the constants α_i, $(i = 1, \cdots, s)$, being chosen to make the integrals as simple as possible, and to assist in realizing the inequalities (5.16), below, and (5.23).

If

$$|(D/\omega)^{-1}f_2(\tau)|_{\text{typical}} \ll |f_2(\tau)|_{\text{typical}}, \qquad 0 \leqq \tau \leqq t, \tag{5.16}$$

the integral on the right side of (5.13) will usually be small compared with that on the left side, and may be omitted.

The condition (5.16) implies that

$$|Df_2/\omega|_{\text{typical}} \gg |f_2|_{\text{typical}},$$

i.e., typically, the proportional change in f_2 during a period of the free motion is large. We describe this as varying rapidly.

Repeated use of the identity (5.13) gives

$$\begin{aligned} e^{-i\omega t}\int_0^t e^{i\omega\tau} f_2(\tau)d\tau = &-\frac{1}{i\omega}\sum_{r=1}^{s}\left(\frac{iD}{\omega}\right)^{-r} f_2(t) + e^{-i\omega t}\frac{1}{i\omega}\sum_{r=1}^{s}\left(\frac{iD}{\omega}\right)^{-r} f_2(0) \\ &+ e^{-i\omega t}\int_0^t e^{i\omega\tau}\left(\frac{iD}{\omega}\right)^{-s} f_2(\tau)d\tau. \end{aligned} \tag{5.17}$$

The α_i implicit in $(D/\omega)^{-s}f_2(\tau)$ may be chosen to make this quantity as small as possible.

Substituting from the identity (5.17) into the equation for x_2, (2.9), we may now write for $s > 0$,

$$x_2 = S_{2,-s}(t) + \{A_{2,-s} + R_{2,-s}(t)\}e^{-i\omega t}, \tag{5.18}$$

with $S_{2,-s}(t)$, $A_{2,-s}$, and $R_{2,-s}(t)$ having definitions similar to those given in equations (5.4)—(5.6).

Writing down x_1 in terms of $S_{1,-s}(t)$, $A_{1,-s}$, $R_{1,-s}(t)$, and using the principles of section 3 when combining x_1 and x_2 to form $\mathbf{r}_\perp(t)$, we obtain for $s > 0$ the further form of the solution

$$\mathbf{r}_\perp(t) = \mathbf{S}_{-s}(t) + [\mathbf{A}_{-s} + \mathbf{R}_{-s}(t), -\omega t], \tag{5.19}$$

with

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

$$
\begin{aligned}
\boldsymbol{S}_{-s}(t) &= \boldsymbol{a}_\perp - \frac{1}{\omega}\boldsymbol{k}\times\left\{\boldsymbol{u}_\perp + \int_0^t \boldsymbol{f}_\perp(\tau)d\tau\right\} \\
&+\frac{1}{\omega^2}\left\{\boldsymbol{k}\times\left(\frac{D}{\omega}\right)^{-1} + \left(\frac{D}{\omega}\right)^{-2} - \boldsymbol{k}\times\left(\frac{D}{\omega}\right)^{-3} - \left(\frac{D}{\omega}\right)^{-4} + \cdots + T_{-s}\left(\frac{D}{\omega}\right)^{-s}\right\}\dot{\boldsymbol{f}}_\perp(t),
\end{aligned} \tag{5.20}
$$

$$
\begin{aligned}
\boldsymbol{A}_{-s} = \frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{u}_\perp - \frac{1}{\omega^2}\Bigg\{&\boldsymbol{k}\times\left(\frac{D}{\omega}\right)^{-1} + \left(\frac{D}{\omega}\right)^{-2} \\
&-\boldsymbol{k}\times\left(\frac{D}{\omega}\right)^{-3} - \left(\frac{D}{\omega}\right)^{-4} + \cdots + T_{-s}\left(\frac{D}{\omega}\right)^{-s}\Bigg\}\boldsymbol{f}_\perp(0),
\end{aligned} \tag{5.21}
$$

$$
\boldsymbol{R}_{-s}(t) = \frac{1}{\omega}\int_0^t \left[\left(\frac{D}{\omega}\right)^{-s} f_\perp(\tau),\ \omega\tau + (1-s)\pi/2\right] d\tau. \tag{5.22}
$$

The general term in the braces is the operator $T_{-p}(D/\omega)^{-p}$ where

$$
T_{-p}\left(\frac{D}{\omega}\right)^{-p} = \begin{cases} (-1)^{(p-1)/2}\boldsymbol{k}\times(D/\omega)^{-p}; & \text{if } p \text{ odd}, \\ -(-1)^{p/2}(D/\omega)^{-p}; & \text{if } p \text{ even}. \end{cases}
$$

The residual term $\boldsymbol{R}_{-s}(t)$ will usually be negligible when

$$
|(D/\omega)^{-s}\boldsymbol{f}_\perp(\tau)|_{\text{typical}} \ll |\boldsymbol{f}_\perp(\tau)|_{\text{typical}}, \qquad 0 \leqq \tau \leqq t. \tag{5.23}
$$

The solution of (1.1) has now been put in three forms (4.3), (5.8) and (5.19), each useful under different conditions. For example, if $\boldsymbol{f}_\perp(t)$ is a polynomial in t of degree m, the use of (5.8), with $n > m$, will give a constant rotation component $[\boldsymbol{A}_n, -\omega t]$ superimposed on a guiding centre $\boldsymbol{S}_n(t)$. A second example would be

$$
\boldsymbol{f}_\perp(t) = \boldsymbol{j}K\cos\omega_i t,
$$

with K = constant and $\boldsymbol{j}$ a constant unit vector perpendicular to $\boldsymbol{k}$. Then

$$
\left(\frac{D}{\omega}\right)^r \boldsymbol{f}_\perp(t) = \boldsymbol{j}K\left(\frac{\omega_i}{\omega}\right)^r \cos(\omega_i t + r\pi/2), \qquad r = 0, \ \pm 1, \pm 2, \cdots. \tag{5.24}
$$

When $|\omega_i/\omega| \ll 1$, (5.24) is small compared with $\boldsymbol{f}_\perp(t)$ for $r > 0$ and the form (5.8) is appropriate. Similarly when $|\omega_i/\omega| \gg 1$ the appropriate form is (5.19) and when $\omega_i \sim \omega$, it is (4.3).

We note that the expression for x_2 given in (2.9) and the expanded versions of this contained in the (5.3) and (5.18) may be obtained by operator methods. We would begin with the differential equation for x_2 written as

$$
D(D+i\omega)x_2 = f_2(t).
$$

Then, noting that the inverse operator $\{D(D+i\omega)\}^{-1}$ may be written in the forms

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

$$\frac{1}{i\omega}\left\{\frac{1}{D}-\frac{1}{D+i\omega}\right\}=\frac{D^{-1}}{i\omega}+\frac{1}{\omega^2}\left\{\frac{1}{1-iD/\omega}\right\}=\frac{D^{-1}}{i\omega}-\frac{\omega}{1D}\left\{\frac{1}{1-\omega/iD}\right\},$$

we use the identity

$$1/(1-x) = 1+x+\cdots x^{n-1}+x^n/(1-x)$$

to expand the operators

$$1/(1-iD/\omega) \text{ and } 1/(1-\omega/iD)$$

and interpret the expansions in the usual manner.

6. Some known results as special cases

In this section three results derived elsewhere in different ways will be shown to follow from the forms of the solution which we have developed.

Alfvén's "Guiding Centre". From the discussion following (4.5) we have, quoting Alfvén, "the displacement of the perpendicular component of the point where the centre of curvature would be if $\boldsymbol{f}_\perp$ vanished for a moment" at the instants 0, t is

$$\text{(6.1)} \qquad \boldsymbol{D}(t) = \boldsymbol{S}(t)-\boldsymbol{S}(0) = -\frac{1}{\omega}\boldsymbol{k}\times\int_0^t \boldsymbol{f}_\perp(\tau)d\tau.$$

This result is given by Alfvén [1] and in velocity form by Alfvén and Fälthammar [2]. They derive their result by considering the continuous force to be equivalent to a series of impulsive forces.

Alfvén's Series. According to (5.12) the residual term $\boldsymbol{R}_n(t)$ in the expression (5.8) may be neglected when $(D/\omega)^n\boldsymbol{f}_\perp(t)$ is sufficiently small. The motion is then uniform motion about a moving centre whose perpendicular component is $\boldsymbol{S}_n(t)$, given by (5.9). The corresponding velocity component of this centre is then

$$\text{(6.2)} \qquad \dot{\boldsymbol{S}}_n(t) = -(1/\omega)\boldsymbol{k}\times\{\boldsymbol{f}_\perp+\sum_{p=0}^{n-1} T_p(D/\omega)^{p+1}\boldsymbol{f}_\perp(t)\}.$$

The components of this vector expression are the series expressions (27) and (28) on p. 18 of Alfvén [1], and mentioned in the introduction. The development presented here also gives a series for the rotation component $\boldsymbol{A}_n$, (5.10), and an expression for the residual term $\boldsymbol{R}_n(t)$, (5.11).

The Polarization Drift. The polarization drift given in texts on plasma physics, Chandrasekhar [4], Thompson [6], is obtained by putting $n = 1$ in the expression, (5.8), for $\boldsymbol{r}_\perp(t)$.

If the characteristic frequency of $\boldsymbol{f}_\perp(t)$ is ν, with $|\nu/\omega| \ll 1$, the inequality

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

(5.12) is satisfied and the residual term may be neglected. This leaves a uniform rotation term $[\boldsymbol{A}_1, -\omega t]$ and a centre term $\boldsymbol{S}_1(t)$.

The perpendicular component of the velocity of the centre is then

$$\dot{\boldsymbol{S}}_1(t) = -\frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{f}_\perp(t)+\frac{1}{\omega^2}D\boldsymbol{f}_\perp(t). \tag{6.3}$$

If $\boldsymbol{f}_\perp$ is due to an electric field $\boldsymbol{E}(t)$ then $\boldsymbol{f}_\perp(t) = (e/m)\boldsymbol{E}_\perp(t)$ and (6.3) may be written

$$\dot{\boldsymbol{S}}_1(t) = (\boldsymbol{E}\times\boldsymbol{B})/B^2+(m/eB^2)\dot{\boldsymbol{E}}_\perp(t). \tag{6.4}$$

The terms on the right-hand side are the familiar electric field drift and polarization drift. The constant rotation component is specified by $\boldsymbol{A}_1(t)$, and $\boldsymbol{R}_1(t)$ gives a specific expression for the neglected terms.

7. Application to the Omegatron

To illustrate the usefulness of the various forms of the solution we analyze the transverse motion produced by a transverse periodic forcing-function

$$\boldsymbol{f}_\perp(t) = \boldsymbol{j}K\cos\omega_i t. \tag{7.1}$$

We take $\boldsymbol{j}$ to be a unit vector perpendicular to $\boldsymbol{k}$ and define a third unit vector $\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{k}$. K and ω_i are constants with $0 \leqq \omega_i \leqq \infty$. Each of the three forms of the solution is used in an appropriate range of ω_i.

A resonance occurs at $\omega_i = \omega$ and this has been used in a precision ion-selector known as an omegatron, Hipple et al. [5]. The particle orbits have been analyzed by Berry [3].

When $\boldsymbol{f}_\perp(t)$ from (7.1) is substituted into (4.5) we obtain two different results for $\boldsymbol{A}(t)$ according as $\omega \neq \omega_i$ or $\omega = \omega_i$. They are, respectively,

$$\begin{aligned}\boldsymbol{A}(t) = \frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{u}_\perp - \frac{K}{2}\Bigg\{&\left(\frac{1-\cos(\omega+\omega_i)t}{(\omega+\omega_i)}+\frac{1-\cos(\omega-\omega_i)t}{(\omega-\omega_i)}\right)\boldsymbol{i}\\ &-\left(\frac{\sin(\omega+\omega_i)t}{(\omega+\omega_i)}+\frac{\sin(\omega-\omega_i)t}{(\omega-\omega_i)}\right)\boldsymbol{j}\Bigg\},\end{aligned} \tag{7.2}$$

and

$$\boldsymbol{A}(t) = \frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{u}_\perp - \frac{K}{2}\left\{\frac{1}{2\omega}\boldsymbol{i} - t\boldsymbol{j} - \frac{\cos 2\omega t}{2\omega}\boldsymbol{i} - \frac{\sin 2\omega t}{2\omega}\boldsymbol{j}\right\}. \tag{7.3}$$

Since, by equation (4.8), $|\dot{\boldsymbol{r}}_\perp(t)| = |\omega A|$, we deduce from (7.2) and (7.3) that, assuming K independent of ω_i, we require $\omega_i \sim \omega$, to obtain the largest transverse speeds. Then the terms containing $1/(\omega-\omega_i)$ are dominant in (7.2), and, considering these alone we find that $|A|$ and there-

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

fore the transverse kinetic energy oscillates at an angular rate $(\omega-\omega_i)$. When $\omega_i = \omega$ the transverse speed increases indefinitely.

The forcing function (7.1) was used as an example in section 5 and the forms recommended in the paragraph following equation (5.24) apply here.

Taking ω to be positive we find:

Case (*a*), $\omega_i/\omega \ll 1$.

$$\boldsymbol{S}_n(t) = \boldsymbol{a}_\perp - \frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{u}_\perp + \frac{K}{\omega^2}\left(\sum_{p=0}^{n}\left(\frac{\omega_i}{\omega}\right)^{p-1}\sin^2(p-1)\pi/2\right)\sin(\omega_i t)\boldsymbol{i} + \frac{K}{\omega^2}\left(\sum_{p=0}^{n}\left(\frac{\omega_i}{\omega}\right)^{p-1}\cos^2(p-1)\pi/2\right)\cos(\omega_i t)\boldsymbol{j},$$

$$\boldsymbol{A}_n = \frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{u}_\perp - \frac{K}{\omega^2}\left(\sum_{p=0}^{n}\left(\frac{\omega_i}{\omega}\right)^{p-1}\cos^2(p-1)\pi/2\right)\boldsymbol{j},$$

and

$$\boldsymbol{R}_n(t) = \left(\frac{\omega_i}{\omega}\right)^n \frac{K}{\omega}\int_0^t [\boldsymbol{j}\cos(\omega_i\tau + n\pi/2),\ \omega\tau + (n+1)\pi/2]d\tau.$$

Owing to the factor $(\omega_i/\omega)^n$ the residual term decreases as n increases and is easily rendered negligible. Here $\boldsymbol{S}_n(t)$ represents an elliptic motion and on this is superimposed the circular motion of radius A_n and period $2\pi/\omega_i$. The axes of the ellipse are parallel to $\boldsymbol{i}$ and $\boldsymbol{j}$ and the semi-axes are the coefficients of $\boldsymbol{i}\sin\omega_i t$ and $\boldsymbol{j}\cos\omega_i t$ in the expression for $\boldsymbol{S}_n(t)$.

Case (*b*), $\omega_i \sim \omega$. Using (7.2), but including only the major oscillating terms, we have

$$\boldsymbol{A}(t) \simeq \frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{u}_\perp - \frac{K}{2}\,\frac{1-\cos(\omega-\omega_i)t}{(\omega-\omega_i)}\,\boldsymbol{i} + \frac{K}{2}\,\frac{\sin(\omega-\omega_i)t}{(\omega-\omega_i)}\,\boldsymbol{j},$$

together with

$$\boldsymbol{S}(t) = \boldsymbol{a}_\perp - (1/\omega)\boldsymbol{k}\times\boldsymbol{u}_\perp + (K/\omega\omega_i)\sin\omega_i t\boldsymbol{i}.$$

The radius of the rotation component, $\boldsymbol{A}(t)$, is a periodic function of time with period $2\pi/(\omega-\omega_i)$. $\boldsymbol{S}(t)$ gives the centre of this rotation. It remains relatively constant compared with the changes in A. The particle spirals at an angular rate ω about an approximately constant centre position with radius expanding and contracting at a rate $(\omega-\omega_i)$.

Case (*c*), $\omega_i = \omega$. $\boldsymbol{A}(t)$ is given by (7.3) and

$$\boldsymbol{S}(t) = \boldsymbol{a}_\perp - (1/\omega)\boldsymbol{k}\times\boldsymbol{u}_\perp + (K/\omega^2)\sin\omega t\,\boldsymbol{i}.$$

After a few periods the particle spirals at an angular rate ω, with increasing radius, about a centre which is approximately constant.

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press

Case (d), $\omega_i \gg \omega$. With $s \geqq 2$.

$$S_{-s}(t) = \boldsymbol{a}_\perp - \frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{u}_\perp - \frac{K}{\omega^2}\left(\sum_{p=2}^{s}\left(\frac{\omega}{\omega_i}\right)^p \sin^2(p\pi/2)\right)\sin\omega_i t\,\boldsymbol{i}$$
$$- \frac{K}{\omega^2}\left(\sum_{p=2}^{s}\left(\frac{\omega}{\omega_i}\right)^p \cos^2(p\pi/2)\right)\cos\omega_i t\,\boldsymbol{j},$$

$$\boldsymbol{A}_{-s} = \frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{u}_\perp + \frac{K}{\omega^2}\left(\sum_{p=2}^{s}\left(\frac{\omega}{\omega_i}\right)^p \cos^2(p\pi/2)\right)\boldsymbol{j},$$

$$\boldsymbol{R}_{-s}(t) = \left(\frac{\omega}{\omega_i}\right)^s \frac{K}{\omega}\int_0^t [\boldsymbol{j}\cos(\omega_i\tau - s\pi/2),\ \omega\tau + (1-s)\pi/2]\,d\tau.$$

The residual term is easily rendered negligible by choosing a sufficiently large value of s. Here $\boldsymbol{S}_n(t)$ traces out an elliptic locus with period $2\pi/\omega_i$. The ellipse has the fixed centre $\boldsymbol{a}_\perp - (\boldsymbol{k}\times\boldsymbol{u}_\perp)/\omega$, axes parallel to $\boldsymbol{i}$ and $\boldsymbol{j}$ and semi-axes given by the coefficients of $\boldsymbol{i}\sin\omega_i t$ and $\boldsymbol{j}\cos\omega_i t$. The size of the ellipse decreases to zero as ω_i tends to infinity. The motion is best described as a uniform circular motion with an elliptic motion superimposed on it.

Acknowledgement

The approach used here in solving the vector differential equation and the notation of (3.6) were suggested by Professor K. C. Westfold. I acknowledge his contribution and thank him for his generous assistance in the writing of this paper.

References

[1] Alfvén, H., *Cosmical Electrodynamics*, Oxford (1950), 12—19.
[2] Alfvén, Hannes and Fälthammar, Carl-Gunne, *Cosmical Electrodynamics*, 2 Ed. Oxford (1963), 18—26.
[3] Berry, C. E., 'Ion trajectories in the Omegatron', *J. App. Phys.* 25 (1954), 28—31.
[4] Chandrasekhar, S., *Plasma Physics*, Phoenix (1962), 20—21.
[5] Hipple, J. A., Sommer, H. and Thomas, H. A., 'A precise method of determining the Faraday by magnetic resonance', *Phys. Rev.* 76 (1949), 1877—1878.
[6] Thompson, W. B., *Plasma Physics*, Addison-Wesley (1962), 153—154.
[7] Westfold, K. C., 'The solution of linear differential equations containing gyroscop'' terms', *Amer. Math. Monthly* 64 (1957), 174—180.

Monash University
Victoria

https://doi.org/10.1017/S1446788700004869 Published online by Cambridge University Press