TWO REMARKS ON THE COMMUTATIVITY OF RINGS

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In **(1)** and **(2)** we proved that under certain conditions a given ring R must be commutative. The conditions used there were "global" in the sense that they were imposed at once on the relation of a given element to *all* the other elements of the ring R.

In this note we replace these global conditions by "local" ones that relate only to two elements of R at a time. We show that the results of **(1)** and **(2)** carry over to this situation.

In **(1)** we proved that if in a ring R with centre Z, $x^{n(x)} \in Z$ for some integer $n(x) \geqslant 1$ for all $x \in R$, then either R is commutative or its commutator ideal is a nil ideal.

The condition $x^{n(x)} \in Z$, of course, means that $x^{n(x)}y = yx^{n(x)}$ for *all* $y \in R$. We prove here

THEOREM 1. *Suppose that R is a ring such that given any two elements $x, y \in R$ then for some integer $n(x, y) \geqslant 1$ which depends on both x and y*

$$x^{n(x,y)}y = yx^{n(x,y)}.$$

Then either R is commutative or its commutator ideal is a nil ideal.

Proof. Suppose that R is not commutative. Let $c \neq 0$ be a typical element in the commutator ideal of R. We want to show that c is nilpotent. Since c is in the commutator ideal of R,

$$c = \sum_{i=1}^{m} (a_i b_i - b_i a_i) + \sum_{i=1}^{n} r_i(d_i e_i - e_i d_i) + \sum_{i=1}^{p} (f_i g_i - g_i f_i)\, s_i + \sum_{i=1}^{q} t_i(h_i k_i - k_i h_i)\, u_i.$$

Let T be the subring of R generated by all the elements $a_i, b_i, d_i, e_i, f_i, g_i, h_i, k_i, r_i, s_i, t_i, u_i$ appearing in the expression for c. Clearly c is a commutator in T. Suppose $\tau \in T$. Then

$$\tau^{n_1} = \tau_1$$

for suitable n_1 commutes with a_1 by the condition imposed on R. Similarly

$$\tau_2 = \tau_1^{n_2} = \tau^{n_1 n_2}$$

commutes with a_2 for some integer n_2; of course τ_2 also commutes with a_1 since τ_1 does. Continuing in this way we arrive at an integer $m \geqslant 1$ so that τ^m commutes with all the a's, b's, etc. appearing in the expression for c and which

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generate T. Thus τ^m commutes with all the elements of T since it commutes with the generators of T; that is, for every $\tau \in T$, $\tau^{n(\tau)}$ is in the centre of T. By **(1)** this means that either T is commutative or its commutator ideal is a nil ideal. Since $c \neq 0$ is in the commutator ideal of T, T is not commutative. So c is nilpotent and the theorem is established.

In **(2)** we proved: let R be a ring with centre Z such that for all $x \in R$ $x^{n(x)} - x \in Z$ for some integer $n(x) > 1$; then R is commutative. The condition $x^{n(x)} - x \in Z$ of course means that

$$(x^{n(x)} - x)y = y(x^{n(x)} - x)$$

for *all* $y \in R$. We localize the condition in the following

THEOREM 2. *If in a ring R for every pair of elements x and y we can find an integer $n(x, y) > 1$ which depends on x and y so that $x^{n(x,y)} - x$ commutes with y, then R is commutative.*

Proof. Let T be the subring of R generated by x and y. Suppose $t \in T$. Thus for some integer $m > 1$, $t_1 = t^m - t$ commutes with x. For some other integer $n > 1$, $t_2 = t^n{}_1 - t_1$ commutes with y. Since t_1 commutes with x, t_2 also commutes with x. Thus t_2 commutes with both x and y, and so with every element in the subring they generate. Thus t_2 is in the centre of T. However

$$t_2 = t_1^n - t_1 = (t^m - t)^n - (t^m - t) = -(t^2p(t) - t)$$

where $p(t)$ is a polynomial with integer coefficients. That is, for every $t \in T$ we can find a polynomial $p(t)$ with integer coefficients so that $t^2p(t) - t$ is in the centre of T. By the principal result of **(3)** T must be commutative. Since both x and y are in T, $xy = yx$, and so R is commutative.

The main theorem of **(3)** can also be generalized in the same fashion as the other two theorems. We state it without proof,

THEOREM 3. *If for every x and y in R we can find a polynomial $p_{x,y}(t)$ with integer coefficients which depend on x and y such that $x^2p_{x,y}(x) - x$ commutes with y, then R is commutative.*

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