ON THE SIMPLEX OF COMPLETELY MONOTONIC FUNCTIONS ON A COMMUTATIVE SEMIGROUP

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Introduction. Bernstein's classical integral representation theorem for completely monotonic functions can be proved most elegantly, on a commutative semigroup with identity, by the integral version of the Kreĭn-Milman theorem [**2**]. The key to this approach is the identification (as exponentials) of the extremal points of the normalized completely monotonic functions. Alternate proofs of this identification are given in § 1. The first (Corollary 1.3) is based on the Kreĭn-Milman theorem and the second (see remarks following Corollary 1.5) is derived from elementary analytic techniques. Other interesting facts about completely monotonic functions are mentioned in passing. For example, we observe that the normalized completely monotonic functions form a simplex (Corollary 1.4). In Corollary 1.6 we note that the product of completely monotonic functions corresponds to the convolution of their representing measures. Thus the normalized completely monotonic functions form an affine semigroup [**3**].

In § 2 we consider extension theorems for exponentials and completely monotonic functions. We motivate and introduce a notion of power closed semigroups (i.e., multiplicative semigroups in which each of its members can be raised to non-negative real powers). In Proposition 2.1 we show that if M is a power closed semigroup and e is a non-zero homomorphism from M into [0, 1] (i.e., e is an exponential), then $e^r(m) = e(m^r)$. This, and a theorem of Ross [**7**] concerning the extension of exponentials, are the principal tools of this section. We show that if a semigroup A admits enough exponentials to separate points, then it is naturally embedded in the power closed semigroup $\exp^2 A$, consisting of the exponentials on the semigroup $\exp A$, of exponentials on A. In this event, every completely monotonic function f on A admits a completely monotonic extension to $\exp^2 A$ (Corollary 2.7), and a unique completely monotonic extension to its power closure, $\tilde{A} \subset \exp^2 A$ (Corollary 2.4). In conclusion, we obtain a rather curious extension theorem (Theorem 2.11) for uniquely power closed semigroups.

1. The simplex of normalized completely monotonic functions. Let A be a commutative semigroup (written additively) with identity 0. For each real-valued function f on A define the function $\Delta_n f$ $(n \geqq 0)$ of the $n + 1$

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variables $x, h_1, h_2, \ldots, h_n \in A$ inductively by:

$$\Delta_0 f(x) = f(x),$$
$$\Delta_n f(x; h_1, \ldots, h_n) = \Delta_{n-1} f(x; h_1, \ldots, h_{n-1}) - \Delta_{n-1} f(x + h_n; h_1, \ldots, h_{n-1}).$$

A real-valued function f on A is said to be *completely monotonic* if $\Delta_n f \geqq 0$ for all non-negative integers n. The set $C_\infty(A)$ $(\equiv C_\infty)$ of completely monotonic functions on A is a convex cone in the linear space $E_\infty(A) \equiv E_\infty \equiv C_\infty - C_\infty$, i.e., $\alpha C_\infty + \beta C_\infty \subset C_\infty$ for $\alpha, \beta > 0$ and $C_\infty \cap -C_\infty = \{0\}$. The topology of simple convergence induces a locally convex linear topology on E_∞ such that for each $a \in A$, the linear functional $\hat{a}$ defined by $\hat{a}(f) = f(a)$ is continuous. Since C_∞ is topologically closed and every completely monotonic function is non-negative and bounded by $f(0)$, Tychonoff's theorem implies that the normalized completely monotonic functions, namely

$$X_\infty(A) \equiv X_\infty \equiv \{f \in C_\infty | f(0) = 1\},$$

form a compact base for C_∞. Thus every non-zero completely monotonic function can be uniquely expressed as a multiple of some $f \in X_\infty$. An *exponential* is defined to be a non-trivial homomorphism e from A into the unit interval [0, 1] under multiplication. Every exponential e is completely monotonic; in fact, $\Delta_n e(x; h_1, \ldots, h_n) = e(x)[1 - e(h_1)] \ldots [1 - e(h_n)]$. The class of all exponentials in A will be denoted by $\exp A$ and the extreme points of X_∞ by ext X_∞. Since the identically one function is both an extremal point and an exponential, we see that neither ext X_∞ nor $\exp A$ is ever void.

That ext $X_\infty \subset \exp A$, is easily established in [**2**] as follows: Let $e \in$ ext X_∞ and define $e_a(x) = e(x + a)$. Then $e_a \in C_\infty$ and $e - e_a \in C_\infty$. Since e is extremal, there exists $\alpha > 0$ such that $e_a = \alpha e$. Evaluation at 0 implies that $e(a) = \alpha$, and the assertion follows. For the sake of completeness we state the following result.

PROPOSITION 1.1. *Every extremal point of X_∞ is an exponential on A.*

The converse of Proposition 1.1 is known [**2**]. An elementary proof of this fact is offered after Corollary 1.5. A quick, but non-elementary, proof follows as a consequence of Theorem 1.2.

Let X be a convex subset of a locally convex space E and let E^* denote the adjoint of E. Recall [**6**] that a regular probability measure μ_x which is supported by X is said to represent $x \in X$ if $\int_X L\, d\mu_x = L(x)$ for all $L \in E^*$. The Kreĭn-Milman theorem asserts that if X is compact, then every $x \in X$ admits a representing measure which is supported by the closure of the extreme points of X.

THEOREM 1.2. *Every normalized completely monotonic function f admits a unique representing measure μ_f which is supported by* $\exp A$, *i.e.*, $f(z) = \int_{\exp A} e(z)\, d\mu_f(e)$.

Proof. Let $f \in X_\infty$. The Kreĭn-Milman theorem, along with Proposition 1.1, establishes the existence of a representing measure μ_f for f such that μ_f is supported by $\mathrm{cl}(\mathrm{ext}\, X_\infty) \subset \mathrm{cl}(\exp A) = \exp A$. To prove uniqueness, let a^1 denote the restriction of the continuous function $\hat{a}$ to the closed set $\exp A$. Since $a^1 \cdot b^1 = (a + b)^1$, it follows that the linear span S of the set $\{a^1 |\, a \in A\}$ is a point-separating subalgebra of the algebra $C[\exp A]$ of all continuous functions on the compact Hausdorff space $\exp A$. Moreover, $0^1 \equiv 1$ implies that S contains the constant functions. Thus S is dense in $C[\exp A]$ by the Stone-Weierstrass theorem. But if ν is any representing measure for f which is supported by $\exp A$, then we must have,

$$\int_{\exp A} \left(\sum \alpha_i a_i^{\ 1}\right) d\mu_f = \int_{\exp A} \left(\sum \alpha_i a_i^{\ 1}\right) d\nu$$

for all finite sums $\sum \alpha_i a_i^{\ 1} \in S$. Hence $\mu_f \equiv \nu$, since both μ_f and ν are regular.

COROLLARY 1.3. *Every exponential on A is an extremal point.*

Proof. Suppose that $e \in \exp A$ and $f_1, f_2 \in X_\infty$ such that $e = \frac{1}{2}f_1 + \frac{1}{2}f_2$. Let μ_i be the representing measure for f_i $(i = 1, 2)$, guaranteed by Theorem 1.2. By uniqueness, $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ is point mass at e so that μ_i is also point mass at e. Thus $f_1 = f_2 = e$ or $e \in \mathrm{ext}\, X_\infty$.

Recall [**6**] that if X is a convex base for a cone C in a linear space, then X is said to be a *simplex* if $C - C$ is lattice-ordered. In the event that $C - C$ is a locally convex linear topological space and both X and its extremal points are compact, then it follows that X is a simplex if and only if every $x \in X$ admits a unique representing measure which is supported by $\mathrm{ext}\, X$ [**6**]. In this event X is called a *resolutive simplex* [**1**] and C is affinely equivalent to the cone of all non-negative regular Borel measures on $\mathrm{ext}\, X$. Theorem 1.2 and Corollary 1.3 now imply the following results.

COROLLARY 1.4. *The normalized completely monotonic functions $X_\infty(A)$ on a commutative semigroup A with identity form a resolutive simplex.*

COROLLARY 1.5. (a) *For every completely monotonic function f it is true that $f^2(x) \leqq f(0)f(2x)$.*

(b) *A non-trivial completely monotonic function e is an exponential if and only if $e^2(x) = e(2x)$.*

Proof. Assertion (a) follows from Theorem 1.2 and Schwarz's inequality since,

$$f(x) = \int_{\exp A} e(x)\, d\mu_f(e)$$
$$\leqq \sqrt{\left(\int_{\exp A} 1\, d\mu_f(e)\right)} \sqrt{\left(\int_{\exp A} e^2(x)\, d\mu_f(e)\right)} = \sqrt{(f(0)f(2x))}\,.$$

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The "only if" part of assertion (b) is clear. For the converse, assume that $e \in C_\infty$ and $e^2(x) = e(2x)$. From Proposition 1.1, we need only prove that $e \in \text{ext}\, X_\infty$. Since $e^2(0) = e(0)$ and e is non-trivial, we have $e \in X_\infty$. Suppose that $e = \frac{1}{2}f_1 + \frac{1}{2}f_2$, where $f_1, f_2 \in X_\infty$. Then from (a),

$$f_1^2(x) + f_2^2(x) \leqq f_1(2x) + f_2(2x) = 2e(2x) = 2e^2(x) = \tfrac{1}{2}[f_1(x) + f_2(x)]^2$$

so that $[f_1(x) - f_2(x)]^2 \leqq 0$ for all x, or $f_1 \equiv f_2$.

A direct proof of (a) proceeds as follows: Without loss of generality we may assume that $f(0) = 1$. For fixed $x \in A$, define the linear functional L on the vector space of real polynomials by $L(t^n) = f(nx)$ $(n \geqq 0)$. It is easy to see that for $0 \leqq k \leqq n$,

$$0 \leqq \Delta_{n-k} f(kx; x, x, \ldots, x) = \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} f[(k+r)x] = L[t^k(1-t)^{n-k}].$$

Hence, if c is any real number and

$$p_n(t) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - c\right)^2 t^k (1-t)^{n-k},$$

then we have $L(p_n(t)) \geqq 0$. Now, using the fact that

$$\sum_{k=0}^{n} \binom{n}{k} k^j z^k = \left(z \frac{d}{dz}\right)^j (1+z)^n$$

(or the known values of the first and second moments of the binomial distribution), we find that

$$p_n(t) = (t-c)^2 + \frac{t(1-t)}{n}.$$

Hence $0 \leqq L[p_n(t)] = L[(t-c)^2] + (1/n)L[t(1-t)]$. Letting $n \to \infty$, we have $L[(t-c)^2] \geqq 0$. Putting $c = f(x)$, we obtain $f(2x) - f^2(x) = L[(t - f(x))^2] \geqq 0$. Of course, $p_n(t)$ is the Bernstein polynomial of degree n for the function $f(t) = (t-c)^2$.

It should be noted that the above argument, coupled with the proof given of Corollary 1.5 (b), yields a direct and elementary proof of Corollary 1.3, namely that every exponential is extremal.

It is clear that exp A is itself a semigroup under multiplication. It is therefore reasonable to expect the same of C_∞ and X_∞.

COROLLARY 1.6. *If $f, g \in C_\infty$ (or X_∞) and if μ_f and μ_g are the respective non-negative regular Borel measures on the compact semigroup,* exp A, *which represent f and g, then $f \cdot g \in C_\infty$ (or X_∞) and $f \cdot g$ is represented by the convolution $\mu_f * \mu_g$ of μ_f and μ_g.*

Proof. Using the notation of the proof of Theorem 1.2 we have:

$$\int_{\exp A} a^1 d(\mu_f * \mu_g) = \int\int_{\exp A \times \exp A} e_1(a)e_2(a)\, d\mu_f(e_1)\, d\mu_g(e_2)$$

$$= \int_{\exp A} e(a)\, d\mu_f(e) \int_{\exp A} e(a)\, d\mu_g(e) = f(a)g(a) \quad \text{for all } a \in A.$$

A direct proof that the product of two completely monotonic functions is completely monotonic proceeds as follows. Let $f, g \in C_\infty$ and hypothesize inductively that

$$\Delta_n(f \cdot g) = \sum \Delta_p(f)\Delta_q(g),$$

where appropriate arguments are assumed, and the summation is finite. Clearly the inductive hypothesis is valid for $n = 0$. For $n = 1$ we have:

$$\Delta_1(fg)(x; h) = f(x)\Delta_1 g(x; h) + g(x + h)\Delta_1 f(x; h).$$

The inductive step from n to $n + 1$ then follows readily from additivity of Δ_1 and the case $n = 1$.

In particular, we note that Corollary 1.6 implies that the normalized completely monotonic functions, X_∞, form an affine semigroup [**3**]. Moreover, the vector lattice E_∞ is a Banach algebra.

As applications of the above theory, two classical theorems of Bernstein and Hausdorff can be recovered (cf. [**2**]). Let $\mathbf{R}^+$ denote the non-negative reals, $\mathbf{N}$ the non-negative integers, and $[0, 1]$ the closed unit interval equipped with its usual topology. It follows that the exponentials on $\mathbf{R}^+$ and $\mathbf{N}$ are all functions of the form $t^{(\cdot)}$ $(0 \leqq t \leqq 1, 0^0 \equiv 1)$. Bernstein's and Hausdorff's theorem takes the following form.

COROLLARY 1.8 (Bernstein (Hausdorff)). *Every completely monotonic function f on $\mathbf{R}^+$ (moment sequence on $\mathbf{N}$) admits a unique integral representation of the form:*

$$f(a) = \int_0^1 t^a\, d\mu(t), \qquad a \in \mathbf{R}^+ \ (a \in \mathbf{N}).$$

In particular, Corollary 1.8 implies that every completely monotonic function f on $\mathbf{N}$ can be uniquely extended to a completely monotonic function on $\mathbf{R}$, a well-known theorem (see [**8**]).

2. Extensions of exponentials and completely monotonic functions. As previously observed, $\exp A$ is a commutative semigroup under pointwise multiplication whose identity is the identically one function. If we adopt the convention that $0^0 = 1$, then $e^0 \equiv 1 \in \exp A$ for $e \in \exp A$. If we define $e^\infty(a) = 1$ when either a or e is the identity and $e^\infty(a) = 0$ otherwise, then $e^\infty \in \exp A$ for $e \in \exp A$. Observe that $e^\infty(a) \neq \lim_{t\to\infty} e^t(a)$, although this function is an exponential. It follows that $\exp A = M$ is a commutative

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semigroup (under multiplication) with identity 1, which admits a parametrization ψ: $(m, r) \to m^{(r)}$ on $M \times [0, \infty]$ into M such that:

(i) $m^{(r+s)} = m^{(r)}m^{(s)}$,

(ii) $m^{(n)} = m^n$, $m^{(0)} = 1$, $1^{(\infty)} = 1$,

(iii) $m^{(\infty)}n = m^{(\infty)}$ for all $m \neq 1$.

Any multiplicative semigroup which admits such a parametrization will be called *power closed*. If M admits only one such parametrization, then we will say that M is *uniquely power closed*. For notational convenience we will sometimes set $m^{(r)} = m^r$ when M is uniquely power closed.

An example of a semigroup which admits a non-unique parametrization ψ is the semigroup of non-negative reals under multiplication. Let λ be any additive homomorphism of $\mathbf{R}^+$ onto $\mathbf{R}$ such that $\lambda(1) = 1$. Then ψ, defined by

$$\psi(m, r) = \begin{cases} m^{\lambda(r)} & \text{for } 0 \neq r \neq \infty, \\ 0 & \text{for } m \neq 1, r = \infty, \\ 1 & \text{otherwise,} \end{cases}$$

is a parametrization of $\mathbf{R}^+$. It is well known that λ, and hence ψ, is not unique.

PROPOSITION 2.1. *If M is a power closed semigroup, then $e^r(m) = e(m^r)$ for all $r \in [0, \infty]$, $m \in M$, and $e \in \exp M$.*

Proof. If $r = \infty$, then $e^\infty(m) = 0$ for $e \neq 1 \neq m$. But if $e \neq 1$, then there exists $n \in M$ such that $e(n) < 1$. But since $m \neq 1$, we have $e(m^\infty) = e(m^\infty n) = e(m^\infty)e(n)$, so that $e(m^\infty) = 0$. Therefore $e^\infty(m) = e(m^\infty)$ if $e \neq 1 \neq m$. If either $e = 1$ or $m = 1$, then clearly $e^\infty(m) = 1 = e(m^\infty)$. Thus we may assume that $0 \leqq r < \infty$. Define the real-valued function F on $\mathbf{R}^+$ by $F(r) = e(m^r)$. Then $F \in \exp \mathbf{R}^+$, and hence $e(m^r) = F(r) = F^r(1) = e^r(m)$.

COROLLARY 2.2. *Every power closed semigroup with enough exponentials to separate points is uniquely power closed.*

Proof. Suppose that ϕ and ψ are two parametrizations. Then for every x and r, Proposition 2.1 implies that $e[\phi(x, r)] = e^r(x) = e[\psi(x, r)]$ for all exponentials e. Thus $\psi(x, r) = \phi(x, r)$.

We leave the converse question open. That is, does every uniquely power closed semigroup have enough exponentials to separate points? Partial results are indicated in Corollary 2.11.

With reference to the notation of § 1 we recall that for each a in the semigroup A, the function a^1 is $\hat{a}|_{\exp A}$; i.e., $a^1(e) = e(a)$ is a member of $\exp^2 A$. If we assume that A admits enough exponentials to separate points, then, the map $a \to a^1$ is a biunique homomorphism of A into $\exp^2 A$. The range of this map will be denoted by A^1. Corollary 2.2 implies that we may identify the exponential $(a^1)^r$ with a^r $(0 \leqq r \leqq \infty)$, in the event that A is power closed. We also make this identification if A is not power closed. When A admits enough exponentials to separate points, the notation $\tilde{A}$ will be introduced to denote all of the members of $\exp^2 A$ which are of the form

$a_1^{r_1}a_2^{r_2}\ldots a_k^{r_k}$, where $a_i \in A$, $0 \leqq r_i \leqq \infty$, $i = 1, 2, \ldots, k$, and $k \in \mathbf{N}$. Note that $\tilde{A} \supset A^1$; in fact, $\tilde{A}$ is the smallest power closed subsemigroup of $\exp^2 A$ in which A^1 is embedded. For this reason we call $\tilde{A}$ the *power closure* of A.

THEOREM 2.3. *Each $e_1 \in \exp A^1$ has a unique extension $e^1 \in \exp \tilde{A}$. Thus $\exp A^1 = \exp \tilde{A}$.*

Proof. Define an extension e^1 of e_1 by

$$e^1(a_1^{r_1}a_2^{r_2}\ldots a_k^{r_k}) = e_1^{r_1}(a_1^1)e_1^{r_2}(a_2^1)\ldots e_1^{r_k}(a_k^1).$$

Clearly, e^1 is an extension of e_1 such that $e^1 \in \exp \tilde{A}$. The uniqueness follows from Proposition 2.1, since we must have

$$e_1^r(a^1) = (e^1)^r(a^1) = e^1(a^r) \quad \text{for all } a \in A.$$

COROLLARY 2.4. *Every completely monotonic function f on A has a unique completely monotonic extension $\tilde{f}$ to $\tilde{A}$, given by*

$$\tilde{f}(a_1^{r_1}a_2^{r_2}\ldots a_k^{r_k}) = \int_{\exp A} e^{r_1}(a_1)\ldots e^{r_k}(a_k)\, d\mu_f(e).$$

Proof. It easily follows that $\tilde{f}$ is a completely monotonic extension of f. The uniqueness of $\tilde{f}$ follows from Theorem 2.3 and the uniqueness of the representing measure μ_f for f.

In the event that A is the non-negative integers under addition, we know that $\exp A$ is isomorphic to $[0, 1]$ under multiplication (zero corresponding to t^∞ for $t \neq 1$). If $0 \leqq r \leqq \infty$, then the function $t \to t^r$ is an exponential on $[0, 1]$. In fact, $\tilde{A}$ is precisely all exponentials of this form. $\exp^2 A$ properly contains $\tilde{A}$, since $\exp^2 A$ contains the additional exponential ψ as defined by: $\psi(t) = 1$ if $t \neq 0$, and $\psi(0) = 0$. The power closure $\tilde{A}$ can be identified with $[0, \infty]$ under addition, so that the remark following Corollary 1.8 is a special case of Corollary 2.3. Note that the identity exponential on $\mathbf{N}$ has two completely monotonic extensions to $\exp^2 A$, the identity, and Φ as defined by: $\Phi(r) = 1$ for all $0 \leqq r \leqq \infty$ and $\Phi(\psi) = 0$.

We now waive the uniqueness requirement for extensions of completely monotonic functions. Our setting will be as follows: A_0 will denote a subsemigroup of the additive commutative semigroup A such that A_0 contains the identity 0 of A. An exponential e on A_0 will be called *monotonic* if $b = a + h$ for $a, b \in A_0$, $h \in A$ implies $e(a) \geqq e(b)$. The following lemma is a consequence of a theorem of Ross [7].

LEMMA 2.5. *An exponential e on A_0 has an extension to an exponential on A if and only if e is monotonic.*

THEOREM 2.6. *If every exponential on A_0 is monotonic, then every completely monotonic function on A_0 has an extension to a completely monotonic function on A.*

Proof. Let the map $\sigma\colon X_\infty(A) \to X_\infty(A_0)$ be defined by $\sigma(f) = f|_{A_0}$. We need only show that the range of σ is all of $X_\infty(A_0)$. The range of σ is clearly convex. Since σ is continuous, its range is also compact. From Lemma 2.5 and the hypothesis, we see that each exponential on A_0 is a member of $\sigma(X_\infty(A))$, and hence $\sigma(X_\infty(A))$ contains the closed convex hull of $\exp A_0^\infty$, which is equal to $X_\infty(A_0)$. The assertion follows.

COROLLARY 2.7. *Every completely monotonic function on A^1 has a completely monotonic extension to* $\exp^2 A$.

Proof. Let $e \in \exp(A^1)$. Define $e_0 \in \exp^3(A)$ by $e_0(a) = e(a^1)$ for all $a \in A$ and define $e' \in \exp^3(A)$ by $e'(h) = h(e_0)$ for all $h \in \exp^2(A)$. Then e' is an extension of e since $e'(a^1) = a^1(e_0) = e_0(a) = e(a^1)$. The assertion follows from Theorem 2.6.

As can be seen from the following example, the requirement that A have enough exponentials to separate points is not sufficient to ensure that every exponential on a subsemigroup A_0 of A can be extended to an exponential on A. Let $A = \mathbf{R}^+$ and $A_0 = \{m + \sqrt{2}n \mid m, n \in \mathbf{N} \cup \{0\}\}$. Define $e \in \exp A_0$ by $e(m + \sqrt{2}n) = (\frac{1}{2})^m(\frac{3}{4})^n$. Since $e(1) < e(\sqrt{2})$, it follows that e cannot be extended to an exponential on $\mathbf{R}^+$. Moreover, A^1 is isomorphic to $(\mathbf{N} \times \mathbf{N}) \cup \{\infty\}$, while $\tilde{A}$ is isomorphic to $(\mathbf{R}^+ \times \mathbf{R}^+) \cup \{\infty\}$. Since A can also be embedded in $\mathbf{R}^+ \cup \{\infty\}$, the above example also shows that the power closure of A is not embeddable in every power closed semigroup which contains a copy of A.

In conclusion, we hereby present some lemmas which lead to an extension theorem for uniquely power closed semigroups.

LEMMA 2.8. *If M is uniquely power closed, then*
(a) $x^2 = x$ *implies* $x^{(r)} = x$ *if* $0 < r < \infty$,
(b) $[x^{(r)}]^{(s)} = x^{(rs)}$ *if* $0 \leqq r < \infty$ *and* $0 \leqq s < \infty$,
(c) $x^{(r)}y^{(r)} = (xy)^{(r)}$ *if* $0 \leqq r \leqq \infty$.

Proof. If (b) were false, then a new power closure $m^{[\cdot]}$ on M could be defined by

$$m^{[t]} = \begin{cases} m^{(t)} & \text{if } m \neq x^{(r)}, \\ x^{(rt)} & \text{if } m = x^{(r)}; \end{cases}$$

the proofs of (a) and (c) follow analogously.

LEMMA 2.9. *If M is a uniquely power closed semigroup such that $m^{(s)} = m$ for some $s \neq 1$, then m is idempotent.*

Proof. The assertion is clearly valid if $s = 0$ or $s = \infty$. If not, then Lemma 2.8 (b) implies that $m^{(s)} = m = m^{(1/s)}$. Thus we may assume that $s > 1$. But since $m^{(s^2)} = [m^{(s)}]^{(s)} = [m^{(1/s)}]^{(s)} = m$ and induction on n implies that $m^{(s^{2n})} = m$, we may assume that $s > 2$. But then

$$m^{(2(s-1))} = m^{(s-2)}m^{(s)} = m^{(s-1)},$$

so that Lemma 2.8 (b) implies that $m^2 = m$.

LEMMA 2.10. *If M is a uniquely power closed semigroup, then $ym = m$ implies that $ym^{(s)} = m^{(s)}$ for all $0 < s < \infty$. That is, y acts as an identity on the subsemigroup $\{m^{(s)} | \ 0 < s < \infty\}$.*

Proof. By induction we have, $y^n m = m$ for all natural numbers n. Thus if $(1/n) < s$, then Lemma 2.8 implies that $ym^{(1/n)} = m^{(1/n)}$, so that $ym^{(1/n)}m^{(s-1/n)} = m^{(1/n)}m^{(s-1/n)}$, and hence $ym^{(s)} = m^{(s)}$.

THEOREM 2.11. *Let M be a uniquely power closed semigroup and $m \in M$. Every exponential e on the subsemigroup $\{m^r | \ 0 \leqq r < \infty\}$ has an extension to an exponential on M.*

Proof. From Lemma 2.5 we need only show that $xm^r = m^s$ implies that $e(m^r) \geqq e(m^s)$. If $e(m)$ is either 0 or 1, the assertion follows from Proposition 2.1 (without the uniqueness of the power closure). Thus we may assume that $0 < e(m) < 1$ and $r > s$. Then Lemma 2.8 implies that

$$[x^{(1/(r-s))}m^{(r/(r-s)-1)}]m = m^{(s/(r-s))},$$

so that

$$[x^{(1/(r-s))}m]m^{(s/(r-s))} = m^{(s/(r-s))}.$$

Multiplying both sides of the above equation by $m^{(1-s/(r-s))}$, setting $z = x^{(1/(r-s))}$ and applying Lemma 2.10 we see that,

$$(zm)m^{(t)} = m^{(t)} \quad \text{for all } t > 0.$$

Moreover, since $(zm)m = m$ we must have $(zm)^2 = (zm)$, so that Lemma 2.8 implies that $(zm)^t = (zm)$ if $0 < t < \infty$.

Let f be a normalized additive isomorphism of $\mathbf{R}$ onto itself which is not of the form $f(t) = t$. Define a new power closure $y^{[\cdot]}$ by

$$\begin{aligned} y^{[u]} &= y^{(u)} && \text{if } y \neq m, \\ m^{[u]} &= m^{(f(u))} && \text{if } f(u) \geqq 0, \\ m^{[u]} &= (zm)z^{(-f(u))} && \text{if } f(u) < 0. \end{aligned}$$

To reach a contradiction, and thereby show that $r \leqq s$, we will show that $y^{[\cdot]}$ is a power closure. To see this, we need only verify that $m^{[u+v]} = m^{[u]}m^{[v]}$ for $u, v \geqq 0$.

Case (i). $f(u), f(v) \geqq 0$. Then $f(u + v) \geqq 0$ so that

$$m^{[u]}m^{[v]} = m^{(f(u))}m^{(f(v))} = m^{(f(u+v))} = m^{[u+v]}.$$

Case (ii). $f(u), f(v) < 0$. Then $f(u + v) < 0$ so that

$$m^{[u]}m^{[v]} = (zm)z^{(-f(u)-f(v))} = m^{[u+v]}.$$

Case (iii). $f(u) \geqq 0, f(v) < 0$, and $f(u + v) > 0$. Then

$$\begin{aligned} m^{[u]}m^{[v]} &= m^{(f(u))}[(zm)z^{(-f(v))}] \\ &= m^{(f(u)+f(v))}[zm]^{(-f(v)+1)} = m^{(f(u)+f(v))}[zm] = m^{[u+v]}. \end{aligned}$$

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Case (iv). $f(u) \geqq 0, f(v) < 0$, and $f(u+v) < 0$. Then

$$m^{[u]}m^{[v]} = m^{(f(u))}[(zm)z^{(-f(v))}]$$
$$= m^{(f(u))}z^{(f(u))}z^{(-f(u+v))}(zm) = z^{(-f(u+v))}(zm) = m^{[u+v]}.$$

Case (v). $f(u+v) = 0$. Since f is biunique, we must have $u = -v$ so that $u = v = 0$, and hence $m^{[u]}m^{[v]} = m^{[0]} = {}^{[u+v]}$.

As previously mentioned, we do not know if every uniquely power closed semigroup has enough exponentials to separate points. However we may assert the following.

COROLLARY 2.12. *Let M be a uniquely power closed semigroup.*

(a) *If m is a non-idempotent element of M and $0 < r, s < \infty$ such that $r \neq s$, then there exists an exponential e such that $e(m^r) \neq e(m^s)$.*

(b) *If m_1 is an idempotent element of M and $m_2 \in M$ such that $m_1 \neq m_2$, then there exists an exponential e such that $e(m_1) \neq e(m_2)$.*

Proof. Part (a) follows directly from Theorem 2.11 since we can extend the exponential e as defined on $\{m^r \mid 0 \leqq r < \infty\}$ by $e(m^r) = (\frac{1}{2})^r$.

If both m_1 and m_2 are idempotent, then part (b) follows even without the assumption of M being power closed. For in this event we let $M_i = \{x \mid xy = m_i$ for some $y \in M\}$ $(i = 1, 2)$. It follows that the characteristic function e_i, of M_i is an exponential. Thus we need only show that either $m_1 \notin M_2$ or $m_2 \notin M_1$. But if $m_1 \in M_2$, then there exists x such that $xm_1 = m_2$, so that $(xm_1)m_1 = m_2$ or $m_1m_2 = m_2$. Analogously, $m_2 \in M_1$ implies $m_1m_2 = m_1$, so that both $m_1 \in M_2$ and $m_2 \in M_1$ imply $m_1 = m_2$. If m_1 is idempotent and m_2 is not, part (a) verifies the assertion since every exponential assumes only the values 0 or 1 on m_1.

REFERENCES

1. E. M. Alfsen, *On the geometry of Choquet simplexes*, Math. Scand. *15* (1964), 97–110.
2. H. Bauer, *Konvexität in Topologischen Vektorraumen*, Lecture notes University of Hamburg, Hamburg, West Germany.
3. H. Cohen and H. S. Collins, *Affine semigroups*, Trans. Amer. Math. Soc. *93* (1959), 97–113.
4. P. J. Davis, *Interpolation and approximation* (Blaisdell, New York-Toronto-London, 1963).
5. K. Fan, *Les fonctions définies-positives et les fonctions complètement monotones. Leurs applications au calcul des probabilités et à la théorie des espaces distanciés*, Mémor. Sci. Math., no. 114 (Gauthier-Villars, Paris, 1950).
6. R. R. Phelps, *Lectures on Choquet's theorem* (Van Nostrand, Princeton, New Jersey, 1966).
7. K. Ross, *A note on extending semicharacters on semigroups*, Proc. Amer. Math. Soc. *10* (1959), 579–583.
8. D. V. Widder, *The Laplace transform*, Princeton Mathematical Series, Vol. 6 (Princeton Univ. Press, Princeton, N.J., 1941).

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