

MAXIMUM PRINCIPLES FOR PARABOLIC EQUATIONS

GIOVANNI PORRU and SALVATORE SERRA

(Received 7 February 1991; revised 12 November 1991)

Communicated by A. J. Pryde

Abstract

Let $u(x, t)$ be a smooth function in the domain $Q = \Omega \times (0, L]$, Ω in \mathbb{R}^n , let Du be the spatial gradient of $u(x, t)$ and let $\nabla u = (Du, u_t)$. If $u(x, t)$ satisfies the parabolic equation $F(u, Du, D^2u) = u_t$, we define $w(x, t)$ by $g(w) = |\nabla u|^{-1}G(\nabla u)$ (g is positive and decreasing, G is concave and homogeneous of degree one) and we prove that $w(x, t)$ attains its maximum value on the parabolic boundary of Q . If $u(x, t)$ satisfies the equation $\Delta u + 2h(q^2)u_i u_j u_{ij} = u_t$ ($q^2 = |Du|^2$, $1 + 2q^2 h(q^2) > 0$) we prove that $qf(u)$ takes its maximum value on the parabolic boundary of Q provided f satisfies a suitable condition. If $u(x, t)$ satisfies the parabolic equation $a^{ij}(Du)u_{ij} - b(x, t, u, Du) = u_t$ (b is concave with respect to (x, t, u)) we define $C(x, y, t, \tau) = u(z, \theta) - \alpha u(x, t) - \beta u(y, \tau)$ ($0 < \alpha$, $0 < \beta$, $\alpha + \beta = 1$, $z = \alpha x + \beta y$, $\theta = \alpha t + \beta \tau$) and we prove that if $C(x, y, t, \tau) \leq 0$ when $x, y, z \in \Omega$ and one of $t, \tau = 0$, and when $t, \tau \in (0, L]$, and one of $x, y, z, \in \partial\Omega$, then it is $C(x, y, t, \tau) \leq 0$ everywhere.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 35 B 50.

1. Introduction

For elliptic and parabolic equations various maximum principles have been known for a long time [11]. In recent years some maximum principles have been obtained for expressions involving the gradient of solutions. In [12] it is proved that if u is a smooth solution of the elliptic equation $F(u, Du, D^2u) = 0$ in a bounded domain Ω of \mathbb{R}^n and v is a fixed direction, then the angle between v and the gradient Du (assumed to be non-vanishing) takes its maximum value on the boundary of Ω . The previous result has been extended to parabolic equations

© 1994 Australian Mathematical Society 0263-6115/94 \$A2.00 + 0.00

(defined in $\Omega \times [0, L] \subset \mathbb{R}^n \times \mathbb{R}$) in [13] in case $Du = (u_1, \dots, u_n)$ (the spatial gradient), and in [1] in case $\nabla u = (u_1, \dots, u_n, u_t)$. Here $u_i (i = 1, \dots, n)$ is the derivative of u with respect to x_i and u_t is the derivative with respect to t , $t \in (0, L]$.

In [10] the elliptic equation $\Delta u + 2h(q^2)u_i u_j u_{ij} = 0$ is investigated. Here Δ is the Laplace operator, the summation convention (from 1 to n) is in effect, $q^2 = u_i u_i$ and $h(q^2)$ is a smooth function satisfying $1 + 2q^2 h(q^2) > 0$. In [10] it is proved that if u is a smooth solution of this equation in a bounded domain Ω then the function $qf(u)$ takes its maximum value on the boundary of Ω provided f satisfies a suitable condition. Similar results were previously obtained in [7, 9, 8] especially for harmonic and p -harmonic functions. (If $h(q^2) = (p - 2)/q^2$ then the previous equation reads as $(q^{p-2}u_i)_i = 0$ and its solutions are the usual p -harmonic functions.)

In Section 3 of this paper we consider a solution u of the equation $u_t = F(u, Du, D^2u)$ and we define a function $w(x, t)$ by $\rho g(w) = G(u_1, \dots, u_n, u_t)$, where $\rho = |\nabla u|$, g is a smooth positive decreasing function and G is a smooth concave function, positively homogeneous of degree one. We prove that $w(x, t)$ assumes its maximum value on $\partial\Omega$. If v is a fixed direction in \mathbb{R}^{n+1} , $G = v \cdot \nabla u$ and $g(w) = \cos(w)$ then we obtain the result of [1]. This maximum principle may be used to investigate the shape of the level sets of $u(x, t)$.

Furthermore we extend the result of [10] to a parabolic equation. Namely, let u satisfy $\Delta u + 2h(q^2)u_i u_j u_{ij} = u_t$, where the left hand side acts on the spatial variables x_1, \dots, x_n only and $q = |Du|$. We prove that if $u(x, t)$ is a solution of this equation in $Q = \Omega \times (0, L]$ and f satisfies a suitable condition, then $qf(u)$ takes its maximum value on the parabolic boundary of Q . We also prove a maximum principle for $qf(v)$ where q and f are the same as before, whereas v is a solution of an associated parabolic equation. As an application we find a new estimate for the gradient of a solution to the classical heat equation.

In order to investigate the convexity of the solutions of certain parabolic equations, the following concavity function is introduced in [5]:

$$(1.1) \quad C(x, y, t) = u(z, t) - \alpha u(x, t) - \beta u(y, t), \quad z = \alpha x + \beta y,$$

where $x, y, z \in \Omega$, $t \in (0, L]$, α and β are positive real numbers satisfying $\alpha + \beta = 1$. If Ω is convex and $C(x, y, t) \leq 0$ for all $x, y \in \Omega$ and $t \in (0, L]$ then the function $u(x, t)$ is convex with respect to x . In [6] it is proved that if $u(x, t)$ is a smooth solution of the parabolic equation

$$(1.2) \quad a^{ij}(Du)u_{ij} - b(x, t, u, Du) = u_t,$$

where $\partial b/\partial u \geq 0$ and b is jointly concave with respect to (x, u) then, if $C(x, y, t)$ is anywhere positive, its maximum value is attained at some point (x, y, t) satisfying: $t = 0$ or one of $x, y, z \in \partial\Omega$ (the boundary of Ω). Consequently, if $C(x, y, t) \leq 0$ when $x, y, z \in \Omega$, and $t = 0$ and when $t \in (0, L]$, and one of $x, y, z \in \partial\Omega$ then necessarily $C(x, y, t) \leq 0$ everywhere.

In Section 4 of the present paper we consider the equation (1.2) where b is jointly concave with respect to (x, t, u) , and, for a solution $u(x, t)$ of this equation we define the (more general) concavity function

$$(1.3) \quad \begin{aligned} C(x, y, t, \tau) &= u(z, \theta) - \alpha u(x, t) - \beta u(y, \tau), \\ z &= \alpha x + \beta y, \\ \theta &= \alpha t + \beta \tau. \end{aligned}$$

We prove that if $C(x, y, t, \tau) \leq 0$ when $x, y, z \in \Omega$, and one of $t, \tau = 0$, and when $t, \tau \in (0, L]$, and one of $x, y, z \in \partial\Omega$, then necessarily $C(x, y, t, \tau) \leq 0$ everywhere. We remark that no use of the condition $\partial b/\partial u \geq 0$ is made.

We close this introduction with some examples of problems described by the equations involved in the present paper. Equation $u_t = F(u, Du, D^2u)$ is a general autonomous evolution equation, and $u_t = \Delta u + h(q^2)u_i u_j u_{ij}$ is a special case of it. When $F(u, Du, D^2u) = K\Delta u$ (K a positive constant) we obtain the classical heat equation for a homogeneous medium in the absence of heat sources [3, p. 41]. If $F(u, Du, D^2u) = \Delta(u^m)$, $m > 1$, the corresponding equation describes the flow through a porous media [4, p. 121]. If $F(u, Du, D^2u) = \Delta u + ug(u^2)$, where g is a suitable smooth function, we find a parabolic dissipative equation [2, p. 26]. Equation (1.2) includes also some non-autonomous equations. This is the case of the heat equation in presence of heat sources [3, p. 41]. The population genetic equation is a special case of (1.2) [3, p. 43]. More examples may be found in [3] and [2].

2. Notation and preliminary results

Throughout this paper we denote by Ω a bounded domain of \mathbb{R}^n , by $\partial\Omega$ its boundary, by Q the domain $Q = \Omega \times (0, L]$ and by (x, t) a point in Q . Subscripts denote partial derivatives; for example, u_i is the derivative of u with respect to x_i . We use the convention that the sum from 1 to n is understood over the repeated indices i, j and k . For the indices r and s we use a slightly different convention: u_r means the derivative of u with respect to x_r if $r = 1, \dots, n$,

and u_t if $r = n + 1$. The same convention is used for the index s . The sum over repeated indices r and s is extended from 1 to $n + 1$. As usual we denote $Du = (u_1, \dots, u_n)$, $\nabla u = (u_1, \dots, u_n, u_t)$.

Now we state three lemmas for later use.

LEMMA 2.1. *If $[a^{ij}]$ is an $n \times n$ symmetric positive definite matrix and $[b^{rs}]$ is an $(n + 1) \times (n + 1)$ negative semidefinite matrix then, for any $(n + 1) \times n$ matrix $[\xi^{ri}]$ we have*

$$\xi^{ri} a^{ij} \xi^{sj} b^{rs} \leq 0.$$

PROOF. The result is well-known. It can be proved easily by diagonalizing the matrix $[a^{ij}]$.

LEMMA 2.2. *Let $[a^{ij}]$ be an $n \times n$ symmetric positive definite matrix, and let u be a $C^2(Q)$ function. If $q^2 = u_i u_i$, then*

$$(2.1) \quad a^{ij} u_{ki} u_{kj} \geq a^{ij} q_i q_j.$$

Furthermore, if $\rho^2 = u_r u_r$ then

$$(2.2) \quad a^{ij} u_{ri} u_{rj} \geq a^{ij} \rho_i \rho_j.$$

PROOF. Let us recall that the sum with respect to i, j and k is from 1 to n , whereas, the sum with respect to r is from 1 to $n + 1$ (consequently, $\rho^2 = q^2 + u_t u_t$). This lemma is also well-known. For completeness let us prove (2.2) (the proof of (2.1) being similar). Let A denote the matrix $[a^{ij}]$, let H be the $(n + 1) \times n$ Hessian matrix $[u_{ri}]$ and let H^T be the transposed matrix of H . Since A is symmetric and positive definite, the matrix HAH^T is symmetric and positive semidefinite, and its trace is the left hand side of (2.2). The right hand side of (2.2) can be written as $a^{ij} u_{ri} u_{sj} (u_r / \rho)(u_s / \rho)$; but this is the quadratic form associated to HAH^T and computed at the unit vector $\nabla u / \rho$. Hence (2.2) is true.

LEMMA 2.3. *Let $[a^{ij}]$ be an $n \times n$ positive semidefinite matrix. Then the $2n \times 2n$ matrix*

$$(2.3) \quad B = \begin{vmatrix} [a^{ij}] & [a^{ij}] \\ [a^{ij}] & [a^{ij}] \end{vmatrix}$$

is positive semidefinite.

PROOF. For a given vector $\zeta \in \mathbb{R}^{2n}$ let ξ be the vector of \mathbb{R}^n defined by the first n coordinates of ζ , and let η be the vector defined by the last n coordinates of ζ . Then, the quadratic form $(B\zeta, \zeta)$ can be written as

$$(B\zeta, \zeta) = a^{ij}\xi_i\xi_j + a^{ij}\xi_i\eta_j + a^{ij}\eta_i\xi_j + a^{ij}\eta_i\eta_j = a^{ij}(\xi_i + \eta_i)(\xi_j + \eta_j).$$

The lemma follows.

3. Maximum principles involving the gradient

THEOREM 3.1. *Let $u(x, t)$ be a smooth solution of the parabolic equation*

$$(3.1) \quad F(u, Du, D^2u) = u_t$$

in Q , where F is a smooth function, Du is the vector of the spatial derivatives $[u_i]$ and $D^2u = [u_{ij}]$. If a^{ij} denote the partial derivatives of F with respect to u_{ij} we assume that $a^{ij} = a^{ji}$ and that for every $\xi \in \mathbb{R}^n$ and some $v > 0$ the condition.

$$(3.2) \quad a^{ij}\xi_i\xi_j \geq v\xi_i\xi_i.$$

holds. If $g(\tau)$ is a smooth positive decreasing function and $\rho^2 = u_r u_r$, we define $w(x, t)$ by

$$(3.3) \quad g(w)\rho = G(u_1, \dots, u_n, u_t),$$

where $G(\eta_1, \dots, \eta_{n+1})$ is a smooth concave function satisfying

$$(3.4) \quad \eta_r \partial G / \partial \eta_r = G.$$

If $G(u_1, \dots, u_n, u_t)$ and ρ are positive corresponding to the solution $u(x, t)$ then the function $w(x, t)$ defined by (3.3), takes its maximum value on the parabolic boundary of Q , that is on $\Omega \times [0] \cup \partial\Omega \times [0, L]$.

PROOF. Recall that we sum from 1 to n with respect to the repeated indices i, j, k and from 1 to $n + 1$ with respect to r, s . By (3.3) we derive

$$(3.5) \quad \dot{g}\rho w_i + g\rho_i = \partial G / \partial \eta_r u_{ri},$$

$$(3.6) \quad \dot{g}\rho w_t + g\rho_t = \partial G / \partial \eta_r u_{rt},$$

where \dot{g} is the derivative of g . By (3.5) we obtain

$$(3.7) \quad \dot{g}\rho w_{ij} + (\dot{g}\rho)_j w_i + \dot{g}\rho_i w_j + g\rho_{ij} = b^{rs} u_{ri} u_{sj} + \partial G / \partial \eta_r u_{rij},$$

where $[b^{rs}]$ is the $(n + 1) \times (n + 1)$ Hessian matrix of G . Since G is concave, the matrix $[b^{rs}]$ is negative semidefinite. By (3.2) the $n \times n$ matrix $[a^{ij}]$ is positive definite; hence by Lemma 2.1 we have $a^{ij}b^{rs}u_{ri}u_{sj} \leq 0$ and (3.7) implies

$$(3.8) \quad \dot{g}\rho a^{ij}w_{ij} + b^i w_i + g a^{ij}\rho_{ij} \leq \partial G / \partial \eta_r a^{ij}u_{rij},$$

where $b^i = a^{ij}((\dot{g}\rho)_j + \dot{g}\rho_j)$.

Since $\rho^2 = u_r u_r$ we have

$$(3.9) \quad \rho\rho_i = u_r u_{ri}, \quad \rho\rho_t = u_r u_{rt},$$

$$(3.10) \quad \rho\rho_{ij} = u_{ri}u_{rj} - \rho_i\rho_j + u_r u_{rij}.$$

By (2.2) $a^{ij}u_{ri}u_{rj} - a^{ij}\rho_i\rho_j \geq 0$. Hence (3.10) implies

$$(3.11) \quad \rho a^{ij}\rho_{ij} \geq u_r a^{ij}u_{rij}.$$

This inequality and (3.8) imply

$$(3.12) \quad \dot{g}\rho a^{ij}w_{ij} + b^i w_i \leq (\partial G / \partial \eta_r - u_r g / \rho) a^{ij}u_{rij}.$$

By equation (3.1) we derive, for $r = 1, \dots, n + 1$,

$$a^{ij}u_{ijr} = u_{ir} - F_u u_r - u_{ir} \partial F / \partial u_i.$$

By inserting the last equation into (3.12) we obtain

$$-\dot{g}\rho a^{ij}w_{ij} - b^i w_i \geq g\rho_t - g\rho F_u - g\rho_i \frac{\partial F}{\partial u_i} - u_{ir} \frac{\partial G}{\partial \eta_r} + G F_u + u_{ir} \frac{\partial F}{\partial u_i} \frac{\partial G}{\partial \eta_r}$$

where (3.9) and (3.4) have been used. Finally, (3.3), (3.5), (3.6) and the last inequality lead to

$$(3.13) \quad a^{ij}w_{ij} + d^i w_i \geq w_t,$$

where $d^i = b^i / (\dot{g}\rho) + \partial F / \partial u_i$. Now the theorem follows by the well-known maximum principle for parabolic inequalities [11, p. 173].

REMARK 3.1. If $F_u = 0$ in equation (3.1), then condition (3.4) can be omitted in Theorem 3.1.

THEOREM 3.2. *Let $u(x, t)$ be a smooth solution of the equation*

$$(3.14) \quad \Delta u + 2h(q^2)u_i u_j u_{ij} = u_t$$

in $Q = \Omega \times (0, L]$, where Δ is the Laplace operator, $q^2 = u_i u_i$ and $h(\tau)$ is a smooth function satisfying $1 + 2q^2 h(q^2) > 0$. We define $T(q^2) = 1 + 2q^2 h(q^2)$ and assume there is a finite γ such that

$$(3.15) \quad 1 + q^2 \dot{T} / T \leq \gamma,$$

where \dot{T} is the derivative of T with respect to q^2 . If $f(\tau)$ is a smooth positive function satisfying

$$(3.16) \quad (\dot{f} f^{-2\gamma})' \geq 0,$$

then the function

$$(3.17) \quad \Phi(x, t) = qf(u)$$

takes its maximum value on $\Omega \times [0] \cup \partial\Omega \times [0, L]$.

PROOF. By (3.17) we derive:

$$(3.18) \quad \Phi_i = q_i f + q f u_i, \quad \Phi_t = q_t f + q f u_t,$$

$$(3.19) \quad \Phi_{ij} = q_{ij} f + q_i f u_j + q_j f u_i + q f u_i u_j + q f u_{ij}.$$

We set $a^{ij} = \delta^{ij} + 2h(q^2)u_i u_j$, where δ^{ij} is the Kronecker delta. The matrix $[a^{ij}]$ is symmetric and its eigenvalues are 1 and $1 + 2q^2 h(q^2)$. Hence it is positive definite by virtue of the assumption $1 + 2q^2 h(q^2) > 0$. With this notation, equation (3.14) reads $a^{ij} u_{ij} = u_t$, and (3.19) implies

$$(3.20) \quad a^{ij} \Phi_{ij} = f a^{ij} q_{ij} + 2 \dot{f} a^{ij} q_i u_j + q f \ddot{a}^{ij} u_i u_j + q f u_t.$$

By the definition of q we derive

$$(3.21) \quad q q_i = u_k u_{ki}, \quad q q_t = u_k u_{kt}$$

and

$$q q_{ij} = u_{ki} u_{kj}, q_i q_j + u_k u_{kij}.$$

The last equation and inequality (2.1) of Lemma 2.2 imply

$$(3.22) \quad q a^{ij} q_{ij} \geq u_k a^{ij} u_{kij}.$$

By the equation $a^{ij}u_{ij} = u_t$ we derive

$$u_k a^{ij}u_{ijk} = u_k u_{tk} - u_k a_k^{ij}u_{ij} = u_k u_{tk} - 2u_k(2\dot{h}q_k u_i u_j + h u_{ik} u_j + h u_i u_{jk})u_{ij}.$$

The latter and equations (3.21) give

$$(3.23) \quad u_k a^{ij}u_{ijk} = q q_t - 2q^2(2\dot{h}(q_k u_k)^2 + 2h q_j q_j).$$

By using (3.23), (3.22) and the last of equations (3.18), (3.20) gives

$$(3.24) \quad a^{ij}\Phi_{ij} \geq \Phi_t - 2f q (2\dot{h}(q_i u_i)^2 + 2h q_i q_i) + 2\dot{f} a^{ij} q_i u_j + q \dot{f} a^{ij} u_i u_j.$$

By (3.18) it follows that

$$(3.25) \quad q_i u_i = f^{-1}\Phi_i u_i - q^3 f^{-1} \dot{f},$$

$$(3.26) \quad \begin{aligned} q_i q_i &= f^{-1}\Phi_i q_i - q f^{-1} \dot{f} q_i u_i \\ &= f^{-1}\Phi_i q_i - q f^{-2} \dot{f} \Phi_i u_i + q^4 f^{-2} \dot{f}^2, \end{aligned}$$

$$(3.27) \quad \begin{aligned} a^{ij} q_i u_j &= a^{ij} (f^{-1}\Phi_i - f^{-1} \dot{f} q u_i) u_j \\ &= f^{-1} a^{ij} \Phi_i u_j - f^{-1} \dot{f} q a^{ij} u_i u_j. \end{aligned}$$

Insertion of (3.25), (3.26) and (3.27) into (3.24) leads to

$$(3.28) \quad a^{ij}\Phi_{ij} + b^i \Phi_i \geq \Phi_t - 2f^{-1} \dot{f}^2 q^5 (2\dot{h}q^2 + 2h) - 2f^{-1} \dot{f}^2 q a^{ij} u_i u_j + q \dot{f} a^{ij} u_i u_j,$$

where b^i are expressions depending on f, h, a^{ij}, u_i, q_i and Φ_i (but not on Φ_t).

Since $2h q^2 + 2h = \dot{T}$ and $a^{ij} u_i u_j = q^2 T$, (3.28) can be rewritten as

$$(3.29) \quad a^{ij}\Phi_{ij} + b^i \Phi_i \geq \Phi_t + q^3 T (\ddot{f} - 2f^{-1} \dot{f}^2 (1 + q^2 \dot{T}/T)).$$

Since, by assumption, $1 + q^2 \dot{T}/T \leq \gamma$, inequality (3.29) implies

$$a^{ij}\Phi_{ij} + b^i \Phi_i - \Phi_t \geq q^3 T (\ddot{f} - 2\gamma f^{-1} \dot{f}^2) = q^3 T f^{2\gamma} (\dot{f} f^{-2\gamma})' \geq 0,$$

where assumption (3.16) has been used in the last step. The theorem follows by the maximum principle for parabolic equations.

THEOREM 3.3. *Under the same notation and assumptions of Theorem 3.2, if $h \leq 0$ and $\gamma = 1$, then the function*

$$(3.30) \quad \Phi(x, t) = q f(v)$$

attains its maximum value on $\Omega \times [0] \cup \partial\Omega \times [0, L]$. Here q and f are the same as in Theorem 3.2, whereas v is any solution of the equation.

$$(3.31) \quad \Delta v + 2h(q^2)u_i u_j v_{ij} = v_t.$$

PROOF. Arguing as in the proof of Theorem 3.2 we obtain

$$(3.32) \quad a^{ij} \Phi_{ij} \geq \Phi_t - 2fq [2\dot{h}(q_i u_i)^2 + 2hq_i q_i] + 2\dot{f} a^{ij} q_i v_j + q \ddot{f} a^{ij} v_i v_j.$$

Our assumption $\gamma = 1$ implies $2\dot{h}q^2 + 2h = \dot{T} \leq 0$. This inequality, together with the Schwarz inequality $(q_i u_i)^2 \leq q_i q_i u_k u_k$ and the assumption $h \leq 0$, makes the quantity between square brackets in (3.32) non-positive, hence it can be deleted.

On the other side, by (3.30) we derive

$$q_i = f^{-1} \Phi_i - q f^{-1} \dot{f} v_i,$$

hence (3.32) implies

$$a^{ij} \Phi_{ij} + b^i \Phi_i - \Phi_t \geq q a^{ij} v_i v_j (\ddot{f} - 2f^{-1} \dot{f}^2) = q a^{ij} v_i v_j f^2 (\dot{f} f^{-2})' \geq 0,$$

where $b^i = -2f^{-1} \dot{f} a^{ij} v_j$ and condition (3.15) (with $\gamma = 1$) has been used in the last step. The theorem follows.

COROLLARY 3.1. *Let u satisfy $u_t = \Delta u$ in $Q = \Omega \times [0, L]$ with $q(x, t) = 0$ on $\partial\Omega \times [0, L]$, and let v be a positive solution of $v_t = \Delta v$ in Q with $v(x, t) = 1$ on $\Omega \times [0]$. Then we have*

$$q(x, t) \leq q(\bar{x}, 0)v(x, t)$$

for all $(x, t) \in Q$ and some $\bar{x} \in \Omega$.

PROOF. The function $f(v) = v^{-1}$ satisfies (3.16) with $\gamma = 1$, hence, by Theorem 3.3, the expression qv^{-1} attains its maximum value on $\Omega \times [0] \cup \partial\Omega \times [0, L]$. But $q = 0$ on $\partial\Omega \times [0, L]$ and $v = 1$ on $\Omega \times [0]$ by assumption. The corollary follows.

4. A maximum principle for the concavity function

If α and β are positive real numbers satisfying $\alpha + \beta = 1$, if λ is a real number, and if $u(x, t)$ is a function defined in Q , let us define

$$(4.1) \quad \begin{aligned} \Phi(x, y, t, \tau) &= e^{\lambda\theta} [u(z, \theta) - \alpha u(x, t) - \beta u(y, \tau)], \\ z &= \alpha x + \beta y, \\ \theta &= \alpha t + \beta \tau, \end{aligned}$$

for all x, y such that $x, y, z \in \Omega$ and for all $t, \tau \in (0, L]$.

THEOREM 4.1. *Let $u(x, t)$ be a smooth function defined in Q and satisfying equation (1.2). Suppose that the $n \times n$ matrix $[a^{ij}(Du)]$ of (1.2) is positive semidefinite for any vector Du , that the function $b(x, t, u, Du)$ is jointly concave with respect to (x, t, u) , and that the derivative $\partial b/\partial u$ is bounded from below. Then the function ϕ defined in (4.1) with λ satisfying*

$$(4.2) \quad \partial b/\partial u - \lambda > 0$$

has no positive maximum for $x, y, z \in \Omega$ and $t, \tau \in (0, L]$.

PROOF. Assume to the contrary that (x, y, t, τ) is a positive maximum for Φ with $x, y, z \in \Omega$ and $t, \tau \in (0, L]$. At this point we have

$$(4.3) \quad \Phi_{x_i} = \alpha e^{\lambda\theta} [u_{z_i}(z, \theta) - u_{x_i}(x, t)] = 0,$$

$$(4.4) \quad \Phi_{y_i} = \beta e^{\lambda\theta} [u_{z_i}(z, \theta) - u_{y_i}(y, \tau)] = 0.$$

Equations (4.3) - (4.4) imply that the spatial gradient Du is the same at the three points $(z, \theta), (x, t), (y, \tau)$. We also have

$$\Phi_t = \lambda\alpha\Phi + e^{\lambda\theta} [\alpha u_\theta(z, \theta) - \alpha u_t(x, t)] \geq 0,$$

$$\Phi_\tau = \lambda\beta\Phi + e^{\lambda\theta} [\beta u_\theta(z, \theta) - \beta u_\tau(y, \tau)] \geq 0,$$

from which it follows that

$$(4.5) \quad \lambda\Phi + e^{\lambda\theta} [u_\theta(z, \theta) - \alpha u_t(x, t) - \beta u_\tau(y, \tau)] \geq 0.$$

At the maximum point (x, y, t, τ) , the Hessian matrix

$$H = \begin{vmatrix} [\Phi_{x_i x_j}] & [\Phi_{x_i y_j}] \\ [\Phi_{y_i x_j}] & [\Phi_{y_i y_j}] \end{vmatrix}$$

is negative semidefinite. By Lemma 2.3, the matrix B defined in (2.3) by using the matrix $[a^{ij}(Du)]$ of the equation (1.2) (computed at any of the points $(z, \theta), (x, t), (y, \tau)$) is positive semidefinite. Consequently, the matrix BH is negative semidefinite and its trace is non positive. Hence

$$(4.6) \quad 0 \geq a^{ij} \Phi_{x_i x_j} + a^{ij} \Phi_{x_i y_j} + a^{ij} \Phi_{y_i x_j} + a^{ij} \Phi_{y_i y_j}.$$

By (4.1) we derive

$$\Phi_{x_i x_j} = e^{\lambda\theta} [\alpha^2 u_{z_i z_j}(z, \theta) - \alpha u_{x_i x_j}(x, t)],$$

$$\Phi_{x_i y_j} = \Phi_{y_i x_j} = e^{\lambda\theta} \alpha \beta u_{z_i z_j}(z, \theta),$$

$$\Phi_{y_i y_j} = e^{\lambda\theta} [\beta^2 u_{z_i z_j}(z, \theta) - \beta u_{y_i y_j}(y, \tau)].$$

Insertion of the last equations into (4.6) leads to

$$0 \geq e^{\lambda\theta} [(\alpha + \beta)^2 a^{ij} u_{z_i z_j}(z, \theta) - \alpha a^{ij} u_{x_i x_j}(x, t) - \beta a^{ij} u_{y_i y_j}(y, \tau)].$$

The latter inequality and the equation (1.2) give

$$(4.7) \quad 0 \geq e^{\lambda\theta} [b(z, \theta, u) + u_\theta(z, \theta) - \alpha b(x, t, u) - \alpha u_t(x, t) - \beta b(y, \tau, u) - \beta u_\tau(y, \tau)],$$

where $Du(z, \theta) = Du(x, t) = Du(y, \tau)$ are suppressed in the expression of b . Using the concavity of b and inequality (4.5), (4.7) implies

$$(4.8) \quad 0 \geq e^{\lambda\theta} [b(z, \theta, u(z, \theta)) - b(z, \theta, \alpha u(x, t) + \beta u(y, \tau))] - \lambda\Phi \\ = (\partial b / \partial u - \lambda)\Phi.$$

Since, by assumption, $\partial b / \partial u - \lambda > 0$ and the value of Φ at the point of maximum (x, y, t, τ) is positive, (4.8) is a contradiction. The theorem is proved.

COROLLARY 4.1. *With the notation and assumptions of Theorem 4.1, if the concavity function $C(x, y, t, \tau)$ defined in (1.3) is non-positive when $x, y, z, \in \Omega$, and one of $t, \tau = 0$, and when $t, \tau \in (0, L]$, and one of $x, y, z \in \partial\Omega$, then $C(x, y, t, \tau) \leq 0$ for all x, y in Ω and t, τ in $(0, L]$.*

PROOF. Since the inequality $C(x, y, t, \tau) \leq 0$ implies $\Phi(x, y, t, \tau) \leq 0$ and vice versa, the corollary follows by Theorem 4.1.

References

- [1] A. Buttu, 'An angle's maximum principle for the gradient of solutions of parabolic equations', *Boll. Un. Mat. Ital. A* **2** (1988), 405–408.
- [2] A. Haraux, *Nonlinear evolution equations-global behaviour of solutions*, Lecture Notes in Math. **841** (Springer, Berlin, 1981).
- [3] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math. **840** (Springer, Berlin, 1981).
- [4] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Math. **1150** (Springer, Berlin, 1985).
- [5] N. Korevaar, 'Capillarity surface convexity above convex domains', *Indiana Univ. Math. J.* **32** (1983), 73–82.

- [6] ———, 'Convex solutions to nonlinear elliptic and parabolic boundary value problems', *Indiana Univ. Math. J.* **32** (1983), 603–614.
- [7] L. P. Payne and G. A. Philippin, 'On some maximum principles involving harmonic functions and their derivatives', *SIAM J. Math. Anal.* **10** (1979), 96–104.
- [8] G. A. Philippin, 'On a free boundary problem in electrostatic', *Math. Methods Appl. Sci.* **12** (1990), 387–392.
- [9] G. A. Philippin and L. P. Payne, 'On the conformal capacity problem', *Sympos. Math.* **30** (1989), 119–136.
- [10] G. Porru and F. Ragnedda, 'Convexity properties for solutions of some second order elliptic semilinear equations', *Appl. Anal.* **37** (1990), 1–18.
- [11] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations* (Prentice-Hall, Englewood Cliffs, 1967).
- [12] C. Pucci, 'An angle's maximum principle for the gradient of solutions of elliptic equations', *Boll. Un. Mat. Ital. A* **1** (1987), 135–139.
- [13] ———, 'A maximum principle related to level surfaces of solutions of parabolic equations', *J. Austral. Math. Soc. (Series A)* **46** (1989), 1–7.

Dipartimento di Matematica
Universita degli studi di Cagliari
Via Ospedale 72
09124 Cagliari
Italy

Dipartimento di Matematica
Universita degli studi di Cagliari
Via Ospedale 72
09124 Cagliari
Italy