J. Austral. Math. Soc. (Series A) **56** (1994), 41–52

MAXIMUM PRINCIPLES FOR PARABOLIC EQUATIONS

GIOVANNI PORRU and SALVATORE SERRA

(Received 7 February 1991; revised 12 November 1991)

Communicated by A. J. Pryde

Abstract

Let $u(x,t)$ be a smooth function in the domain $Q = \Omega \times (0, L]$, Ω in $\mathbb{R}^n$, let Du be the spatial gradient of $u(x,t)$ and let $\nabla u = (Du, u_t)$. If $u(x,t)$ satisfies the parabolic equation $F(u, Du, D^2u) = u_t$, we define $w(x,t)$ by $g(w) = |\nabla u|^{-1}G(\nabla u)$ (g is positive and decreasing, G is concave and homogeneous of degree one) and we prove that $w(x,t)$ attains its maximum value on the parabolic boundary of Q. If $u(x,t)$ satisfies the equation $\Delta u + 2h(q^2)u_iu_ju_{ij} = u_t (q^2 = |Du|^2, 1 + 2q^2h(q^2) > 0)$ we prove that $qf(u)$ takes its maximum value on the parabolic boundary of Q provided f satisfies a suitable condition. If $u(x,t)$ satisfies the parabolic equation $a^{ij}(Du)u_{ij} - b(x, t, u, Du) = u_t$ (b is concave with respect to (x, t, u)) we define $C(x, y, t, \tau) = u(z, \theta) - \alpha u(x, t) - \beta u(y, \tau)$ ($0 < \alpha$, $0 < \beta$, $\alpha + \beta = 1$, $z = \alpha x + \beta y$, $\theta = \alpha t + \beta\tau$) and we prove that if $C(x, y, t, \tau) \le 0$ when $x, y, z \in \Omega$ and one of $t, \tau = 0$, and when $t, \tau \in (0, L]$, and one of $x, y, z, \in \partial\Omega$, then it is $C(x, y, t, \tau) \le 0$ everywhere.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 35 B 50.

1. Introduction

For elliptic and parabolic equations various maximum principles have been known for a long time [11]. In recent years some maximum principles have been obtained for expressions involving the gradient of solutions. In [12] it is proved that if u is a smooth solution of the elliptic equation $F(u, Du, D^2u) = 0$ in a bounded domain Ω of $\mathbb{R}^n$ and ν is a fixed direction, then the angle between ν and the gradient Du (assumed to be non-vanishing) takes its maximum value on the boundary of Ω. The previous result has been extended to parabolic equations

© 1994 Australian Mathematical Society 0263-6115/94 $A2.00 + 0.00

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

(defined in $\Omega \times [0, L] \subset \mathbb{R}^n \times \mathbb{R}$) in [13] in case $Du = (u_1, \dots, u_n)$ (the spatial gradient), and in [1] in case $\nabla u = (u_1, \dots, u_n, u_t)$. Here $u_i (i = 1, \dots, n)$ is the derivative of u with respect to x_i and u_t is the derivative with respect to t, $t \in (0, L]$.

In [10] the elliptic equation $\Delta u + 2h(q^2)u_i u_j u_{ij} = 0$ is investigated. Here Δ is the Laplace operator, the summation convention (from 1 to n) is in effect, $q^2 = u_i u_i$ and $h(q^2)$ is a smooth function satisfying $1 + 2q^2 h(q^2) > 0$. In [10] it is proved that if u is a smooth solution of this equation in a bounded domain Ω then the function $qf(u)$ takes its maximum value on the boundary of Ω provided f satisfies a suitable condition. Similar results were previously obtained in [7, 9, 8] especially for harmonic and p-harmonic functions. (If $h(q^2) = (p-2)/q^2$ then the previous equation reads as $(q^{p-2}u_i)_i = 0$ and its solutions are the usual p-harmonic functions.)

In Section 3 of this paper we consider a solution u of the equation $u_t = F(u, Du, D^2u)$ and we define a function $w(x, t)$ by $\rho g(w) = G(u_1, \dots, u_n, u_t)$, where $\rho = |\nabla u|$, g is a smooth positive decreasing function and G is a smooth concave function, positively homogeneous of degree one. We prove that $w(x, t)$ assumes its maximum value on $\partial\Omega$. If ν is a fixed direction in $\mathbb{R}^{n+1}$, $G = \nu \cdot \nabla u$ and $g(w) = \cos(w)$ then we obtain the result of [1]. This maximum principle may be used to investigate the shape of the level sets of $u(x, t)$.

Furthermore we extend the result of [10] to a parabolic equation. Namely, let u satisfy $\Delta u + 2h(q^2)u_i u_j u_{ij} = u_t$, where the left hand side acts on the spatial variables $x_1, \dots, x_n$ only and $q = |Du|$. We prove that if $u(x, t)$ is a solution of this equation in $Q = \Omega \times (0, L]$ and f satisfies a suitable condition, then $qf(u)$ takes its maximum value on the parabolic boundary of Q. We also prove a maximum principle for $qf(v)$ where q and f are the same as before, whereas v is a solution of an associated parabolic equation. As an application we find a new estimate for the gradient of a solution to the classical heat equation.

In order to investigate the convexity of the solutions of certain parabolic equations, the following concavity function is introduced in [5]:

$$C(x, y, t) = u(z, t) - \alpha u(x, t) - \beta u(y, t), \qquad z = \alpha x + \beta y, \tag{1.1}$$

where $x, y, z \in \Omega$, $t \in (0, L]$, α and β are positive real numbers satisfying $\alpha + \beta = 1$. If Ω is convex and $C(x, y, t) \leq 0$ for all $x, y \in \Omega$ and $t \in (0, L]$ then the function $u(x, t)$ is convex with respect to x. In [6] it is proved that if $u(x, t)$ is a smooth solution of the parabolic equation

$$a^{ij}(Du)u_{ij} - b(x, t, u, Du) = u_t, \tag{1.2}$$

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

where $\partial b/\partial u \geq 0$ and b is jointly concave with respect to (x, u) then, if $C(x, y, t)$ is anywhere positive, its maximum value is attained at some point (x, y, t) satisfying: $t = 0$ or one of $x, y, z \in \partial\Omega$ (the boundary of Ω). Consequently, if $C(x, y, t) \leq 0$ when $x, y, z \in \Omega$, and $t = 0$ and when $t \in (0, L]$, and one of $x, y, z \in \partial\Omega$ then necessarily $C(x, y, t) \leq 0$ everywhere.

In Section 4 of the present paper we consider the equation (1.2) where b is jointly concave with respect to (x, t, u), and, for a solution $u(x, t)$ of this equation we define the (more general) concavity function

$$\begin{aligned} C(x, y, t, \tau) &= u(z, \theta) - \alpha u(x, t) - \beta u(y, \tau), \\ z &= \alpha x + \beta y, \\ \theta &= \alpha t + \beta \tau. \end{aligned} \tag{1.3}$$

We prove that if $C(x, y, t, \tau) \leq 0$ when $x, y, z \in \Omega$, and one of $t, \tau = 0$, and when $t, \tau \in (0, L]$, and one of $x, y, z, \in \partial\Omega$, then necessarily $C(x, y, t, \tau) \leq 0$ everywhere. We remark that no use of the condition $\partial b/\partial u \geq 0$ is made.

We close this introduction with some examples of problems described by the equations involved in the present paper. Equation $u_t = F(u, Du, D^2u)$ is a general autonomous evolution equation, and $u_t = \Delta u + h(q^2)u_i u_j u_{ij}$ is a special case of it. When $F(u, Du, D^2u) = K\Delta u$ (K a positive constant) we obtain the classical heat equation for a homogeneous medium in the absence of heat sources [3, p. 41]. If $F(u, Du, D^2u) = \Delta(u^m)$, $m > 1$, the corresponding equation describes the flow through a porous media [4, p. 121]. If $F(u, Du, D^2u) = \Delta u + ug(u^2)$, where g is a suitable smooth function, we find a parabolic dissipative equation [2, p. 26]. Equation (1.2) includes also some non-autonomous equations. This is the case of the heat equation in presence of heat sources [3, p. 41]. The population genetic equation is a special case of (1.2) [3, p. 43]. More examples may be found in [3] and [2].

2. Notation and preliminary results

Throughout this paper we denote by Ω a bounded domain of $\mathbb{R}^n$, by $\partial\Omega$ its boundary, by Q the domain $Q = \Omega \times (0, L]$ and by (x, t) a point in Q. Subscripts denote partial derivatives; for example, u_i is the derivative of u with respect to x_i. We use the convention that the sum from 1 to n is understood over the repeated indices i, j and k. For the indices r and s we use a slightly different convention: u_r means the derivative of u with respect to x_r if $r = 1, \dots, n$,

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

and u_t if $r = n + 1$. The same convention is used for the index s. The sum over repeated indices r and s is extended from 1 to $n + 1$. As usual we denote $Du = (u_1, \dots, u_n)$, $\nabla u = (u_1, \dots, u_n, u_t)$.

Now we state three lemmas for later use.

LEMMA 2.1. *If* $[a^{ij}]$ *is an* $n \times n$ *symmetric positive definite matrix and* $[b^{rs}]$ *is an* $(n + 1) \times (n + 1)$ *negative semidefinite matrix then, for any* $(n + 1) \times n$ *matrix* $[\xi^{ri}]$ *we have*

$$\xi^{ri} a^{ij} \xi^{sj} b^{rs} \leq 0.$$

PROOF. The result is well-known. It can be proved easily be diagonalizing the matrix $[a^{ij}]$.

LEMMA 2.2. *Let* $[a^{ij}]$ *be an* $n \times n$ *symmetric positive definite matrix, and let* u *be a* $C^2(Q)$ *function. If* $q^2 = u_i u_i$, *then*

$$a^{ij} u_{ki} u_{kj} \geq a^{ij} q_i q_j. \tag{2.1}$$

Furthermore, if $\rho^2 = u_r u_r$ *then*

$$a^{ij} u_{ri} u_{rj} \geq a^{ij} \rho_i \rho_j. \tag{2.2}$$

PROOF. Let us recall that the sum with respect to i, j and k is from 1 to n, whereas, the sum with respect to r is from 1 to $n + 1$ (consequently, $\rho^2 = q^2 + u_t u_t$). This lemma is also well-known. For completeness let us prove (2.2) (the proof of (2.1) being similar). Let A denote the matrix $[a^{ij}]$, let H be the $(n + 1) \times n$ Hessian matrix $[u_{ri}]$ and let H^T be the transposed matrix of H. Since A is symmetric and positive definite, the matrix HAH^T is symmetric and positive semidefinite, and its trace is the left hand side of (2.2). The right hand side of (2.2) can be written as $a^{ij} u_{ri} u_{sj} (u_r/\rho)(u_s/\rho)$; but this is the quadratic form associated to HAH^T and computed at the unit vector $\nabla u/\rho$. Hence (2.2) is true.

LEMMA 2.3. *Let* $[a^{ij}]$ *be an* $n \times n$ *positive semidefinite matrix. Then the* $2n \times 2n$ *matrix*

$$B = \begin{vmatrix} [a^{ij}] & [a^{ij}] \\ [a^{ij}] & [a^{ij}] \end{vmatrix} \tag{2.3}$$

is positive semidefinite.

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

PROOF. For a given vector $\zeta \in \mathbb{R}^{2n}$ let ξ be the vector of $\mathbb{R}^n$ defined by the first n coordinates of ζ, and let η be the vector defined by the last n coordinates of ζ. Then, the quadratic form $(B\zeta, \zeta)$ can be written as

$$(B\zeta, \zeta) = a^{ij}\xi_i\xi_j + a^{ij}\xi_i\eta_j + a^{ij}\eta_i\xi_j + a^{ij}\eta_i\eta_j = a^{ij}(\xi_i + \eta_i)(\xi_j + \eta_j).$$

The lemma follows.

3. Maximum principles involving the gradient

THEOREM 3.1. *Let $u(x, t)$ be a smooth solution of the parabolic equation*

$$F(u, Du, D^2u) = u_t \tag{3.1}$$

in Q, where F is a smooth function, Du is the vector of the spatial derivatives $[u_i]$ and $D^2u = [u_{ij}]$. If a^{ij} denote the partial derivatives of F with respect to u_{ij} we assume that $a^{ij} = a^{ji}$ and that for every $\xi \in \mathbb{R}^n$ and some $\nu > 0$ the condition.

$$a^{ij}\xi_i\xi_j \geq \nu\xi_i\xi_i. \tag{3.2}$$

holds. If $g(\tau)$ is a smooth positive decreasing function and $\rho^2 = u_r u_r$ we define $w(x, t)$ by

$$g(w)\rho = G(u_1, \dots, u_n, u_t), \tag{3.3}$$

where $G(\eta_1, \dots, \eta_{n+1})$ is a smooth concave function satisfying

$$\eta_r \partial G/\partial\eta_r = G. \tag{3.4}$$

If $G(u_1, \dots, u_n, u_t)$ and ρ are positive corresponding to the solution $u(x, t)$ then the function $w(x, t)$ defined by (3.3), takes its maximum value on the parabolic boundary of Q, that is on $\Omega \times [0] \cup \partial\Omega \times [0, L]$.

PROOF. Recall that we sum from 1 to n with respect to the repeated indices i, j, k and from 1 to $n + 1$ with respect to r, s. By (3.3) we derive

$$\dot{g}\rho w_i + g\rho_i = \partial G/\partial\eta_r u_{ri}, \tag{3.5}$$

$$\dot{g}\rho w_t + g\rho_t = \partial G/\partial\eta_r u_{rt}, \tag{3.6}$$

where $\dot{g}$ is the derivative of g. By (3.5) we obtain

$$\dot{g}\rho w_{ij} + (\dot{g}\rho)_j w_i + \dot{g}\rho_i w_j + g\rho_{ij} = b^{rs}u_{ri}u_{sj} + \partial G/\partial\eta_r u_{rij}, \tag{3.7}$$

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

where $[b^{rs}]$ is the $(n+1)\times(n+1)$ Hessian matrix of G. Since G is concave, the matrix $[b^{rs}]$ is negative semidefinite. By (3.2) the $n\times n$ matrix $[a^{ij}]$ is positive definite; hence by Lemma 2.1 we have $a^{ij}b^{rs}u_{ri}u_{sj}\le 0$ and (3.7) implies

$$\dot{g}\rho a^{ij}w_{ij}+b^i w_i+g a^{ij}\rho_{ij}\le \partial G/\partial\eta_r a^{ij}u_{rij}, \tag{3.8}$$

where $b^i=a^{ij}\left((\dot{g}\rho)_j+\dot{g}\rho_j\right)$.

Since $\rho^2=u_r u_r$ we have

$$\rho\rho_i=u_r u_{ri},\qquad \rho\rho_t=u_r u_{rt}, \tag{3.9}$$

$$\rho\rho_{ij}=u_{ri}u_{rj}-\rho_i\rho_j+u_r u_{rij}. \tag{3.10}$$

By (2.2) $a^{ij}u_{ri}u_{rj}-a^{ij}\rho_i\rho_j\ge 0$. Hence (3.10) implies

$$\rho a^{ij}\rho_{ij}\ge u_r a^{ij}u_{rij}. \tag{3.11}$$

This inequality and (3.8) imply

$$\dot{g}\rho a^{ij}w_{ij}+b^i w_i\le(\partial G/\partial\eta_r-u_r g/\rho)a^{ij}u_{rij}. \tag{3.12}$$

By equation (3.1) we derive, for $r=1,\dots,n+1$,

$$a^{ij}u_{ijr}=u_{tr}-F_u u_r-u_{ir}\partial F/\partial u_i.$$

By inserting the last equation into (3.12) we obtain

$$-\dot{g}\rho a^{ij}w_{ij}-b^i w_i\ge g\rho_t-g\rho F_u-g\rho_i\frac{\partial F}{\partial u_i}-u_{tr}\frac{\partial G}{\partial\eta_r}+GF_u+u_{ir}\frac{\partial F}{\partial u_i}\frac{\partial G}{\partial\eta_r}$$

where (3.9) and (3.4) have been used. Finally, (3.3), (3.5), (3.6) and the last inequality lead to

$$a^{ij}w_{ij}+d^i w_i\ge w_t, \tag{3.13}$$

where $d^i=b^i/(\dot{g}\rho)+\partial F/\partial u_i$. Now the theorem follows by the well-known maximum principle for parabolic inequalities [11, p. 173].

REMARK 3.1. If $F_u=0$ in equation (3.1), then condition (3.4) can be omitted in Theorem 3.1.

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

THEOREM 3.2. *Let $u(x,t)$ be a smooth solution of the equation*

$$\Delta u + 2h(q^2)u_i u_j u_{ij} = u_t \tag{3.14}$$

in $Q = \Omega \times (0, L]$, where Δ is the Laplace operator, $q^2 = u_i u_i$ and $h(\tau)$ is a smooth function satisfying $1 + 2q^2 h(q^2) > 0$. We define $T(q^2) = 1 + 2q^2 h(q^2)$ and assume there is a finite γ such that

$$1 + q^2 \dot{T}/T \le \gamma, \tag{3.15}$$

where $\dot{T}$ is the derivative of T with respect to q^2. If $f(\tau)$ is a smooth positive function satisfying

$$(\dot{f} f^{-2\gamma})^{\cdot} \ge 0, \tag{3.16}$$

then the function

$$\Phi(x,t) = q f(u) \tag{3.17}$$

takes its maximum value on $\Omega \times [0] \cup \partial\Omega \times [0, L]$.

PROOF. By (3.17) we derive:

$$\Phi_i = q_i f + q\dot{f}u_i, \qquad \Phi_t = q_t f + q\dot{f}u_t, \tag{3.18}$$

$$\Phi_{ij} = q_{ij} f + q_i \dot{f} u_j + q_j \dot{f} u_i + q\ddot{f} u_i u_j + q\dot{f} u_{ij}. \tag{3.19}$$

We set $a^{ij} = \delta^{ij} + 2h(q^2)u_i u_j$, where δ^{ij} is the Kronecker delta. The matrix $[a^{ij}]$ is symmetric and its eigenvalues are 1 and $1 + 2q^2 h(q^2)$. Hence it is positive definite by virtue of the assumption $1 + 2q^2 h(q^2) > 0$. With this notation, equation (3.14) reads $a^{ij}u_{ij} = u_t$, and (3.19) implies

$$a^{ij}\Phi_{ij} = f a^{ij} q_{ij} + 2\dot{f} a^{ij} q_i u_j + q\ddot{f} a^{ij} u_i u_j + q\dot{f} u_t. \tag{3.20}$$

By the definition of q we derive

$$q q_i = u_k u_{ki}, \quad q q_t = u_k u_{kt} \tag{3.21}$$

and

$$q q_{ij} = u_{ki} u_{kj}, q_i q_j + u_k u_{kij}.$$

The last equation and inequality (2.1) of Lemma 2.2 imply

$$q a^{ij} q_{ij} \ge u_k a^{ij} u_{kij}. \tag{3.22}$$

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

By the equation $a^{ij}u_{ij} = u_t$ we derive

$$u_k a^{ij} u_{ijk} = u_k u_{tk} - u_k a^{ij}_k u_{ij} = u_k u_{tk} - 2u_k(2\dot{h}qq_k u_i u_j + hu_{ik}u_j + hu_i u_{jk})u_{ij}.$$

The latter and equations (3.21) give

$$u_k a^{ij} u_{ijk} = qq_t - 2q^2(2\dot{h}(q_k u_k)^2 + 2hq_j q_j). \tag{3.23}$$

By using (3.23), (3.22) and the last of equations (3.18), (3.20) gives

$$a^{ij}\Phi_{ij} \geq \Phi_t - 2fq\left(2\dot{h}(q_i u_i)^2 + 2hq_i q_i\right) + 2\dot{f}a^{ij}q_i u_j + q\ddot{f}a^{ij}u_i u_j. \tag{3.24}$$

By (3.18) it follows that

$$q_i u_i = f^{-1}\Phi_i u_i - q^3 f^{-1}\dot{f}, \tag{3.25}$$

$$\begin{aligned} q_i q_i &= f^{-1}\Phi_i q_i - qf^{-1}\dot{f}q_i u_i \\ &= f^{-1}\Phi_i q_i - qf^{-2}\dot{f}\Phi_i u_i + q^4 f^{-2}\dot{f}^2, \end{aligned} \tag{3.26}$$

$$\begin{aligned} a^{ij}q_i u_j &= a^{ij}\left(f^{-1}\Phi_i - f^{-1}\dot{f}qu_i\right)u_j \\ &= f^{-1}a^{ij}\Phi_i u_j - f^{-1}\dot{f}qa^{ij}u_i u_j. \end{aligned} \tag{3.27}$$

Insertion of (3.25), (3.26) and (3.27) into (3.24) leads to

$$a^{ij}\Phi_{ij} + b^i\Phi_i \geq \Phi_t - 2f^{-1}\dot{f}^2q^5\left(2\dot{h}q^2 + 2h\right) - 2f^{-1}\dot{f}^2qa^{ij}u_i u_j + q\ddot{f}a^{ij}u_i u_j, \tag{3.28}$$

where b^i are expressions depending on f, h, a^{ij}, u_i, q_i and Φ_i (but not on Φ_t).

Since $2\dot{h}q^2 + 2h = \dot{T}$ and $a^{ij}u_i u_j = q^2 T$, (3.28) can be rewritten as

$$a^{ij}\Phi_{ij} + b^i\Phi_i \geq \Phi_t + q^3 T\left(\ddot{f} - 2f^{-1}\dot{f}^2(1 + q^2\dot{T}/T)\right). \tag{3.29}$$

Since, by assumption, $1 + q^2\dot{T}/T \leq \gamma$, inequality (3.29) implies

$$a^{ij}\Phi_{ij} + b^i\Phi_i - \Phi_t \geq q^3 T\left(\ddot{f} - 2\gamma f^{-1}\dot{f}^2\right) = q^3 T f^{2\gamma}\left(\dot{f}f^{-2\gamma}\right)^{\cdot} \geq 0,$$

where assumption (3.16) has been used in the last step. The theorem follows by the maximum principle for parabolic equations.

THEOREM 3.3. *Under the same notation and assumptions of Theorem 3.2, if $h \leq 0$ and $\gamma = 1$, then the function*

$$\Phi(x, t) = qf(v) \tag{3.30}$$

attains its maximum value on $\Omega \times [0] \cup \partial\Omega \times [0, L]$. Here q and f are the same as in Theorem 3.2, whereas v is any solution of the equation.

$$\Delta v + 2h(q^2)u_i u_j v_{ij} = v_t. \tag{3.31}$$

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

PROOF. Arguing as in the proof of Theorem 3.2 we obtain

$$a^{ij}\Phi_{ij} \geq \Phi_t - 2fq\left[2\dot{h}(q_i u_i)^2 + 2hq_i q_i\right] + 2\dot{f}a^{ij}q_i v_j + q\ddot{f}a^{ij}v_i v_j. \tag{3.32}$$

Our assumption $\gamma = 1$ implies $2\dot{h}q^2 + 2h = \dot{T} \leq 0$. This inequality, together with the Schwarz inequality $(q_i u_i)^2 \leq q_i q_i u_k u_k$ and the assumption $h \leq 0$, makes the quantity between square brackets in (3.32) non-positive, hence it can be deleted.

On the other side, by (3.30) we derive

$$q_i = f^{-1}\Phi_i - qf^{-1}\dot{f}v_i,$$

hence (3.32) implies

$$a^{ij}\Phi_{ij} + b^i\Phi_i - \Phi_t \geq qa^{ij}v_i v_j\left(\ddot{f} - 2f^{-1}\dot{f}^2\right) = qa^{ij}v_i v_j f^2\left(\dot{f}f^{-2}\right)^{\cdot} \geq 0,$$

where $b^i = -2f^{-1}\dot{f}a^{ij}v_j$ and condition (3.15) (with $\gamma = 1$) has been used in the last step. The theorem follows.

COROLLARY 3.1. *Let u satisfy $u_t = \Delta u$ in $Q = \Omega \times [0, L]$ with $q(x, t) = 0$ on $\partial\Omega \times [0, L]$, and let v be a positive solution of $v_t = \Delta v$ in Q with $v(x, t) = 1$ on $\Omega \times [0]$. Then we have*

$$q(x, t) \leq q(\bar{x}, 0)v(x, t)$$

for all $(x, t) \in Q$ and some $\bar{x} \in \Omega$.

PROOF. The function $f(v) = v^{-1}$ satisfies (3.16) with $\gamma = 1$, hence, by Theorem 3.3, the expression qv^{-1} attains its maximum value on $\Omega \times [0] \cup \partial\Omega \times [0, L]$. But $q = 0$ on $\partial\Omega \times [0, L]$ and $v = 1$ on $\Omega \times [0]$ by assumption. The corollary follows.

4. A maximum principle for the concavity function

If α and β are positive real numbers satisfying $\alpha + \beta = 1$, if λ is a real number, and if $u(x, t)$ is a function defined in Q, let us define

$$\begin{aligned} \Phi(x, y, t, \tau) &= e^{\lambda\theta}\left[u(z, \theta) - \alpha u(x, t) - \beta u(y, \tau)\right], \\ z &= \alpha x + \beta y, \\ \theta &= \alpha t + \beta\tau, \end{aligned} \tag{4.1}$$

for all x, y such that $x, y, z \in \Omega$ and for all $t, \tau \in (0, L]$.

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

THEOREM 4.1. *Let $u(x, t)$ be a smooth function defined in Q and satisfying equation* (1.2). *Suppose that the $n \times n$ matrix $[a^{ij}(Du)]$ of* (1.2) *is positive semidefinite for any vector Du, that the function $b(x, t, u, Du)$ is jointly concave with respect to (x, t, u), and that the derivative $\partial b/\partial u$ is bounded from below. Then the function ϕ defined in* (4.1) *with λ satisfying*

$$\partial b/\partial u - \lambda > 0 \tag{4.2}$$

has no positive maximum for $x, y, z \in \Omega$ and $t, \tau \in (0, L]$.

PROOF. Assume to the contrary that (x, y, t, τ) is a positive maximum for Φ with $x, y, z \in \Omega$ and $t, \tau \in (0, L]$. At this point we have

$$\Phi_{x_i} = \alpha e^{\lambda\theta}\left[u_{z_i}(z, \theta) - u_{x_i}(x, t)\right] = 0, \tag{4.3}$$

$$\Phi_{y_i} = \beta e^{\lambda\theta}\left[u_{z_i}(z, \theta) - u_{y_i}(y, \tau)\right] = 0. \tag{4.4}$$

Equations (4.3) - (4.4) imply that the spatial gradient Du is the same at the three points (z, θ), (x, t), (y, τ). We also have

$$\Phi_t = \lambda\alpha\Phi + e^{\lambda\theta}\left[\alpha u_\theta(z, \theta) - \alpha u_t(x, t)\right] \geq 0,$$
$$\Phi_\tau = \lambda\beta\Phi + e^{\lambda\theta}\left[\beta u_\theta(z, \theta) - \beta u_\tau(y, \tau)\right] \geq 0,$$

from which it follows that

$$\lambda\Phi + e^{\lambda\theta}\left[u_\theta(z, \theta) - \alpha u_t(x, t) - \beta u_\tau(y, \tau)\right] \geq 0. \tag{4.5}$$

At the maximum point (x, y, t, τ), the Hessian matrix

$$H = \begin{vmatrix} [\Phi_{x_i x_j}] & [\Phi_{x_i y_j}] \\ [\Phi_{y_i x_j}] & [\Phi_{y_i y_j}] \end{vmatrix}$$

is negative semidefinite. By Lemma 2.3, the matrix B defined in (2.3) by using the matrix $[a^{ij}(Du)]$ of the equation (1.2) (computed at any of the points (z, θ), (x, t), (y, τ)) is positive semidefinite. Consequently, the matrix BH is negative semidefinite and its trace is non positive. Hence

$$0 \geq a^{ij}\Phi_{x_i x_j} + a^{ij}\Phi_{x_i y_j} + a^{ij}\Phi_{y_i x_j} + a^{ij}\Phi_{y_i y_j}. \tag{4.6}$$

By (4.1) we derive

$$\Phi_{x_i x_j} = e^{\lambda\theta}\left[\alpha^2 u_{z_i z_j}(z, \theta) - \alpha u_{x_i x_j}(x, t)\right],$$
$$\Phi_{x_i y_j} = \Phi_{y_i x_j} = e^{\lambda\theta}\alpha\beta u_{z_i z_j}(z, \theta),$$
$$\Phi_{y_i y_j} = e^{\lambda\theta}\left[\beta^2 u_{z_i z_j}(z, \theta) - \beta u_{y_i y_j}(y, \tau)\right].$$

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

Insertion of the last equations into (4.6) leads to

$$0 \geq e^{\lambda\theta}\left[(\alpha+\beta)^2 a^{ij} u_{z_i z_j}(z,\theta) - \alpha a^{ij} u_{x_i x_j}(x,t) - \beta a^{ij} u_{y_i y_j}(y,\tau)\right].$$

The latter inequality and the equation (1.2) give

$$(4.7)\quad 0 \geq e^{\lambda\theta}\big[b(z,\theta,u) + u_\theta(z,\theta) - \alpha b(x,t,u) - \alpha u_t(x,t) \\ - \beta b(y,\tau,u) - \beta u_\tau(y,\tau)\big],$$

where $Du(z,\theta) = Du(x,t) = Du(y,\tau)$ are suppressed in the expression of b. Using the concavity of b and inequality (4.5), (4.7) implies

$$(4.8)\qquad 0 \geq e^{\lambda\theta}\big[b(z,\theta,u(z,\theta)) - b(z,\theta,\alpha u(x,t)+\beta u(y,\tau))\big] - \lambda\Phi \\ = (\partial b/\partial u - \lambda)\Phi.$$

Since, by assumption, $\partial b/\partial u - \lambda > 0$ and the value of Φ at the point of maximum (x, y, t, τ) is positive, (4.8) is a contradiction. The theorem is proved.

COROLLARY 4.1. *With the notation and assumptions of Theorem* 4.1, *if the concavity function* $C(x,y,t,\tau)$ *defined in* (1.3) *is non-positive when* $x, y, z, \in \Omega$, *and one of* $t, \tau = 0$, *and when* $t, \tau \in (0, L]$, *and one of* $x, y, z \in \partial\Omega$, *then* $C(x,y,t,\tau) \leq 0$ *for all* x, y *in* Ω *and* t, τ *in* $(0, L]$.

PROOF. Since the inequality $C(x,y,t,\tau) \leq 0$ implies $\Phi(x,y,t,\tau) \leq 0$ and vice versa, the corollary follows by Theorem 4.1.

References

[1] A. Buttu, 'An angle's maximum principle for the gradient of solutions of parabolic equations', *Boll. Un. Mat. Ital. A* **2** (1988), 405–408.

[2] A. Haraux, *Nonlinear evolution equations-global behaviour of solutions*, Lecture Notes in Math. **841** (Springer, Berlin, 1981).

[3] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math. **840** (Springer, Berlin, 1981).

[4] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Math. **1150** (Springer, Berlin, 1985).

[5] N. Korevaar, 'Capillarity surface convexity above convex domains', *Indiana Univ. Math. J.* **32** (1983), 73–82.

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728

[6] ———, 'Convex solutions to nonlinear elliptic and parabolic boundary value problems', *Indiana Univ. Math. J.* **32** (1983), 603–614.

[7] L. P. Payne and G. A. Philippin, 'On some maximum principles involving harmonic functions and their derivatives', *SIAM J. Math. Anal.* **10** (1979), 96–104.

[8] G. A. Philippin, 'On a free boundary problem in electrostatic', *Math. Methods Appl. Sci.* **12** (1990), 387–392.

[9] G. A. Philippin and L. P. Payne, 'On the conformal capacity problem', *Sympos. Math.* **30** (1989), 119–136.

[10] G. Porru and F. Ragnedda, 'Convexity properties for solutions of some second order elliptic semilinear equations', *Appl. Anal.* **37** (1990), 1–18.

[11] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations* (Prentice-Hall, Englewood Ciffs, 1967).

[12] C. Pucci, 'An angle's maximum principle for the gradient of solutions of elliptic equations', *Boll. Un. Mat. Ital. A* **1** (1987), 135–139.

[13] ———, 'A maximum principle related to level surfaces of solutions of parabolic equations', *J. Austral. Math. Soc. (Series A)* **46** (1989), 1–7.

Dipartimento di Matematica
Universita degli studi di Cagliari
Via Ospedale 72
09124 Cagliari
Italy

Dipartimento di Matematica
Universita degli studi di Cagliari
Via Ospedale 72
09124 Cagliari
Italy

Downloaded from https://www.cambridge.org/core. IP address: 18.210.22.132, on 23 Oct 2019 at 15:42:45, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700034728