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# On Berman–Gibbs stability and K-stability of $\mathbb{Q}$-Fano varieties

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Abstract

The notion of Berman–Gibbs stability was originally introduced by Berman for $\mathbb{Q}$-Fano varieties $X$. We show that the pair $(X, -K_X)$ is K-stable (respectively K-semistable) provided that $X$ is Berman–Gibbs stable (respectively semistable).

## 1. Introduction

One of the most important problems for the study of $\mathbb{Q}$-Fano varieties $X$ (i.e., projective log-terminal varieties with $-K_X$ ample $\mathbb{Q}$-Cartier) is to determine whether the pairs $(X, -K_X)$ are K-stable or not (for the notion of K-stability, see § 2.1). Recently, Berman introduced a new stability of $X$, which he calls Gibbs stability, and its variants. The main purpose of this paper is to show that, slightly modifying the definition (we rename it as *Berman–Gibbs stability*), it implies the K-stability in Donaldson's [Don02] and Tian's [Tia97] sense. In particular, by [CDS15a, CDS15b, CDS15c, Tia15], it implies the existence of a Kähler–Einstein metric if $X$ is smooth and the base field is the complex number field. We remark that Berman showed in [Ber13, Theorem 7.3] that strongly Gibbs stable Fano manifolds defined over the complex number field admit Kähler–Einstein metrics, where the notion of strong Gibbs stability is stronger than the notion of Berman–Gibbs stability. Now we define the notion of Berman–Gibbs stability. (We remark that the notion of Berman–Gibbs stability is slightly weaker than the notion of uniform Gibbs stability. For detail, see [Ber13, § 7].)

Definition 1.1. Let $X$ be a projective variety and $L$ be a globally generated Cartier divisor on $X$. Set $N := h^0(X, \mathcal{O}_X(L))$ and $\phi := \phi_{|L|} : X \to \mathbb{P}^{N-1}$, where $\phi_{|L|}$ is a morphism defined by the complete linear system $|L|$. Consider the morphism $\Phi : X^N \to (\mathbb{P}^{N-1})^N$ defined by the copies of $\phi$, that is, $\Phi(x_1, \ldots, x_N) := (\phi(x_1), \ldots, \phi(x_N))$ for $x_1, \ldots, x_N \in X$. Let $\mathrm{Det}_N \subset (\mathbb{P}^{N-1})^N$ be the divisor defined by the equation $\det(x_{ij})_{1\leqslant i,j\leqslant N} = 0$, where

$$(x_{11} : \cdots : x_{1N}; \cdots ; x_{N1} : \cdots : x_{NN})$$

are the multi-homogeneous coordinates of $(\mathbb{P}^{N-1})^N$. We set the divisor $D_{X,L} \subset X^N$ defined by $D_{X,L} := \Phi^*\mathrm{Det}_N$.

*Remark* 1.2. The divisor $D_{X,L} \subset X^N$ is defined uniquely by $X$ and the linear equivalence class of $L$. In particular, the definition is independent of the choice of the basis of $H^0(X, \mathcal{O}_X(L))$.

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Definition 1.3 [Ber13, (7.2)]. Let $X$ be a $\mathbb{Q}$-Fano variety. For $k \in \mathbb{Z}_{>0}$ with $-kK_X$ Cartier and globally generated, we set $N := N_k := h^0(X, \mathcal{O}_X(-kK_X))$ and $D_k := D_{X,-kK_X} \subset X^N$. Set

$$\gamma(X) := \liminf_{\substack{k\to\infty \\ -kK_X\colon \text{Cartier}}} \left(\operatorname{lct}_{\Delta_X}\left(X^N, \frac{1}{k}D_k\right)\right),$$

where $\Delta_X (\simeq X)$ is the diagonal, that is,

$$\Delta_X := \{(x, \dots, x) \in X^N \mid x \in X\} \subset X^N,$$

and $\operatorname{lct}_{\Delta_X}(X^N, (1/k)D_k)$ is the log-canonical threshold (see [Laz04, § 9]) of the pair $(X^N, (1/k)D_k)$ around $\Delta_X$, that is,

$$\operatorname{lct}_{\Delta_X}\left(X^N, \frac{1}{k}D_k\right) := \sup\left\{c \in \mathbb{Q}_{>0} \,\middle|\, \left(X^N, \frac{c}{k}D_k\right) : \begin{array}{c}\text{log-canonical}\\ \text{around } \Delta_X\end{array}\right\}.$$

We say that $X$ is *Berman–Gibbs stable* (respectively *Berman–Gibbs semistable*) if $\gamma(X) > 1$ (respectively $\gamma(X) \geqslant 1$).

We show in this paper that Berman–Gibbs stability implies K-stability for any $\mathbb{Q}$-Fano variety. More precisely, we show the following.

Theorem 1.4 (Main theorem). *Let $X$ be a $\mathbb{Q}$-Fano variety. If $X$ is Berman–Gibbs stable (respectively Berman–Gibbs semistable), then the pair $(X, -K_X)$ is K-stable (respectively K-semistable).*

Now we explain how this article is organized. In § 2.1, we recall the notion and basic properties of K-stability. In § 2.2, we recall the notion and basic properties of multiplier ideal sheaves, which is a powerful tool to determine how much the singularities of given divisors or given ideal sheaves are mild. In § 3, we determine whether the projective line $\mathbb{P}^1$ is Berman–Gibbs stable or not. We will see that $\mathbb{P}^1$ is Berman–Gibbs semistable but is not Berman–Gibbs stable. In § 4, we prove the key propositions in order to prove Theorem 1.4. We will prove in Proposition 4.2 that Berman–Gibbs stability of $X$ implies that the singularity of a given certain ideal sheaf on $X \times \mathbb{A}^1$ is somewhat mild. The strategy of the proof of Proposition 4.2 is to see their multiplier ideal sheaves in detail. In § 5, we prove Theorem 1.4. By combining the results in [OS12], Proposition 4.2, and by some numerical arguments, we can prove Theorem 1.4.

Throughout this paper, we work in the category of algebraic (separated and of finite type) scheme over a fixed algebraically closed field $\Bbbk$ of characteristic zero. A *variety* means a reduced and irreducible algebraic scheme. For the theory of minimal model program, we refer the readers to [KM98]; for the theory of multiplier ideal sheaves, we refer the readers to [Laz04]. For varieties $X_1, \dots, X_N$, let $p_j : \prod_{1\leqslant i\leqslant N} X_i \to X_j$ be the $j$th projection morphism for any $1 \leqslant j \leqslant N$.

## 2. Preliminaries

In this section, we correct some definitions.

### 2.1 K-stability

We quickly recall the definition and basic properties of K-stability. For detail, for example, see [Oda13] and references therein.

DEFINITION 2.1 (See [Tia97, Don02, RT07, Oda13, LX14]). Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$.

(1) A *flag ideal* $\mathscr{I}$ is an ideal sheaf $\mathscr{I} \subset \mathcal{O}_{X\times\mathbb{A}^1_t}$ of the form

$$\mathscr{I} = I_M + I_{M-1}t + \cdots + I_1 t^{M-1} + (t^M) \subset \mathcal{O}_{X\times\mathbb{A}^1_t},$$

where $\mathcal{O}_X \supset I_1 \supset \cdots \supset I_M$ is a sequence of coherent ideal sheaves.

(2) Let $\mathscr{I}$ be a flag ideal and let $s \in \mathbb{Q}_{>0}$. A *normal $\mathbb{Q}$-semi test configuration* $(\mathcal{B}, \mathcal{L})/\mathbb{A}^1$ *of* $(X, -K_X)$ *obtained by* $\mathscr{I}$ *and* $s$ is defined by the following datum:

- $\Pi : \mathcal{B} \to X \times \mathbb{A}^1$ is the blowing up along $\mathscr{I}$, and let $E$ be the exceptional divisor, that is, $\mathcal{O}_{\mathcal{B}}(-E) := \mathscr{I}\mathcal{O}_{\mathcal{B}}$;
- $\mathcal{L} := \Pi^* p_1^*(-K_X) - sE$,

and we require the following conditions:

- $\mathcal{B}$ is normal and the morphism $\Pi$ is not an isomorphism;
- $\mathcal{L}$ is semiample over $\mathbb{A}^1$.

(3) Let $\pi : (\mathcal{B}, \mathcal{L}) \to \mathbb{A}^1$ be a normal $\mathbb{Q}$-semi test configuration of $(X, -K_X)$ obtained by $\mathscr{I}$ and $s$. For a sufficiently divisible positive integer $k$, the multiplicative group $\mathbb{G}_m$ naturally acts on $(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(k\mathcal{L}))$ and the morphism $\pi$ is $\mathbb{G}_m$-equivariant, where the action $\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1$ is in a standard way $(a, t) \mapsto at$. Let $w(k)$ be the total weight of the induced action on $(\pi_*\mathcal{O}_{\mathcal{B}}(k\mathcal{L}))|_{\{0\}}$ and set $N_k := h^0(X, \mathcal{O}_X(-kK_X))$. Then $N_k$ (respectively $w(k)$) is a polynomial in variable $k$ of degree $n$ (respectively at most $n+1$) for $k \gg 0$. Consider the expansion

$$\frac{w(k)}{kN_k} = F_0 + F_1 k^{-1} + F_2 k^{-2} + \cdots.$$

Let $\mathrm{DF}(\mathcal{B}, \mathcal{L}) := -F_1$ be the *Donaldson–Futaki invariant* of $(\mathcal{B}, \mathcal{L})/\mathbb{A}^1$. We set $\mathrm{DF}_0 := 2(n+1)^2((-K_X)^{\cdot n})\mathrm{DF}(\mathcal{B}, \mathcal{L})$ for simplicity.

(4) The pair $(X, -K_X)$ is said to be *K-stable* (respectively *K-semistable*) if $\mathrm{DF}(\mathcal{B}, \mathcal{L}) > 0$ (respectively $\mathrm{DF}(\mathcal{B}, \mathcal{L}) \geqslant 0$) holds for any normal $\mathbb{Q}$-semi test configuration $(\mathcal{B}, \mathcal{L})/\mathbb{A}^1$ of $(X, -K_X)$ obtained by $\mathscr{I}$ and $s$.

The following is a fundamental result.

THEOREM 2.2 [OS12, Oda13]. *Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$, $(\mathcal{B}, \mathcal{L})/\mathbb{A}^1$ be a normal $\mathbb{Q}$-semi test configuration of $(X, -K_X)$ obtained by $\mathscr{I}$ and $s$, and $(\bar{\mathcal{B}}, \bar{\mathcal{L}})/\mathbb{P}^1$ be its natural compactification to $\mathbb{P}^1$, that is, $\Pi : \bar{\mathcal{B}} \to X \times \mathbb{P}^1$ be the blowing up along $\mathscr{I}$ and $\bar{\mathcal{L}} := \Pi^* p_1^*(-K_X) - sE$ on $\bar{\mathcal{B}}$. Then the following holds.*

(1) *For a sufficiently divisible positive integer $k$, we have*

$$w(k) = \chi(\bar{\mathcal{B}}, \mathcal{O}_{\bar{\mathcal{B}}}(k\bar{\mathcal{L}})) - \chi(\bar{\mathcal{B}}, \Pi^* p_1^* \mathcal{O}_X(-kK_X)) + O(k^{n-1}).$$

*In particular, we have*

$$\lim_{k\to\infty} \frac{w(k)}{kN_k} = \frac{(\bar{\mathcal{L}}^{\cdot n+1})}{(n+1)((-K_X)^{\cdot n})}.$$

(2) *We have*

$$\begin{aligned}\mathrm{DF}_0 &= \frac{n}{n+1}(\bar{\mathcal{L}}^{\cdot n+1}) + (\bar{\mathcal{L}}^{\cdot n} \cdot K_{\bar{\mathcal{B}}/\mathbb{P}^1}) \\ &= -\frac{1}{n+1}(\bar{\mathcal{L}}^{\cdot n+1}) + (\bar{\mathcal{L}}^{\cdot n} \cdot K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1} - sE).\end{aligned}$$

(3) *We have* $(\bar{\mathcal{L}}^{\cdot n}\cdot E)>0$.

(4) *If* $K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1}-sE\geqslant 0$, *then* $\mathrm{DF}_0>0$.

*Proof.* Parts (1) and (2) follow from [Oda13, Proof of Theorem 3.2], part (3) follows from [OS12, Lemma 4.5], and part (4) follows from [OS12, Proposition 4.4]. □

## 2.2 Multiplier ideal sheaves

We recall the definition and basic properties of multiplier ideal sheaves.

DEFINITION 2.3. Let $Y$ be a normal $\mathbb{Q}$-Gorenstein variety, $\mathfrak{a}_1,\dots,\mathfrak{a}_l\subset\mathcal{O}_Y$ be coherent ideal sheaves and $c_1,\dots,c_l\in\mathbb{Q}_{\geqslant 0}$. The *multiplier ideal sheaf* $\mathcal{I}(Y,\mathfrak{a}_1^{c_1}\cdots\mathfrak{a}_l^{c_l})\subset\mathcal{O}_Y$ of the pair $(Y,\mathfrak{a}_1^{c_1}\cdots\mathfrak{a}_l^{c_l})$ is defined by the following. Take a common log resolution $\mu:\hat{Y}\to Y$ of $\mathfrak{a}_1,\dots,\mathfrak{a}_l$, i.e., $\hat{Y}$ is smooth, $\mathfrak{a}_i\mathcal{O}_{\hat{Y}}=\mathcal{O}_{\hat{Y}}(-F_i)$ and $\mathrm{Exc}(\mu)$, $\mathrm{Exc}(\mu)+\sum_{1\leqslant i\leqslant l}F_i$ are divisors with simple normal crossing supports. Then we set

$$\mathcal{I}(Y,\mathfrak{a}_1^{c_1}\cdots\mathfrak{a}_l^{c_l}):=\mu_*\mathcal{O}_{\hat{Y}}\biggl(\biggl\lceil K_{\hat{Y}/Y}-\sum_{1\leqslant i\leqslant l}c_iF_i\biggr\rceil\biggr),$$

where $\lceil K_{\hat{Y}/Y}-\sum_{1\leqslant i\leqslant l}c_iF_i\rceil$ is the smallest $\mathbb{Z}$-divisor which contains $K_{\hat{Y}/Y}-\sum_{1\leqslant i\leqslant l}c_iF_i$.

The following proposition can be proved in essentially the same way as in the proofs in [Laz04, § 9]. We omit the proof.

PROPOSITION 2.4 (See [Laz04, § 9]). *Under the hypotheses of Definition 2.3, we have the following.*

(1) $\mathcal{I}(Y,\mathfrak{a}_1^{c_1}\cdots\mathfrak{a}_l^{c_l})$ *does not depend on the choice of* $\mu$.

(2) *For an effective Cartier divisor* $D$ *on* $Y$, *we have*

$$\mathcal{I}(Y,\mathcal{O}_Y(-D)^1\mathfrak{a}_1^{c_1}\cdots\mathfrak{a}_l^{c_l})=\mathcal{I}(Y,\mathfrak{a}_1^{c_1}\cdots\mathfrak{a}_l^{c_l})\otimes\mathcal{O}_Y(-D).$$

(3) *If coherent ideal sheaves* $\mathfrak{b}_1,\dots,\mathfrak{b}_l\subset\mathcal{O}_Y$ *satisfy that* $\mathfrak{a}_i\subset\mathfrak{b}_i$ *for all* $1\leqslant i\leqslant l$, *then*

$$\mathcal{I}(Y,\mathfrak{a}_1^{c_1}\cdots\mathfrak{a}_l^{c_l})\subset\mathcal{I}(Y,\mathfrak{b}_1^{c_1}\cdots\mathfrak{b}_l^{c_l}).$$

(4) *Let* $Y'$ *be another normal* $\mathbb{Q}$*-Gorenstein variety,* $\mathfrak{b}_1,\dots,\mathfrak{b}_{l'}\subset\mathcal{O}_{Y'}$ *be coherent ideal sheaves and* $c'_1,\dots,c'_{l'}\in\mathbb{Q}_{\geqslant 0}$. *Then we have*

$$\begin{aligned}&\mathcal{I}(Y\times Y',p_1^{-1}\mathfrak{a}_1^{c_1}\cdots p_1^{-1}\mathfrak{a}_l^{c_l}\cdot p_2^{-1}\mathfrak{b}_1^{c'_1}\cdots p_2^{-1}\mathfrak{b}_{l'}^{c'_{l'}})\\&\quad=p_1^{-1}\mathcal{I}(Y,\mathfrak{a}_1^{c_1}\cdots\mathfrak{a}_l^{c_l})\cdot p_2^{-1}\mathcal{I}(Y',\mathfrak{b}_1^{c'_1}\cdots\mathfrak{b}_{l'}^{c'_{l'}}).\end{aligned}$$

The following theorem is a singular version of Mustaţă's summation formula [Mus02, Corollary 1.4] due to Takagi.

THEOREM 2.5 [Tak06, Theorem 3.2]. *Let* $Y$ *be a normal* $\mathbb{Q}$*-Gorenstein variety, let* $\mathfrak{a}_0,\mathfrak{a}_1,\dots,\mathfrak{a}_l\subset\mathcal{O}_Y$ *be coherent ideal sheaves and let* $c_0$, $c\in\mathbb{Q}_{\geqslant 0}$. *Then we have*

$$\mathcal{I}\biggl(Y,\mathfrak{a}_0^{c_0}\cdot\biggl(\sum_{i=1}^{l}\mathfrak{a}_i\biggr)^{c}\biggr)=\sum_{\substack{c_1+\cdots+c_l=c\\ c_1,\dots,c_l\in\mathbb{Q}_{\geqslant 0}}}\mathcal{I}\biggl(Y,\mathfrak{a}_0^{c_0}\cdot\prod_{i=1}^{l}\mathfrak{a}_i^{c_i}\biggr).$$

## 3. The projective line case

In this section, we see whether the projective line $\mathbb{P}^1$ is Berman–Gibbs stable or not. For any $k \in \mathbb{Z}_{>0}$, we have $N_k = 2k+1$ and the morphism associated to the complete linear system $|-kK_{\mathbb{P}^1}|$ is the $(2k)$th Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^{2k}$. If the multi-homogeneous coordinates of $(\mathbb{P}^1)^{2k+1}$ are denoted by

$$(t_{1,0} : t_{1,1}; \cdots ; t_{2k+1,0} : t_{2k+1,1}),$$

then the divisor $D_k \subset (\mathbb{P}^1)^{2k+1}$ corresponds to the following section:

$$\det \begin{pmatrix} t_{1,0}^{2k} & t_{1,0}^{2k-1} t_{1,1}^{1} & \cdots & t_{1,0}^{1} t_{1,1}^{2k-1} & t_{1,1}^{2k} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ t_{2k+1,0}^{2k} & t_{2k+1,0}^{2k-1} t_{2k+1,1}^{1} & \cdots & t_{2k+1,0}^{1} t_{2k+1,1}^{2k-1} & t_{2k+1,1}^{2k} \end{pmatrix}.$$

The above matrix is so-called the Vandermonde matrix. Thus, around $0 \in \mathbb{A}^{2k+1}_{u_1,\ldots,u_{2k+1}} \subset (\mathbb{P}^1)^{2k+1}$, the divisor $D_k \subset \mathbb{A}^{2k+1}_{u_1,\ldots,u_{2k+1}}$ is defined by the polynomial $f_k \in \Bbbk[u_1, \ldots, u_{2k+1}]$, where

$$f_k := \prod_{1 \leqslant i < j \leqslant 2k+1} (u_i - u_j).$$

By Lemma 3.1, $\mathrm{lct}_0(\mathbb{A}^{2k+1}, (f_k = 0)) = 2/(2k+1)$. Thus

$$\mathrm{lct}_{\Delta_{\mathbb{P}^1}}((\mathbb{P}^1)^N, (1/k)D_k) = 2k/(2k+1).$$

Hence $\gamma(\mathbb{P}^1) = 1$. As a consequence, the projective line $\mathbb{P}^1$ is Berman–Gibbs semistable but is not Berman–Gibbs stable.

Lemma 3.1 [Mus06]. *For $g \geqslant 2$, we have*

$$\mathrm{lct}_0\biggl(\mathbb{A}^g_{u_1,\ldots,u_g}, \biggl(\prod_{1 \leqslant i < j \leqslant g} (u_i - u_j) = 0\biggr)\biggr) = 2/g.$$

*Proof.* Set $D := (\prod_{1 \leqslant i < j \leqslant g}(u_i - u_j) = 0) \subset \mathbb{A}^g$. Let $\tau : V \to \mathbb{A}^g$ be the blowing up along the line $(u_1 = \cdots = u_g)$ and let $F$ be its exceptional divisor. For $c \in \mathbb{Q}_{>0}$, the discrepancy $a(F, \mathbb{A}^g, cD)$ is equal to $g - 2 - cg(g-1)/2$. Thus $\mathrm{lct}_0(\mathbb{A}^g, D) \leqslant 2/g$. Hence it is enough to show that $\mathrm{lct}(\mathbb{A}^g, D) \geqslant 2/g$.

Let $H_{ij} \subset \mathbb{A}^g$ be the hyperplane defined by $u_i - u_j = 0$ and set $\mathcal{A} := \{H_{ij}\}_{1 \leqslant i,j \leqslant g, i \neq j}$. We set

$$L(\mathcal{A}) := \biggl\{ W \subset \mathbb{A}^g \,\bigg|\, {}^{\exists}\mathcal{A}' \subset \mathcal{A}; W = \bigcap_{H \in \mathcal{A}'} H \biggr\}.$$

For $W \in L(\mathcal{A})$, set $s(W) := \#\{H \in \mathcal{A} \,|\, W \subset H\}$ and $r(W) := \mathrm{codim}_{\mathbb{A}^g} W$. By [Mus06, Corollary 0.3],

$$\mathrm{lct}(\mathbb{A}^g, D) = \min_{W \in L(\mathcal{A}) \setminus \{\mathbb{A}^g\}} \left\{ \frac{r(W)}{s(W)} \right\}.$$

Pick any $W \in L(\mathcal{A}) \setminus \{\mathbb{A}^g\}$ and set $r := r(W)$. It is enough to show that $s(W) \leqslant r(r+1)/2$. If $r = 1$, then $s(W) = 1$. Thus we can assume that $r \geqslant 2$. There exist distinct $H_{i_1 j_1}, \ldots, H_{i_r j_r} \in \mathcal{A}$ such that $W = H_{i_1 j_1} \cap \cdots \cap H_{i_r j_r}$.

Assume that $i_1, j_1 \notin \{i_2, j_2, \ldots, i_r, j_r\}$. For any $H_{ij} \in L(\mathcal{A})$, if $W \subset H_{ij}$ then $H_{i_1 j_1} = H_{ij}$ or $H_{i_2 j_2} \cap \cdots \cap H_{i_r j_r} \subset H_{ij}$. Thus $s(W) = 1 + s(H_{i_2 j_2} \cap \cdots \cap H_{i_r j_r}) \leqslant 1 + r(r-1)/2 < r(r+1)/2$ by induction on $r$. Hence we can assume that $(i_0 :=) i_1 = i_2$.

Assume that $i_0, j_1, j_2 \notin \{i_3, j_3, \ldots, i_r, j_r\}$. For any $H_{ij} \in L(\mathcal{A})$, if $W \subset H_{ij}$ then $H_{i_0 j_1} \cap H_{i_0 j_2} \subset H_{ij}$ or $H_{i_3 j_3} \cap \cdots \cap H_{i_r j_r} \subset H_{ij}$. Thus $s(W) = s(H_{i_0 j_1} \cap H_{i_0 j_2}) + s(H_{i_3 j_3} \cap \cdots \cap H_{i_r j_r}) \leqslant 2 \cdot 3/2 + (r-1)(r-2)/2 < r(r+1)/2$ by induction on $r$. Hence we can assume that $i_3 \in \{i_0, j_1, j_2\}$. If $i_3 = j_1$, then $H_{i_0 j_1} \cap H_{j_1 j_3} = H_{i_0 j_1} \cap H_{i_0 j_3}$. By replacing $H_{j_1 j_3}$ to $H_{i_0 j_3}$, we can assume that $(i_0 =) i_1 = i_2 = i_3$.

We repeat this process. (We note that, for any $1 \leqslant j \leqslant r-1$, $j(j+1)/2 + (r-j)(r-j+1)/2 < r(r+1)/2$.) We can assume that $(i_0 =) i_1 = \cdots = i_r$. For any $H_{ij} \in L(\mathcal{A})$, the condition $W \subset H_{ij}$ is equivalent to the condition $\{i, j\} \subset \{i_0, j_1, \ldots, j_r\}$. Thus $s(W) = r(r+1)/2$. Therefore we have proved that $s(W) \leqslant r(r+1)/2$. □

## 4. Key propositions

In this section, we see the key propositions in order to prove Theorem 1.4. Throughout the section, let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$ and let $(\mathcal{B}, \mathcal{L})/\mathbb{A}^1$, $\mathscr{I}$, $s$, and so on are as in § 2.1.

LEMMA 4.1. *Let $k$ be a sufficiently divisible positive integer.*

(1) *Set $I_0 := \mathcal{O}_X$ (cf. [RT07, §§ 3 and 4]). We also set*

$$\tilde{I}_j := \sum_{\substack{j_1 + \cdots + j_{ks} = j \\ 0 \leqslant j_1, \ldots, j_{ks} \leqslant M}} I_{j_1} \cdots I_{j_{ks}}$$

*for all $0 \leqslant j \leqslant Mks$. Then $\mathscr{I}^{ks} = \tilde{I}_{Mks} + \tilde{I}_{Mks-1} t + \cdots + \tilde{I}_1 t^{Mks-1} + (t^{Mks})$. Consider the filtration*

$$H^0(X, \mathcal{O}_X(-kK_X)) = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_{Mks} \supset 0$$

*defined by $\mathcal{F}_j := H^0(X, \mathcal{O}_X(-kK_X) \cdot \tilde{I}_j)$. Set $m := \sum_{j=1}^{Mks} \dim \mathcal{F}_j$. Then $m = NMks + w$ holds, where $w = w(k)$ and $N = N_k$ are as in Definition 2.1(3).*

(2) *Let $\tilde{I}_{i,j} \subset \mathcal{O}_{X_i}$ be the copies of $\tilde{I}_j \subset \mathcal{O}_X$ $(X_i := X)$ for all $1 \leqslant i \leqslant N$ and set*

$$J_j := \sum_{\substack{j_1 + \cdots + j_N = j \\ 0 \leqslant j_1, \ldots, j_N \leqslant Mks}} p_1^{-1} \tilde{I}_{1, j_1} \cdots p_N^{-1} \tilde{I}_{N, j_N} \subset \mathcal{O}_{X^N}$$

*for all $0 \leqslant j \leqslant NMks$. Then $\mathcal{O}_{X^N}(-D_k) \subset J_m$ holds.*

*Proof.* (1) By [RT07, §§ 3 and 4], $(\pi_* \mathcal{O}_{\mathcal{B}}(k\mathcal{L}))|_{\{0\}}$ is equal to

$$H^0(X \times \mathbb{A}^1_t, \mathcal{O}(-kK_{X \times \mathbb{A}^1}) \cdot \mathscr{I}^{ks}) / t \cdot H^0(X \times \mathbb{A}^1_t, \mathcal{O}(-kK_{X \times \mathbb{A}^1}) \cdot \mathscr{I}^{ks})$$

and is also equal to

$$\mathcal{F}_{Mks} \oplus \bigoplus_{j=1}^{Mks} t^j \cdot (\mathcal{F}_{Mks-j} / \mathcal{F}_{Mks-j+1}).$$



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Thus $w=\sum_{j=1}^{Mks}(-j)(\dim\mathcal{F}_{Mks-j}-\dim\mathcal{F}_{Mks-j+1})=-Mks\dim\mathcal{F}_0+\sum_{j=1}^{Mks}\dim\mathcal{F}_j$. This implies that $m=NMks+w$.

(2) Choose a basis $s_1,\ldots,s_N\in H^0(X,\mathcal{O}_X(-kK_X))$ along the filtration $\{\mathcal{F}_j\}_{0\leqslant j\leqslant Mks}$. For $1\leqslant j\leqslant N$, set

$$f(j):=\max\{0\leqslant i\leqslant Mks\,|\,s_j\in\mathcal{F}_i\}.$$

Let $s_{i1},\ldots,s_{iN}\in H^0(X_i,\mathcal{O}_{X_i}(-kK_{X_i}))$ be the $i$th copies of $s_1,\ldots,s_N$ for all $1\leqslant i\leqslant N$. Then the divisor $D_k\subset X^N$ corresponds to the section

$$\sum_{\sigma\in\mathfrak{S}_N}\mathrm{sgn}\sigma\cdot s_{1\sigma(1)}\cdots s_{N\sigma(N)}\in H^0(X^N,\mathcal{O}_{X^N}(-kK_{X^N})),$$

where $\mathfrak{S}_N$ is the $N$th symmetric group. Take any $\sigma\in\mathfrak{S}_N$. Since $s_{i,j}\in p_i^{-1}\tilde{I}_{i,f(j)}$, we have

$$s_{1\sigma(1)}\cdots s_{N\sigma(N)}\in p_1^{-1}\tilde{I}_{1,f(\sigma(1))}\cdots p_N^{-1}\tilde{I}_{N,f(\sigma(N))}.$$

Note that $\sum_{i=1}^N f(\sigma(i))=\sum_{i=1}^N f(i)=\sum_{j=0}^{Mks}j(\dim\mathcal{F}_j-\dim\mathcal{F}_{j+1})=m$, where $\mathcal{F}_{Mks+1}:=0$. Thus $\mathcal{O}_{X^N}(-D_k)\subset J_m$. □

Proposition 4.2. *Assume that a positive rational number $\gamma\in\mathbb{Q}_{>0}$ satisfies that, for a sufficiently divisible positive integer $k$, the pair $(X^N,(\gamma/k)D_k)$ is log-canonical around $\Delta_X$. Then for any $\varepsilon\in(0,1)\cap\mathbb{Q}$ and any sufficiently big positive integer $P$, the structure sheaf $\mathcal{O}_{X\times\mathbb{A}^1}$ is contained in the sheaf*

$$\mathcal{I}(X\times\mathbb{A}^1,(t)^{(1-\varepsilon)(1+\gamma w/(kN))+P}\cdot\mathscr{I}^{(1-\varepsilon)\gamma s})\otimes\mathcal{O}_{X\times\mathbb{A}^1}(P\cdot(t=0))$$

*(that is, the pair $(X\times\mathbb{A}^1,(t)^{(1+\gamma w/(kN))}\cdot\mathscr{I}^{\gamma s})$ is 'sub-log-canonical'), where $w=w(k)$ and $N=N_k$ are as in Definition 2.1(3).*

*Proof.* We set

$$\Theta:=\left\{\vec{j}=(j_1,\ldots,j_N)\,\middle|\,\begin{smallmatrix}j_1+\cdots+j_N=m,\\ 0\leqslant j_1,\ldots,j_N\leqslant Mks\end{smallmatrix}\right\},$$

$$A:=\left\{\vec{\alpha}=(\alpha_{\vec{j}})_{\vec{j}\in\Theta}\,\middle|\,\begin{smallmatrix}\sum_{\vec{j}\in\Theta}\alpha_{\vec{j}}=(1-\varepsilon)\gamma/k,\\ \forall\alpha_{\vec{j}}\in\mathbb{Q}_{\geqslant0}\end{smallmatrix}\right\},$$

$$B:=\left\{\vec{\beta}=(\beta_0,\ldots,\beta_{Mks})\,\middle|\,\begin{smallmatrix}\beta_0,\ldots,\beta_{Mks}\in\mathbb{Q}_{\geqslant0},\\ \sum_{j=0}^{Mks}\beta_j=(1-\varepsilon)\gamma/k\end{smallmatrix}\right\},$$

$$\Xi:=\left\{\vec{\xi}=(\xi_0,\ldots,\xi_{Mks})\,\middle|\,\begin{smallmatrix}\xi_0,\ldots,\xi_{Mks}\in\mathbb{Q}_{\geqslant0},\\ \sum_{j=0}^{Mks}\xi_j=(1-\varepsilon)\gamma/k,\\ \sum_{j=0}^{Mks}j\xi_j\geqslant(1-\varepsilon)\gamma m/(kN)\end{smallmatrix}\right\}$$

for simplicity.

Claim 4.3. *We have the equality*

$$\mathcal{O}_X=\sum_{\vec{\xi}\in\Xi}\mathcal{I}\left(X,\prod_{i=0}^{Mks}\tilde{I}_i^{\xi_i}\right).$$

*Proof of Claim 4.3.* By Proposition 2.4, Theorem 2.5 and Lemma 4.1, around $\Delta_X$, we have

$$\begin{aligned}\mathcal{O}_{X^N} &= \mathcal{I}(X^N, \mathcal{O}_{X^N}(-D_k)^{(1-\varepsilon)\gamma/k}) \\ &\subset \mathcal{I}(X^N, J_m^{(1-\varepsilon)\gamma/k}) \\ &= \mathcal{I}\bigg(X^N, \bigg(\sum_{\vec{j}\in\Theta} p_1^{-1}\tilde{I}_{1,j_1}\cdots p_N^{-1}\tilde{I}_{N,j_N}\bigg)^{(1-\varepsilon)\gamma/k}\bigg) \\ &= \sum_{\vec{\alpha}\in A}\mathcal{I}\bigg(X^N, \prod_{\vec{j}\in\Theta}(p_1^{-1}\tilde{I}_{1,j_1}\cdots p_N^{-1}\tilde{I}_{N,j_N})^{\alpha_{\vec{j}}}\bigg) \\ &= \sum_{\vec{\alpha}\in A} p_1^{-1}\mathcal{I}\bigg(X_1, \prod_{\vec{j}\in\Theta}\tilde{I}_{1,j_1}^{\alpha_{\vec{j}}}\bigg)\cdots p_N^{-1}\mathcal{I}\bigg(X_N, \prod_{\vec{j}\in\Theta}\tilde{I}_{N,j_N}^{\alpha_{\vec{j}}}\bigg).\end{aligned}$$

Restricting to $\Delta_X$, we have

$$\mathcal{O}_X = \sum_{\vec{\alpha}\in A}\mathcal{I}\bigg(X, \prod_{\vec{j}\in\Theta}\tilde{I}_{j_1}^{\alpha_{\vec{j}}}\bigg)\cdots\mathcal{I}\bigg(X, \prod_{\vec{j}\in\Theta}\tilde{I}_{j_N}^{\alpha_{\vec{j}}}\bigg).$$

Fix an arbitrary $\vec{\alpha}\in A$. Since

$$\sum_{\vec{j}\in\Theta}\alpha_{\vec{j}}j_1 + \cdots + \sum_{\vec{j}\in\Theta}\alpha_{\vec{j}}j_N = (1-\varepsilon)\gamma m/k,$$

we have $\sum_{\vec{j}\in\Theta}\alpha_{\vec{j}}j_q \geqslant (1-\varepsilon)\gamma m/(kN)$ for some $1\leqslant q\leqslant N$. We set

$$\xi_i := \sum_{\vec{j}\in\Theta;\ j_q=i}\alpha_{\vec{j}}$$

for $0\leqslant i\leqslant Mks$. Then $\vec{\xi} := (\xi_0,\ldots,\xi_{Mks})\in\Xi$ and

$$\mathcal{I}\bigg(X, \prod_{\vec{j}\in\Theta}\tilde{I}_{j_q}^{\alpha_{\vec{j}}}\bigg) = \mathcal{I}\bigg(X, \prod_{i=0}^{Mks}\tilde{I}_i^{\xi_i}\bigg).$$

Therefore we have proved Claim 4.3. $\square$

By Proposition 2.4(4) and Claim 4.3, we have

$$\mathcal{O}_{X\times\mathbb{A}^1}(-P\cdot(t=0)) = \sum_{\vec{\xi}\in\Xi}\mathcal{I}\bigg(X\times\mathbb{A}^1, (t)^{1-\varepsilon+P}\cdot\prod_{i=0}^{Mks}\tilde{I}_i^{\xi_i}\bigg).$$

For any $\vec{\xi}\in\Xi$, since $(1-\varepsilon)(1+\gamma m/(kN)) + P - \sum_{i=0}^{Mks} i\xi_i \leqslant 1-\varepsilon+P$, we have

$$\begin{aligned}&\mathcal{I}\bigg(X\times\mathbb{A}^1, (t)^{1-\varepsilon+P}\cdot\prod_{i=0}^{Mks}\tilde{I}_i^{\xi_i}\bigg) \\ &\qquad\subset \mathcal{I}\bigg(X\times\mathbb{A}^1, (t)^{(1-\varepsilon)(1+\gamma m/(kN))+P-\sum_{i=0}^{Mks} i\xi_i}\cdot\prod_{i=0}^{Mks}\tilde{I}_i^{\xi_i}\bigg).\end{aligned}$$

On the other hand, by Lemma 4.1(1) and Theorem 2.5, we have



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$$\begin{aligned}
&\mathcal{I}(X\times\mathbb{A}^1,(t)^{(1-\varepsilon)(1+\gamma w/(kN))+P}\cdot\mathscr{I}^{(1-\varepsilon)\gamma s})\\
&\quad=\mathcal{I}\left(X\times\mathbb{A}^1,(t)^{(1-\varepsilon)(1-\gamma(Ms-m/(kN)))+P}\cdot\left(\sum_{i=0}^{Mks}(t)^{Mks-i}\tilde{I}_i\right)^{(1-\varepsilon)\gamma/k}\right)\\
&\quad=\sum_{\vec{\beta}\in B}\mathcal{I}\left(X\times\mathbb{A}^1,(t)^{(1-\varepsilon)(1-\gamma(Ms-m/(kN)))+P}\cdot\prod_{i=0}^{Mks}\left((t)^{Mks-i}\tilde{I}_i\right)^{\beta_i}\right)\\
&\quad=\sum_{\vec{\beta}\in B}\mathcal{I}\left(X\times\mathbb{A}^1,(t)^{(1-\varepsilon)(1+\gamma m/(kN))+P-\sum_{i=0}^{Mks}i\beta_i}\cdot\prod_{i=0}^{Mks}\tilde{I}_i^{\beta_i}\right).
\end{aligned}$$

Since $\Xi\subset B$, we have proved Proposition 4.2. □

## 5. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$ and set $\gamma:=\gamma(X)$. We assume that $\gamma\geqslant 1$. Let $(\mathcal{B},\mathcal{L})/\mathbb{A}^1$ be a normal $\mathbb{Q}$-semi test configuration of $(X,-K_X)$ obtained by $\mathscr{I}$ and $s$, and let $E$, $\bar{\mathcal{B}}$, $\bar{\mathcal{L}}$ and so on be as in § 2.1. Let $\{E_\lambda\}_{\lambda\in\Lambda}$ be the set of $\Pi$-exceptional prime divisors. We note that $\Lambda\neq\emptyset$, since the morphism $\Pi$ is not an isomorphism. We set

$$\begin{aligned}
\sum_{\lambda\in\Lambda}a_\lambda E_\lambda&:=K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1},\\
\sum_{\lambda\in\Lambda}b_\lambda E_\lambda&:=\Pi^*X_0-\hat{X}_0,\\
\sum_{\lambda\in\Lambda}c_\lambda E_\lambda&:=E
\end{aligned}$$

as in [OS12], where $X_0$ is the fiber of $p_2: X\times\mathbb{P}^1\to\mathbb{P}^1$ at $0\in\mathbb{P}^1$ and $\hat{X}_0$ is the strict transform of $X_0$ in $\bar{\mathcal{B}}$. We note that $b_\lambda$, $c_\lambda\in\mathbb{Z}_{>0}$ and $a_\lambda-b_\lambda+1>0$ for any $\lambda\in\Lambda$ since the pair $(X\times\mathbb{P}^1,X_0)$ is purely-log-terminal. We set

$$d:=\max_{\lambda\in\Lambda}\left\{\frac{\gamma sc_\lambda-(a_\lambda-b_\lambda+1)}{\gamma b_\lambda}\right\}.$$

By Theorem 2.2(4), we can assume that $d>0$.

Claim 5.1. *We have the inequality:*

$$\frac{-(\bar{\mathcal{L}}^{\cdot n+1})}{(n+1)((-K_X)^{\cdot n})}\geqslant d.$$

*Proof of Claim 5.1.* For any sufficiently small positive rational numbers $\varepsilon$ and $\varepsilon'$, by Proposition 4.2, the coefficient of

$$K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1}-(1-\varepsilon)(1+(\gamma-\varepsilon')w/(kN))\Pi^*X_0-(1-\varepsilon)(\gamma-\varepsilon')sE$$

at $E_\lambda$ is strictly bigger than $-1$ for any $\lambda\in\Lambda$ and for any sufficiently divisible positive integer $k$. Thus, by Theorem 2.2(1), we have

$$-1\leqslant a_\lambda-\left(1-\gamma\frac{-(\bar{\mathcal{L}}^{\cdot n+1})}{(n+1)((-K_X)^{\cdot n})}\right)b_\lambda-\gamma sc_\lambda$$

for any $\lambda\in\Lambda$. Hence we have proved Claim 5.1. □

By Claim 5.1, we have the inequalities:

$$\begin{aligned}\mathrm{DF}_0 &= \frac{-(\bar{\mathcal{L}}^{\cdot n+1})}{n+1} + \left(\bar{\mathcal{L}}^{\cdot n}\cdot\sum_{\lambda\in\Lambda}(a_\lambda - sc_\lambda)E_\lambda\right)\\ &\geqslant \left(\bar{\mathcal{L}}^{\cdot n}\cdot d\Pi^* X_0 + \sum_{\lambda\in\Lambda}(a_\lambda - sc_\lambda)E_\lambda\right)\\ &= d(\bar{\mathcal{L}}^{\cdot n}\cdot \hat{X}_0) + \left(\bar{\mathcal{L}}^{\cdot n}\cdot\sum_{\lambda\in\Lambda}(db_\lambda + a_\lambda - sc_\lambda)E_\lambda\right)\\ &\geqslant \left(\bar{\mathcal{L}}^{\cdot n}\cdot\sum_{\lambda\in\Lambda}(db_\lambda + a_\lambda - sc_\lambda)E_\lambda\right).\end{aligned}$$

For any $\lambda\in\Lambda$,

$$\begin{aligned}db_\lambda + a_\lambda - sc_\lambda &\geqslant \frac{1}{\gamma}(\gamma sc_\lambda - (a_\lambda - b_\lambda + 1)) + a_\lambda - sc_\lambda\\ &= \frac{\gamma-1}{\gamma}(a_\lambda - b_\lambda + 1) + b_\lambda - 1 \geqslant \frac{\gamma-1}{\gamma}(a_\lambda - b_\lambda + 1)\end{aligned}$$

holds. Hence

$$\mathrm{DF}_0 \geqslant \frac{\gamma-1}{\gamma}\left(\bar{\mathcal{L}}^{\cdot n}\cdot\sum_{\lambda\in\Lambda}(a_\lambda - b_\lambda + 1)E_\lambda\right).$$

By Theorem 2.2(3), $(\bar{\mathcal{L}}^{\cdot n}\cdot\sum_{\lambda\in\Lambda}(a_\lambda - b_\lambda + 1)E_\lambda) > 0$ holds. Therefore, $\mathrm{DF}_0 \geqslant 0$ holds. Moreover, if $\gamma > 1$, then $\mathrm{DF}_0 > 0$ holds.

As a consequence, we have proved Theorem 1.4.

*Remark* 5.2. Berman pointed out to the author that there is an analogy between the argument after Claim 5.1 and the argument in [Ber12, Lemma 3.4]. In fact, the argument in [Ber12, Lemma 3.4] gives the inequality

$$\frac{\mathrm{DF}_0}{((-K_X)^{\cdot n})} \geqslant \frac{-(\bar{\mathcal{L}}^{\cdot n+1})}{(n+1)((-K_X)^{\cdot n})} - d_0,$$

where

$$d_0 := \max\left\{0, \max_{\lambda\in\Lambda}\left\{\frac{sc_\lambda - a_\lambda}{b_\lambda}\right\}\right\}.$$

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