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Hecke characters associated to Drinfeld modular forms

Gebhard Böckle and Tommaso Centeleghe

Compositio Math. **151** (2015), 2006–2058.

[doi:10.1112/S0010437X15007290](https://doi.org/10.1112/S0010437X15007290)



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Gebhard Böckle

With Appendix B written jointly with Tommaso Centeleghe

ABSTRACT

In this article we explain how the results in our previous article on ‘algebraic Hecke characters and compatible systems of mod p Galois representations over global fields’ allow one to attach a Hecke character to every cuspidal Drinfeld modular eigenform from its associated crystal that was constructed in earlier work of the author. On the technical side, we prove along the way a number of results on endomorphism rings of τ -sheaves and crystals. These are needed to exhibit the close relation between Hecke operators as endomorphisms of crystals on the one side and Frobenius automorphisms acting on étale sheaves associated to crystals on the other. We also present some partial results on the ramification of Hecke characters associated to Drinfeld modular eigenforms. An important phenomenon absent from the case of classical modular forms is that ramification can also result from places of modular curves of good but non-ordinary reduction. In an appendix, jointly with Centeleghe we prove some basic results on p -adic Galois representations attached to GL_2 -type cuspidal automorphic forms over global fields of characteristic p .

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Received 10 March 2013, accepted in final form 2 December 2014, published online 1 June 2015.

2010 Mathematics Subject Classification 11F52, 11F80 (primary), 11R37, 11R39, 11R58, 11F33, 11G09 (secondary).

Keywords: Drinfeld modular forms, Galois representations, Hecke characters, ramification, τ -sheaves, modular Jacobians, p -divisible groups.

The first named author was supported by the Deutsche Forschungsgemeinschaft (SFB/TR 45, SPP 1489, FG 1920). In addition, both named authors are supported by DFG SPP 1489 in a joint project with the FNR Luxembourg.

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1. Introduction

Let p be a prime number, q a power of p and denote by \mathbb{F}_q the field of q elements. For a pair (F, ∞) , where F is a global function field with constant field \mathbb{F}_q and ∞ is a fixed place of F , we define F_∞ as the completion of F at ∞ and \mathbb{C}_∞ as the topological closure of an algebraic closure of F_∞ . The Drinfeld symmetric space $\Omega = \mathbb{P}^1(\mathbb{C}_\infty) \setminus \mathbb{P}^1(F_\infty)$ serves as an analog of the complex upper half plane. It carries a structure of a rigid analytic space over F_∞ , and $\mathrm{GL}_2(F_\infty)$ acts via rigid analytic automorphisms by the standard formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az + b)/(cz + d)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\infty)$ and $z \in \Omega$, cf. [Dri76] or [Gek86].

Let A be the set of elements of F regular away from ∞ . Then for any congruence subgroup $\Gamma \subset \mathrm{GL}_2(A)$, the quotient $\Gamma \backslash \Omega$ can be identified with $\mathfrak{X}_\Gamma(\mathbb{C}_\infty) \setminus \mathfrak{C}_\Gamma$, for a smooth projective curve \mathfrak{X}_Γ defined over a finite extension F_Γ of F and a finite subset \mathfrak{C}_Γ of $\mathfrak{X}_\Gamma(\mathbb{C}_\infty)$ called the set of *cusps*, cf. [GvdP80, IV §1]. For any $k, \ell \in \mathbb{Z}$, one defines the space $M_{k,\ell}(\Gamma, \mathbb{C}_\infty)$ of Drinfeld modular forms of weight k , type ℓ and level Γ , cf. [Gos80a] or [Gek88, (5.7)], as the \mathbb{C}_∞ vector space of rigid analytic functions $f : \Omega \rightarrow \mathbb{C}_\infty$ that satisfy

$$f(\gamma z) = (cz + d)^k \det(\gamma)^{-\ell} f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, z \in \Omega$$

and have a Taylor series expansion around all cusps, cf. [Gek88, (5.7)].

Unless A has class number one, it is preferable to work in an adelic setting: by \mathbb{A}_F^∞ we denote the adèles of F away from ∞ , by \widehat{A} the profinite completion of A . We fix a compact open subgroup \mathcal{K} of $\mathrm{GL}_2(\widehat{A})$. Then $\mathrm{GL}_2(F) \backslash (\Omega \times (\mathrm{GL}_2(\mathbb{A}_F^\infty) / \mathcal{K}))$ can be identified with $\mathfrak{X}_\mathcal{K}(\mathbb{C}_\infty) \setminus \mathfrak{C}_\mathcal{K}$ where $\mathfrak{X}_\mathcal{K}$ is a finite disjoint union of smooth projective curves defined over a finite extension $F_\mathcal{K}$ of F and $\mathfrak{C}_\mathcal{K}$ is a finite subset of $\mathfrak{X}_\mathcal{K}(\overline{F})$. Moreover, by strong approximation, one can find finitely many subgroups Γ_i of $\mathrm{GL}_2(F)$ commensurable with and containing some fixed finite congruence subgroup Γ of $\mathrm{GL}_2(A)$ such that one has a disjoint decomposition $\mathfrak{X}_\mathcal{K}(\mathbb{C}_\infty) \setminus \mathfrak{C}_\mathcal{K} = \bigsqcup_i \Gamma_i \backslash \Omega$, see [Böc04, ch. 4].

One now generalizes in an obvious way the definition of $M_{k,\ell}(\Gamma, \mathbb{C}_\infty)$ to obtain a space $M_{k,\ell}(\mathcal{K}, \mathbb{C}_\infty)$ of Drinfeld modular forms of weight k , type ℓ and level \mathcal{K} , cf. [Böc04, ch. 5]. Extending the definitions of Hecke operators from the case where A has class number one, cf. [Gek88, §7], one defines a set of commuting Hecke operators $T_{\mathfrak{n}}$ on $M_{k,\ell}(\mathcal{K}, \mathbb{C}_\infty)$ for every ideal \mathfrak{n} of A that is prime to the conductor of \mathcal{K} , see [Böc04, ch. 6]. The non-adelic setting would only allow the definition of Hecke operators for principal ideals \mathfrak{n} .

By the commutativity of the Hecke operators, the space $M_{k,\ell}(\mathcal{K}, \mathbb{C}_\infty)$ can be decomposed simultaneously into a direct sum of generalized eigenspaces for all $T_{\mathfrak{n}}$. In particular, for every set of compatible eigenvalues $(a_{\mathfrak{n}})_{\mathfrak{n}}$ there exists a common eigenform $f \in M_{k,\ell}(\mathcal{K}, \mathbb{C}_\infty)$ so that $T_{\mathfrak{n}} f = a_{\mathfrak{n}} f$. Since all forms can be realized as global sections of suitable line bundles on $\mathfrak{X}_\mathcal{K}$,

defined over $F_{\mathcal{K}}$ (after possibly shrinking \mathcal{K}), see [Gos80b, Theorem 1.79], the field $F(a_n \mid \mathfrak{n} \leq A)$ is a finite extension of F .¹

A main difference to the classical case of elliptic modular forms is that $T_{\mathfrak{q}^m} = (T_{\mathfrak{q}})^m$ as endomorphisms of $M_{k,\ell}(\mathcal{K}, \mathbb{C}_{\infty})$ for any prime $0 \neq \mathfrak{q}$ of A , e.g. [Gek86, VIII §1]. This property led Serre, cf. [Gos80c, p. 414], to ask whether the Hecke eigensystem of any eigenform is described by a Hecke character. The question has since remained open. In the following we shall focus on the Hecke stable subspace of cusp forms $S_{k,\ell}(\mathcal{K}, \mathbb{C}_{\infty})$, i.e., forms in $M_{k,\ell}(\mathcal{K}, \mathbb{C}_{\infty})$ whose Taylor expansion have vanishing constant term at every cusp, because Serre's question is, in principle, well understood for the complementary space of Eisenstein series.

In [Böc04], based on the function field methods developed in [BP09] and inspired by the situation for classical (elliptic) modular forms, we attach a compatible system of \mathfrak{p} -adic Galois representation to every cuspidal Drinfeld modular Hecke eigenform. For classical elliptic modular forms one uses étale, Betti and de Rham cohomology and various comparison theorems to obtain such a result, cf. [Con]. The importance of the results of [BP09], the construction of a theory of function field crystals, lies in the fact that these crystals can play the role of a motive for cohomology that has étale, de Rham and Betti realizations at once. This makes it possible in [Böc04] to realize the space $S_{k,\ell}(\mathcal{K}, \mathbb{C}_{\infty})$ as a kind of de Rham (or analytic) realization of a certain crystal whose étale realizations form a compatible family of \mathfrak{p} -adic Galois representations. Moreover, this crystal carries a naturally defined Hecke action, compatible with the one from [Böc04, ch. 6].

As in the classical case, the transition from Hecke eigenforms to Galois representations is characterized by an Eichler–Shimura relation; see formula (7). *Unlike* in the classical situation the relation reads $T_x = \text{Frob}_x$; this is a consequence of Verschiebung acting by zero on characteristic p coefficients.² Thus the Galois representations attached to cuspidal Hecke eigenforms have abelian image. As pointed out by Goss, this is a further indication of the existence of a Hecke character for every cuspidal Hecke eigenform that could at once answer Serre's question and give rise to the compatible system of Galois representations. The main result of this article is the construction of this Hecke character. Furthermore, we provide results on its conductor and thus on the ramification of the attached Galois representations.

As we will see in Proposition 7.3, the Hecke characters defined by Gross in [Gro82] and Goss in [Gos92] are too restrictive. An adequate class of Hecke characters was defined in [Böc13]. There we also prove a correspondence between such Hecke characters and suitable compatible systems of abelian Galois representations. The latter correspondence is based on previous results in the number field case due to Khare, e.g. [Kha05]. Let us recall the main definition from [Böc13]: Denote by \mathbb{A}_F^* the group of ideles of F with their usual topology. An *algebraic Hecke character* of F is a continuous homomorphism

$$\chi : \mathbb{A}_F^* \longrightarrow (F^{\text{alg}})^*$$

for the discrete topology on the algebraic closure F^{alg} of F , such that there exists a finite set of field homomorphisms $\Sigma \subset \text{Hom}(F, F^{\text{alg}})$ and a corresponding tuple of integers $(n_{\sigma})_{\sigma \in \Sigma} \in \mathbb{Z}^{\Sigma}$ such that for all $\alpha \in F^*$ one has $\chi(\alpha) = \prod_{\sigma \in \Sigma} \sigma(\alpha)^{n_{\sigma}}$. For instance, if $F = \mathbb{F}_q(t)$, then the elements of

¹The Hecke stability of these spaces of sections is not explained in [Gos80b]; see [Böc04, §11.1] for a similar argument.

²From here on all Hecke operators T_x are indexed over the places $x \neq \infty$ of our base field, since such operators generate the relevant Hecke algebras. We reserve \mathfrak{n} to denote conductors and \mathfrak{p} the prime of the coefficients.

Σ are defined by mapping t to an arbitrary element of $F^{\text{alg}} \setminus \mathbb{F}_q^{\text{alg}}$. Unlike in [Gos92] or [Gro82], we do not require that there exists an infinite subfield of F that is fixed by the elements in Σ .

For a place x of F , let ϖ_x denote a uniformizer of the completion of F at x , and denote by $(1, \dots, 1, \varpi_x, 1, \dots, 1)$ the idele in \mathbb{A}_F^* with ϖ_x at x and 1 elsewhere. We normalize class field theory, so that a geometric Frobenius Frob_x at x is mapped to the uniformizer ϖ_x at x . In §6, we prove the following theorem.

THEOREM 1.1. *Let f be a cuspidal Drinfeld modular Hecke eigenform for (F, ∞) for some level, weight and type. Then there is a unique Hecke character χ_f associated to f such that, for almost all primes x of $\text{Max } A$, the Hecke operator T_x at x satisfies*

$$T_x f = \chi_f((1, \dots, 1, \varpi_x, 1, \dots, 1))f.$$

The Hecke character χ_f allows one to recover the compatible system of Galois representations attached to f by following [Böc13, §2.5], based on [Ser68, §2.5]. From this perspective, Theorem 1.1 is rather imprecise if compared with the corresponding results known for the compatible system attached to a classical modular forms, which is: for a classical Hecke eigenform f of level N , the associated ℓ -adic Galois representation is unramified outside the primes q dividing ℓN , and for all such q the Hecke polynomial is equal to the characteristic polynomial of a Frobenius automorphism at q . In Example 8.26 we show that the analogous statement is not true in general for Galois representations attached to Drinfeld modular forms f . Their ramification is controlled by the infinity type and the conductor of χ_f . We have no results on the infinity type of χ_f . However we do have partial results on the conductor of χ_f for forms f of weight 2 and level \mathfrak{n} , which can be ramified outside the level \mathfrak{n} , as well as partial results on the ramification of \mathfrak{p} -adic Galois representations for forms f of arbitrary weight, via congruences to weight 2; see Theorem 8.8 and Propositions 8.7, 8.18 and 8.22.

The behavior in the weight 2 case is for the experts not unexpected: the Galois action on the p -adic Tate module of a non-isotrivial elliptic curve over a global function field of characteristic p is not only ramified at places of bad reduction, but also at places of good but supersingular reduction. Similarly, for cuspidal Drinfeld modular Hecke eigenforms f of weight 2, the associated Hecke character χ_f can be ramified at places where the p -rank of the underlying modular curve decreases under reduction.

Let us now summarize the role of this article, given [Böc04] and [Böc13], and in doing so, also give an overview of its individual sections: the main result of [Böc04] is the construction of a crystal, for any given sufficiently small level and any weight, that serves as a motive with a Hecke action and whose analytic realization is (essentially) $S_{k,\ell}(\mathcal{K}, \mathbb{C}_\infty)$, similar to Scholl’s motive [Sch90] for classical modular forms. This crystal is shown to be flat and uniformizable in the sense of Definition 4.11. Moreover, for the crystal as a whole, an Eichler–Shimura relation is proved in [Böc04, §13.4] which leads to an Eichler–Shimura relation of its étale realizations. It follows that the étale realizations form a strictly compatible system of \mathfrak{p} -adic Galois representations.

To describe crystals with extra endomorphisms in an axiomatic way, we shall, in Definition 5.1 of the present article, introduce the notion of a Hecke crystal. We expect that Hecke crystals will arise from Drinfeld modular forms of any rank, and not only in the rank two case considered here and treated in [Böc04].³ Given a Hecke crystal we shall explain how to decompose it under its

³ In the higher rank cases, a satisfactory compactification of the underlying moduli spaces has just recently been obtained by Pink, cf. [Pin13].

Hecke action into generalized eigenspaces in a way similar to the decomposition of $S_{k,\ell}(\mathcal{K}, \mathbb{C}_\infty)$. This will be carried out in §§ 2–5. We find this decomposition an important point, which in [Böc04] was not treated adequately. It gives a bijective correspondence between Hecke eigenforms f and simple subquotients of the crystal, and it attaches to every simple subquotient, and hence to f , a strictly compatible system $(\rho_{f,\mathfrak{p}})_{\mathfrak{p}}$ of one-dimensional \mathfrak{p} -adic Galois representations.

Section 2 recalls parts of the theory of [BP09] on τ -sheaves and function field crystals. We take a direct, but slightly different, approach from [BP09] to the localization procedure that leads from τ -sheaves to crystals; see the bottom of page 2011. Section 3 describes Galois representations arising from crystals. It introduces the notion of a Galois abelian crystal and recalls the main result from [Böc13]. Section 4 clarifies results on endomorphisms of crystals and their étale and Betti realizations. Such endomorphisms will arise from the Hecke action on Hecke crystals. Section 5 attaches a set of Hecke characters to any Hecke crystal. Finally in § 6 we recall the Hecke crystals constructed in [Böc04] and we prove Theorem 1.1, restated as Corollary 6.7. The proof is at this point rather straightforward: the results from [Böc13] applied to the strictly compatible system $(\rho_{f,\mathfrak{p}})_{\mathfrak{p}}$ yield the Hecke character χ_f . In § 7 we illustrate Theorem 1.1 by reworking some examples from [Böc04], or [Böc14, ch. 12, § 7].

Finally, § 8 presents results on the ramification of the Hecke character χ_f , or a Galois representation $\rho_{f,\mathfrak{p}}$, attached to a cuspidal Drinfeld Hecke eigenform f of some level \mathfrak{n} at a place x not dividing \mathfrak{n} . We show that if $\rho_{f,\mathfrak{p}}$ ramifies at such an x , then the p -rank of the corresponding modular curve has to decrease under reduction to x , and the mod \mathfrak{p} Hecke eigenvalue for T_x has to vanish. The case of weight 2 is directly accessible via the p -torsion of the Jacobian of the modular curve. For the study of higher weight cases, we introduce congruence methods, standard in the number theoretic setting, to perform a reduction of any weight to weight 2. We illustrate our results by Example 8.26. In an appendix, we explain the precise relation between the p -rank of a curve and the ramification of the p -adic Tate module of its Jacobian via results from [deJ98].

Key notation used throughout the article. We end this introduction with a small primer of the main notation used throughout the article. We note that some of the notation becomes more restrictive as the article proceeds; we hope that this list will avoid possible confusion.

- X, Y denote noetherian schemes over the field \mathbb{F}_q . From Convention 2.11 onward until the end of § 2 and from Convention 4.3 onward, X always denotes a smooth geometrically irreducible curve over \mathbb{F}_q . For the (absolute) q -power Frobenius on X we write σ_X , or usually simply σ .
- For any field κ , we write G_κ for $\text{Gal}(\kappa^{\text{sep}}/\kappa)$ of κ , where κ^{sep} is a separable closure of κ .
- If X is of finite type over \mathbb{F}_q and $x \in X$ is a closed point, then by k_x we denote the residue field at x , by d_x its degree over \mathbb{F}_q and by q_x its order. Moreover, we write $\text{Frob}_x \in G_{k_x}$ for the geometric Frobenius automorphism at x , i.e., the inverse of the automorphism $\alpha \mapsto \alpha^{q^{d_x}}$ of k_x^{sep} .
- $\eta = \text{Spec } K$ denotes the generic point of X , if X is integral.
- A is an \mathbb{F}_q -algebra of essentially finite type, i.e., the localization of a finitely generated \mathbb{F}_q -algebra, and $C = \text{Spec } A$. The ring A plays the role of a *coefficient ring*. By $\text{Max } A \subset \text{Spec } A$ we denote the subset of maximal ideals. In the introduction and from Convention 2.6 onward, A is a Dedekind domain with constant field \mathbb{F}_q and fraction field F . In that case, by $A_{\mathfrak{p}}$ and $F_{\mathfrak{p}}$ we denote the completions at $\mathfrak{p} \in \text{Max}(A)$.
- In §§ 4 and 5, L denotes a complete valued field containing K .

- In §§ 6 and 8 we assume X to be projective, we mark a closed point ∞ on X , and we take $C = X \setminus \{\infty\} = \text{Spec } A$. We use F to denote $\text{Frac}(A) = \mathbb{F}_q(X)$ and omit K from the notation.

2. On τ -sheaves, crystals and good representatives

In this section, we first recall the notions of τ -sheaf and crystal from [BP09]. Our main interest will lie in flat crystals, flatness being a property which is shared by the crystals we will ultimately consider. From results in [BP09] we know that over the generic points of the base, flat crystals possess a locally free representative. If the base is a curve, work of Gardeyn, see [Gar01], allows one to extend the locally free representative from the generic fiber to an open subscheme of the curve.

Throughout this section, \mathbb{F}_q denotes the finite field of q elements for q a power of a prime p . By X, Y we denote noetherian schemes over \mathbb{F}_q . For the q -power Frobenius on X we write σ (or σ_X). We write A for an \mathbb{F}_q -algebra essentially of finite type and C for $\text{Spec } A$.

We recall some notions from [BP09] which are modeled on the notions F -sheaf or shtuka of Drinfeld, cf. [Dri87] and t -motive of Anderson, cf. [And86].

DEFINITION 2.1. A τ -sheaf $\mathcal{F} = (\mathcal{F}, \tau)$ on X over A (or simply a τ -sheaf on X) is a coherent sheaf of $\mathcal{O}_{X \times C}$ -modules endowed with an $\mathcal{O}_{X \times C}$ -linear homomorphism

$$\tau : (\sigma \times \text{id})^* \mathcal{F} \rightarrow \mathcal{F}.$$

A morphism of τ -sheaves is an $\mathcal{O}_{X \times C}$ -linear morphism which respects the action of τ .

A τ -sheaf \mathcal{F} is called *locally free of rank* $r \in \mathbb{N}$ if the underlying sheaf \mathcal{F} is a locally free $\mathcal{O}_{X \times C}$ -module of finite rank r .

The category $\mathbf{Coh}_\tau(X, A)$ of all τ -sheaves on X over A with the above notion of morphism is an A -linear abelian category. Kernels, cokernels, etc. of a morphism φ are computed as the corresponding object in the category of coherent $\mathcal{O}_{X \times C}$ -modules with the induced τ -action. The morphism sets in $\mathbf{Coh}_\tau(X, A)$ are denoted $\text{Hom}_\tau(_, _)$.

The simplest non-zero τ -sheaf on X over A is the unit τ -sheaf $\mathbb{1}_{X,A}$ whose underlying sheaf is $\mathcal{O}_{X \times C}$ and where τ is the adjoint map to $\sigma \times \text{id} : \mathcal{O}_{X \times C} \rightarrow (\sigma \times \text{id})_* \mathcal{O}_{X \times C}$. It is the unit object for the natural operation of tensor product on τ -sheaves.

For any \mathbb{F}_q -morphism $f : Y \rightarrow X$ there is a contravariant functor $f^* : \mathbf{Coh}_\tau(X, A) \rightarrow \mathbf{Coh}_\tau(Y, A)$, *base change*, which to any τ -sheaves \mathcal{F} on X attaches the τ -sheaf on Y with underlying sheaf $(f \times \text{id}_C)^* \mathcal{F}$ and induced τ -action. In particular, we obtain such a functor for the locally closed immersion $i_x : x = \text{Spec}(k_x) \rightarrow X$ of any point x of X . In the same way, a covariant functor *direct image* $f_* : \mathbf{Coh}_\tau(Y, A) \rightarrow \mathbf{Coh}_\tau(X, A)$ is defined which to \mathcal{G} on Y attaches $f_* \mathcal{G}$ with underlying sheaf $(f \times \text{id}_C)_* \mathcal{G}$ and induced τ -action. Another functor that will be of importance to us is *change of coefficients*: for any A -algebra A' we have a functor $_ \otimes_A A' : \mathbf{Coh}_\tau(X, A) \rightarrow \mathbf{Coh}_\tau(X, A')$.

For $n \in \mathbb{N}$ one defines the endomorphism $\tau^n : (\sigma^n \times \text{id})^* \mathcal{F} \rightarrow \mathcal{F}$ inductively via $\tau^1 = \tau$ and $\tau^{n+1} = \tau^n \circ (\sigma^n \times \text{id})^* \tau$. The endomorphism τ^n is also a homomorphism of τ -sheaves $\tau^n : (\sigma^n \times \text{id})^* \mathcal{F} \rightarrow \mathcal{F}$ where $(\sigma^n \times \text{id})^* \mathcal{F} = ((\sigma^n \times \text{id})^* \mathcal{F}, (\sigma^n \times \text{id})^* \tau)$.

The main concept introduced in [BP09] is that of a crystal. The category $\mathbf{Crys}(X, A)$ of A -crystals on X is obtained from the category of τ -sheaves by a localization procedure. In the following we give a definition of this localization which is slightly different from but equivalent to the one in [BP09, Proposition 3.39]: the new category has the same *objects* as the category of

τ -sheaves, but more morphisms (and also more isomorphisms). For τ -sheaves $\underline{\mathcal{F}}$ and $\underline{\mathcal{G}}$ on X we define the set of *morphisms* from $\underline{\mathcal{F}}$ to $\underline{\mathcal{G}}$ in **Crys** as

$$\text{Hom}_{\mathbf{Crys}}(\underline{\mathcal{F}}, \underline{\mathcal{G}}) = \left(\bigcup_{n \in \mathbb{N}} \text{Hom}_{\tau}((\sigma^n \times \text{id})^* \underline{\mathcal{F}}, \underline{\mathcal{G}}) \right) / \sim,$$

where the equivalence relation \sim is defined as follows: morphisms $\varphi : (\sigma^n \times \text{id})^* \underline{\mathcal{F}} \rightarrow \underline{\mathcal{G}}$ and $\psi : (\sigma^m \times \text{id})^* \underline{\mathcal{F}} \rightarrow \underline{\mathcal{G}}$ are equivalent if there exists $\ell \geq \max\{m, n\}$ such that

$$\varphi \circ (\sigma^n \times \text{id})^*(\tau^{\ell-n}) = \psi \circ (\sigma^m \times \text{id})^*(\tau^{\ell-m}).$$

An alternative way to think of morphisms in **Crys** is to display them as diagrams:

$$\underline{\mathcal{F}} \xleftarrow{\tau^n} (\sigma^n \times \text{id})^* \underline{\mathcal{F}} \xrightarrow{\varphi} \underline{\mathcal{G}}.$$

Composition of morphisms in **Crys** is defined in the obvious way, i.e., the composite of $\varphi : (\sigma^n \times \text{id})^* \underline{\mathcal{F}} \rightarrow \underline{\mathcal{G}}$ and $\psi : (\sigma^m \times \text{id})^* \underline{\mathcal{G}} \rightarrow \underline{\mathcal{H}}$ is defined as

$$\psi \circ (\sigma^m \times \text{id})^* \varphi : (\sigma^{m+n} \times \text{id})^* \underline{\mathcal{F}} \rightarrow \underline{\mathcal{H}}.$$

One can verify that **Crys**(X, A) forms an A -linear abelian category. The functors f^* , $-\otimes_A A'$ and $R^i f_*$, for f proper, pass to well-defined functors on **Crys**(\dots).

It is perhaps instructive to explain the meaning of this localization procedure by giving the proof of the following result (from [BP09]).

PROPOSITION 2.2. *A τ -sheaf $\underline{\mathcal{F}}$ is the zero object in **Crys**(X, A) if and only if $\tau_{\underline{\mathcal{F}}}$ is nilpotent, i.e., there exists $n \in \mathbb{N}$ such that $\tau_{\underline{\mathcal{F}}}^n = 0$.*

Proof. Suppose first that $\tau_{\underline{\mathcal{F}}}^n = 0$. We need to show that in **Crys**(X, A) the morphisms $0 \rightarrow \underline{\mathcal{F}}$ and $\underline{\mathcal{F}} \rightarrow 0$ are isomorphisms. By considering both of their composites, we are reduced to proving that the zero map $0 : \underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}$ is equivalent to the identity $\text{id}_{\underline{\mathcal{F}}}$. But this follows from $\tau^n \circ 0 = 0 = \tau^n \circ \text{id}_{\underline{\mathcal{F}}}$. Suppose conversely that $\underline{\mathcal{F}}$ is isomorphic to zero in **Crys**(X, A). Then the zero map and the identity on $\underline{\mathcal{F}}$ must be equivalent. The latter means that there exists $n \in \mathbb{N}$ with $\tau_{\underline{\mathcal{F}}}^n = \tau_{\underline{\mathcal{F}}}^n \circ \text{id}_{\underline{\mathcal{F}}} = \tau_{\underline{\mathcal{F}}}^n \circ 0 = 0$. □

The following results we quote from [BP09, ch. 3 and § 4.1].

PROPOSITION 2.3. (a) *The category **Crys**(\dots) is obtained from **Coh** $_{\tau}$ (\dots) by localization at nil-isomorphisms, i.e., morphism in **Coh** $_{\tau}$ (\dots) whose kernel and cokernel are nilpotent.*

(b) *Two τ -sheaves $\underline{\mathcal{F}}, \underline{\mathcal{G}}$ are isomorphic in **Crys**(X, A) if and only if there exists a τ -sheaf $\underline{\mathcal{H}}$ and nil-isomorphisms $\underline{\mathcal{F}} \xleftarrow{\varphi} \underline{\mathcal{H}} \xrightarrow{\psi} \underline{\mathcal{G}}$.*

(c) *For any τ -sheaf $\underline{\mathcal{F}}$, the morphism $\tau^n : (\sigma^n \times \text{id})^* \underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}$ is a nil-isomorphism, and (so) $\underline{\mathcal{F}}$ and $\text{Im}(\tau_{\underline{\mathcal{F}}}^n)$ represent the same object in **Crys**(\dots).*

(d) *Any crystal has a representative on which τ is injective, namely $\text{Im}(\tau_{\underline{\mathcal{F}}}^n)$ for $n \gg 0$.*

(e) *For any morphism $f : Y \rightarrow X$, the functor $f^* : \mathbf{Crys}(X, A) \rightarrow \mathbf{Crys}(Y, A)$ is exact.*

(f) *For any open immersion $j : U \hookrightarrow X$, there is a functor $j_! : \mathbf{Crys}(U, A) \rightarrow \mathbf{Crys}(X, A)$, extension by zero, characterized by $j^* j_! = \text{id}$ and $i^* j_! = 0$ for $i : Z \hookrightarrow X$ a closed complement to j .*

We remark that in [BP09, § 3.4] the characterization in (a) was used to define **Crys**(X, A).

DEFINITION 2.4 [BP09, Definition 4.2.6]. The category $\mathbf{Crys}^{\text{flat}}(X, A)$ of flat A -crystals is the full additive subcategory of $\mathbf{Crys}(X, A)$ consisting of those crystals for which the crystal $\text{Tor}_i^A(_, A')$ vanishes for all $i \geq 1$ and all coefficient changes $A \rightarrow A'$.

Flat crystals are preserved under pullback and change of coefficients. After restriction to a suitable finite locally closed stratification of the base X , they are, in some sense, representable by τ -sheaves whose underlying sheaf is locally free. In particular cases, the latter will be important in this article. We refer to [BP09, ch. 7] for further details.

Example 2.5. Let $i_x : x \hookrightarrow X$ be the immersion of a closed point and let $\underline{\mathcal{F}}$ be a flat crystal on x . Then $i_{x*}\underline{\mathcal{F}}$ is a flat crystal on X . This shows the need for a stratification as mentioned above.

Convention 2.6. From now on, for the remainder of this article, we fix an absolutely irreducible affine smooth curve C with constant field \mathbb{F}_q . Its coordinate ring $A = \Gamma(C, \mathcal{O}_C)$ is a Dedekind domain. By $F = \text{Frac}(A)$ we denote its fraction field.

Coefficient rings with a name different from A need not be of the above type.

PROPOSITION 2.7 [BP09, Proposition 7.5.8]. Let $X = \text{Spec } K$ for a field K . Let $\underline{\mathcal{F}}$ be a flat A -crystal on X and denote by $\underline{\mathcal{G}}$ a τ -sheaf that represents $\underline{\mathcal{F}}$. Then, if $\tau_{\mathcal{G}}$ is injective, the sheaf \mathcal{G} is locally free. In particular, for any choice of $\underline{\mathcal{G}}$ the image of $\tau_{\mathcal{G}}^n : (\sigma^n \times \text{id})^*\underline{\mathcal{G}} \rightarrow \underline{\mathcal{G}}$ is locally free for n sufficiently large.

We introduce the following notions which go back to [Gar01, § 1.I].

DEFINITION 2.8. A τ -sheaf $\underline{\mathcal{F}}$ on X is said to be *good* if it is locally free and if the base change τ_x of τ to any $x \in X$ is injective.

A τ -sheaf $\underline{\mathcal{F}}$ on X is said to be *generically good* if its base change to any generic point is good.

Proposition 2.7 can be restated as follows.

COROLLARY 2.9. Any flat A -crystal on $\text{Spec } K$ has a generically good representative.

It is not difficult to deduce from Propositions 2.3(b) and 2.7 that the generic ranks of a generically good representative are independent of any choices.

DEFINITION 2.10. Suppose X is integral. The rank of $\underline{\mathcal{F}} \in \mathbf{Crys}^{\text{flat}}(X, A)$ is the rank over $\eta \times C$ of any representative of $\underline{\mathcal{F}}_{\eta}$ as in Proposition 2.7.

To obtain stronger representability results for flat crystals, we need to recall further concepts from Gardeyn [Gar01], and we need to restrict X to the curve case.

Convention 2.11. For the remainder of this section, we let X denote a smooth geometrically irreducible curve with constant field \mathbb{F}_q . Its closed points will be denoted by x , its generic point by η and the corresponding immersions by $i_x : x \hookrightarrow X$ and $i_{\eta} : \eta \hookrightarrow X$, respectively.

DEFINITION 2.12. Let $\underline{\mathcal{F}}$ be a good τ -sheaf on η .

A *model* of $\underline{\mathcal{F}}$ is a locally free generically good τ -sheaf on X whose generic fiber agrees with $\underline{\mathcal{F}}_{\eta}$.

The model is a *good model* if it is good as a τ -sheaf.

A model $\underline{\mathcal{H}}$ of $\underline{\mathcal{F}}$ is said to be *maximal* if $\text{Hom}_{\tau}(_, \underline{\mathcal{H}})$ represents the functor $\underline{\mathcal{G}} \mapsto \text{Hom}_{\tau}(\underline{\mathcal{G}}_{\eta}, \underline{\mathcal{F}})$.

Due to its universal property, one should think of a maximal model as a τ -sheaf analog of the Néron model in the context of abelian varieties.

Remark 2.13. Our definition of maximal model is slightly different from the one given by Gardeyn. In [Böc04, ch. 8] it is shown that the two definitions agree, and, moreover, that the maximal model, if it exists, is also characterized as being the largest coherent τ -subsheaf of $i_{\eta*}\underline{\mathcal{F}}$.

We quote the following result from [Gar01, Proposition 1.13].

THEOREM 2.14 (Gardeyn).

- (a) Every good τ -sheaf on η admits a maximal model. It is unique, up to unique isomorphism.
- (b) Any good model is maximal.

For any generically good τ -sheaf $\underline{\mathcal{F}}$ on X we write $\underline{\mathcal{F}}^{\max}$ for the maximal model of $\underline{\mathcal{F}}$. By the universal property of $\underline{\mathcal{F}}^{\max}$ we have the following corollary.

COROLLARY 2.15. *There is a canonical homomorphism $\underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}^{\max}$.*

Example 2.16. Even if $\underline{\mathcal{F}}$ is generically good and its underlying sheaf is locally free, so that $\varphi : \underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}^{\max}$ is injective, the monomorphism φ need not be bijective. Let $X = \text{Spec } R$ for R a discrete valuation ring with uniformizer π . If $\underline{\mathcal{F}}$ is locally free and generically good on $\text{Spec } R$, then so is $\pi^n \underline{\mathcal{F}}$ for any $n \in \mathbb{N}_0$. Now if $\underline{\mathcal{F}}$ is good, then $(\pi^n \underline{\mathcal{F}})^{\max} = \underline{\mathcal{F}}$, and hence in this case φ is not bijective.

In light of the above and Example 2.5, the following representability result is optimal.

THEOREM 2.17. *Let $\underline{\mathcal{F}}$ be in $\mathbf{Crys}^{\text{flat}}(X, A)$, and also denote by $\underline{\mathcal{F}}$ a τ -sheaf representative which is generically good and on which $\tau_{\mathcal{F}}$ is injective, cf. Proposition 2.3(d) and Corollary 2.9. Let $\varphi : \underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}^{\max}$ be the canonical homomorphism. Then there exists $S \subset X$ of dimension zero such that the restriction $\varphi|_{X \setminus S}$ is an isomorphism. In particular, $\underline{\mathcal{F}}$ has a locally free representative on a dense open subscheme of X . Moreover, $\text{Ker}(\varphi)$ lies in $\mathbf{Crys}^{\text{flat}}(S, A)$.*

Proof. We consider the exact sequence

$$0 \longrightarrow \underline{\mathcal{K}} \longrightarrow \underline{\mathcal{F}} \xrightarrow{\varphi} \underline{\mathcal{F}}^{\max} \longrightarrow \underline{\mathcal{C}} \longrightarrow 0$$

in $\mathbf{Coh}_{\tau}(X, A)$ with $\underline{\mathcal{K}} = \text{Ker}(\varphi)$ and $\underline{\mathcal{C}} = \text{Coker}(\varphi)$. By the construction of φ and the generic goodness of the chosen representative, φ_{η} is an isomorphism. We deduce that the sheaves $\underline{\mathcal{K}}$ and $\underline{\mathcal{C}}$ are supported on $S \times C$ for some zero-dimensional $S \subset X$. This proves all but the last assertion.

For the last assertion, we may pass to the stalk at any closed point x of S . By Proposition 2.3(e), this is an exact operation on crystals, so that

$$0 \longrightarrow \underline{\mathcal{K}}_x \longrightarrow \underline{\mathcal{F}}_x \xrightarrow{\varphi_x} \underline{\mathcal{F}}_x^{\max} \longrightarrow \underline{\mathcal{C}}_x \longrightarrow 0$$

is exact in $\mathbf{Crys}(x, A)$. Now, $\underline{\mathcal{F}}_x^{\max}$ has a locally free representative on $x \times C$, which is a Dedekind scheme. Therefore the τ -sheaf kernel of $\underline{\mathcal{F}}_x^{\max} \rightarrow \underline{\mathcal{C}}_x$, which is a representative of $\text{Im } \varphi_x$, is locally free on $x \times C$ and thus flat as an A -crystal. Since $\underline{\mathcal{F}}_x$ is also a flat A -crystal, the short exact sequence $0 \rightarrow \underline{\mathcal{K}}_x \rightarrow \underline{\mathcal{F}}_x \rightarrow \text{Im } \varphi_x \rightarrow 0$ yields the flatness of $\underline{\mathcal{K}}_x$ by standard arguments. \square

Remark 2.18. If $\underline{\mathcal{F}}$ has a locally free representative with τ injective, then a representative as in the previous theorem is given by $\text{Im}(\tau_{\underline{\mathcal{F}}}^n)$ for n sufficiently large. In this case, it is easy to deduce that φ has trivial kernel, since $\underline{\mathcal{F}}$ is torsion-free.

From Theorem 2.17 it is straightforward to deduce the following result.

COROLLARY 2.19. *Any $\underline{\mathcal{F}} \in \mathbf{Crys}^{\text{flat}}(X, A)$ has a good representative on a dense open subset of X .*

3. The generic Galois representation of a flat crystal

In this section we collect basic properties of Galois representations attached to flat crystals. Our main focus is on the case where the base is a curve. There we explain how any A -crystal gives rise to a strictly compatible family of \mathfrak{p} -adic Galois representations and we shall present a satisfactory description of the ramification locus of the members of this family. We heavily borrow from [BP09, Gar01, Kat73] and [TW96]. If the compatible system is abelian, we deduce from [Böc13] that its semisimplification arises from a direct sum of Hecke characters.

As in §2, by X, Y we denote noetherian schemes over \mathbb{F}_q . Recall also from 2.6 that A is throughout the coordinate ring of a smooth affine geometrically connected curve C over \mathbb{F}_q .

To prepare the following result, recall that to any coherent sheaf \mathcal{F} on X we can associate an étale sheaf $\mathcal{F}_{\text{ét}}$ on the small étale site as the functor that maps any étale morphism $u : U \rightarrow X$ to $\Gamma(U, u^*\mathcal{F})$. For any morphism $f : Y \rightarrow X$ we have $(f^*\mathcal{F})_{\text{ét}} = f_{\text{ét}}^*(\mathcal{F}_{\text{ét}})$, canonically. Moreover, for any étale sheaf F one has a canonical isomorphism $\sigma^*F \cong F$, cf. [SGA4½, Rapport §1]. Thus $\mathcal{F} \rightarrow \mathcal{F}_{\text{ét}}$ induces for any τ -sheaf $\underline{\mathcal{F}}$ a pair $(\mathcal{F}_{\text{ét}}, \tau_{\text{ét}} : \mathcal{F}_{\text{ét}} \rightarrow \mathcal{F}_{\text{ét}})$. As in [BP09, ch. 10], we define the étale sheaf of τ -invariant sections $\underline{\mathcal{F}}^{\text{ét}}$ as the contravariant functor on étale morphisms $u : U \rightarrow X$ by

$$\underline{\mathcal{F}}^{\text{ét}}(U) := (\Gamma(U, u^*\mathcal{F}))^\tau := \text{Ker}(1 - \tau_{\text{ét}} : \Gamma(U, \mathcal{F}_{\text{ét}}(U)) \rightarrow \Gamma(U, \mathcal{F}_{\text{ét}}(U))).$$

We can now state the following important result of Katz, cf. [Kat73, Proposition 4.1.1].

THEOREM 3.1 (Katz). *Suppose X is normal and integral. Then the map $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{\text{ét}}$ defines an equivalence of categories from the full subcategory category of $\mathbf{Coh}_\tau(X, \mathbb{F}_q)$ on pairs (\mathcal{F}, τ) , such that \mathcal{F} is locally free and τ is an isomorphism, to the category of lisse \mathbb{F}_q -sheaves on the small étale site over X , i.e., to the category of continuous representations of the fundamental group $\pi_1(X)$ on vector spaces over \mathbb{F}_q of finite dimension.*

In this correspondence, the rank of the lisse sheaf $\underline{\mathcal{F}}^{\text{ét}}$ over \mathbb{F}_q is equal to the rank of \mathcal{F} as an \mathcal{O}_X -module, and the pullback of $\underline{\mathcal{F}}^{\text{ét}}$ to a closed point $x \in X$ is the étale sheaf associated to $i_x^\mathcal{F}$.*

Let from now on X be normal and integral with generic point $\eta = \text{Spec } K$, and let A satisfy Convention 2.6 so that it is a Dedekind domain of finite type over \mathbb{F}_q . Suppose that $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X, A)$ is generically good with generic rank r . Fix a maximal ideal \mathfrak{p} of A and $n \in \mathbb{N}$, and consider the functor

$$(\text{Spec } K' \xrightarrow{u} \text{Spec } K) \mapsto \Gamma(\text{Spec } K', u^*(\underline{\mathcal{F}}_\eta) \otimes_A A/\mathfrak{p}^n)^\tau \tag{1}$$

for finite separable morphism $K \hookrightarrow K'$. If τ is an isomorphism on $\underline{\mathcal{F}}_\eta \otimes_A A/\mathfrak{p}$, and thus also on $\underline{\mathcal{F}}_\eta \otimes_A A/\mathfrak{p}^n$, it follows by Theorem 3.1 that this defines a lisse étale sheaf of \mathbb{F}_q -vector spaces over $\text{Spec } K$ of rank equal to $r \dim_{\mathbb{F}_q} A/\mathfrak{p}^n$ which is the rank of $\mathcal{F}_\eta \otimes_A A/\mathfrak{p}^n$ considered as a \mathcal{O}_X -module. The étale sheaf is an A/\mathfrak{p}^n -module in the obvious way, and via induction over n , considering the growth of ranks, one finds that (1) defines a lisse étale sheaf of A/\mathfrak{p}^n -modules of rank r . In other words, $\underline{\mathcal{F}}$ defines a Galois representation

$$\rho_{\underline{\mathcal{F}}, \mathfrak{p}^n} : G_K \rightarrow \text{GL}_r(A/\mathfrak{p}^n).$$

For increasing n , these representations form an inverse system which in the limit defines a representation

$$\rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty} : G_K \rightarrow \text{GL}_r(A_\mathfrak{p}),$$

where $A_\mathfrak{p}$ is the completion of A at \mathfrak{p} . Correspondingly, we write $F_\mathfrak{p}$ for the completion of F at \mathfrak{p} .

Remark 3.2. The representation $\rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}$ depends only on the generic fiber of $\underline{\mathcal{F}}$.

From Katz’s theorem, we also deduce the following corollary.

COROLLARY 3.3. *Let $\underline{\mathcal{F}}$ be a good τ -sheaf on X . Set $D = \text{Supp}(\text{Coker}(\tau : (\sigma \times \text{id})^* \mathcal{F} \rightarrow \mathcal{F}))$ and for \mathfrak{p} a maximal ideal of A define $S_{\mathfrak{p}}^{\mathcal{F}} \subset X$ as the image of the intersection $D \cap (X \times \{\mathfrak{p}\})$ under the projection $X \times C \rightarrow X$. Then $\rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}$ is unramified on $X \setminus S_{\mathfrak{p}}^{\mathcal{F}}$.*

A good τ -sheaf is the maximal model of its generic fiber, and can therefore be constructed out of the generic fiber, cf. Theorem 2.14 for curves and [Böc04, ch. 8] for any regular X . In this sense, the properties we deduce on the ramification of $\rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}$ are properties which depend only on the generic fiber of $\underline{\mathcal{F}}$, in agreement with Remark 3.2.

Proof of Corollary 3.3. We need to show that for every n the assignment

$$(U \xrightarrow{g} X) \mapsto \Gamma(U, g^* \underline{\mathcal{F}} \otimes_A A/\mathfrak{p}^n)^\tau$$

defines a lisse A/\mathfrak{p}^n -sheaf of rank $r = \text{rank } \mathcal{F}$ on $X \setminus S_{\mathfrak{p}}^{\mathcal{F}}$. By induction on n and using the obvious A/\mathfrak{p}^n -action it will suffice to show that it defines a lisse \mathbb{F}_q -sheaf of rank equal to $r \cdot \dim_{\mathbb{F}_q} A/\mathfrak{p}^n$ on $X \setminus S_{\mathfrak{p}}^{\mathcal{F}}$. For this we may apply Theorem 3.1 to the restriction of $\underline{\mathcal{F}} \otimes_A A/\mathfrak{p}^n$ to $U = X \setminus S_{\mathfrak{p}}^{\mathcal{F}}$ where $\mathcal{F} \otimes_A A/\mathfrak{p}^n$ is a locally free sheaf of \mathcal{O}_X -modules of rank $r \cdot \dim_{\mathbb{F}_q} A/\mathfrak{p}^n$ and τ is onto and hence an isomorphism. \square

DEFINITION 3.4. We say $\underline{\mathcal{F}}$ as in Corollary 3.3 is *generically lisse at \mathfrak{p}* if $S_{\mathfrak{p}}^{\mathcal{F}}$ is properly contained in X .

DEFINITION 3.5. By $\underline{\mathcal{F}}_{\mathfrak{p}^n}^{\text{ét}}$ we denote the étale sheaf on X defined by

$$(U \xrightarrow{g} X) \mapsto \Gamma(U, g^* \underline{\mathcal{F}} \otimes_A A/\mathfrak{p}^n)^\tau.$$

The functor defined by (1) is invariant under nil-isomorphism. From Corollary 2.9 we know that any flat crystal possesses a generically good representative. Denoting by $\mathbf{Rep}_{A_{\mathfrak{p}}}(G_K)$ the category of continuous representations of G_K on free finitely generated $A_{\mathfrak{p}}$ -modules, we have the following corollary.

COROLLARY 3.6. *The assignment in (1) induces a functor*

$$\mathbf{Crys}^{\text{flat}}(X, A) \rightarrow \mathbf{Rep}_{A_{\mathfrak{p}}}(G_K) : \underline{\mathcal{F}} \mapsto \rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}.$$

With regards to strict compatibility of the representations just constructed, we need the following result, which is essentially from [BP09, ch. 10].

PROPOSITION 3.7. *Suppose $\underline{\mathcal{F}}$ is good at a closed point $x \in X$ which does not lie in $S_{\mathfrak{p}}^{\mathcal{F}}$ (cf. Corollary 3.3). Then*

$$\det_{A_{\mathfrak{p}}}(z^{d_x} \text{id} - \rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}(\text{Frob}_x)) = \det_A(z \text{id} - \tau_x | \mathcal{F}_x) \in A[z^{d_x}].$$

Proof. Under our hypotheses, $\mathcal{F}_x \otimes_A A/\mathfrak{p}^n$ is locally free of rank r over $k_x \otimes A/\mathfrak{p}^n$. Since x is not in $S_{\mathfrak{p}}^{\mathcal{F}}$ the map $\tau_x : \sigma^* \mathcal{F}_x \rightarrow \mathcal{F}_x$ is an isomorphism. In the notation of [BP09], this implies that $\underline{\mathcal{F}}_x \otimes_A A/\mathfrak{p}^n$ is semisimple so that naive and crystalline L -functions agree, cf. [BP09, §§ 8.1 and 9.3] for the notation. In this situation, [BP09, Proposition 10.6.3] yields

$$\begin{aligned} \det_{A/\mathfrak{p}^n}(\text{id} - z^{d_x} \rho_{\underline{\mathcal{F}}, \mathfrak{p}^n}(\text{Frob}_x)) &= \det_{A/\mathfrak{p}^n}(\text{id} - z \tau_x \otimes_A A/\mathfrak{p}^n | \mathcal{F}_x \otimes_A A/\mathfrak{p}^n) \in A/\mathfrak{p}^n[z] \\ &\equiv \det_A(\text{id} - z \tau_x | \mathcal{F}_x) \pmod{\mathfrak{p}^n}. \end{aligned}$$

Moreover, all three polynomials are of degree rd_x with leading coefficient a unit. Passing to the inverse limit over n , replacing z by z^{-1} and multiplying by z^{rd_x} yields the asserted result. \square

For the following result, we recall the definition of a strictly compatible system from [Böc13, Definition 2.16] adapted to the present context: let \overline{X} be a smooth projective geometrically irreducible curve over \mathbb{F}_q and let $\overline{C} \supset C$ be a smooth compactification.

DEFINITION 3.8. An F -rational strictly compatible system of n -dimensional representations of G_K for a ramification divisor $D \subset \overline{X} \times \overline{C}$ with defect set $T \subset \overline{C}$ and ramification set $S \subset \overline{X}$ consists of:

- (i) a continuous semisimple representation

$$\rho_{\mathfrak{p}} : G_K \rightarrow \mathrm{GL}_n(F_{\mathfrak{p}})$$

for each $\mathfrak{p} \in \overline{C} \setminus T$, which is unramified outside $S \cup S_{\mathfrak{p}}^D$ where $S_{\mathfrak{p}}^D = \{x \in \overline{X} \mid (x, \mathfrak{p}) \in D\}$; and

- (ii) a monic polynomial $f_x \in F[z]$ of degree n for each $x \in \overline{X} \setminus S$,

such that S and T are finite, D has neither vertical nor horizontal components and for all $\mathfrak{p} \in \overline{C} \setminus T$ and $x \in \overline{X} \setminus (S \cup S_{\mathfrak{p}}^D)$ one has $\mathrm{CharPol}_{\rho_{\mathfrak{p}}(\mathrm{Frob}_x)} = f_x$.

Remark 3.9. Insisting that D has neither vertical nor horizontal components implies that S and T are unique. Moreover, the sets $S_{\mathfrak{p}}^D$ for $\mathfrak{p} \notin T$ and $\{\mathfrak{p} \in \overline{C} \mid (x, \mathfrak{p}) \in D\}$ for $s \notin S$ are all finite.

COROLLARY 3.10. Suppose X is a smooth curve and $\mathcal{F} \in \mathbf{Crys}^{\mathrm{flat}}(X, A)$. Then the representations $(\rho_{\mathcal{F}, \mathfrak{p}^\infty})_{\mathfrak{p}}$, where \mathfrak{p} ranges over the maximal ideals of A at which \mathcal{F} is generically lisse in the sense of Definition 3.4, form a strictly compatible system of Galois representations.

Proof. In the following, denote by \overline{X} the smooth compactification of X . Recall first that $\rho_{\mathcal{F}, \mathfrak{p}^\infty}$ only depends on \mathcal{F}_η , which is represented by a good τ -sheaf on η . Next, by Theorem 2.14 we can extend \mathcal{F}_η to a maximal model on \overline{X} . Let U be the maximal open subset of \overline{X} on which \mathcal{F} is good, see Corollary 2.19. Clearly $U \subset \overline{X}$ is dense open, so that $S = \overline{X} \setminus U$ is finite. Define $D_{\mathcal{F}}$ as

$$((\overline{X} \setminus U) \times \overline{C}) \cup (\overline{X} \times (\overline{C} \setminus C)) \cup \mathrm{Supp}(\mathrm{Coker}(\tau : (\sigma \times \mathrm{id})^* \mathcal{F}|_U \rightarrow \mathcal{F}|_U))$$

and T as

$$(\overline{C} \setminus C) \cup \{\mathfrak{p} \in C \mid U \times \{\mathfrak{p}\} \subset D_{\mathcal{F}}\}.$$

The set T is finite because $\mathrm{Coker} \tau$ has a support of codimension at least 1 in $\overline{X} \times C$. Finally, for \mathfrak{p} not in T define $S_{\mathfrak{p}}^{\mathcal{F}} (= S_{\mathfrak{p}}^{D_{\mathcal{F}}})$ as in Corollary 3.3, i.e., as the image of the projection onto U of the intersection of $D_{\mathcal{F}}$ with $U \times \{\mathfrak{p}\}$. By our definition of T , the sets $S_{\mathfrak{p}}^{\mathcal{F}}$ are finite. We deduce from Corollary 3.3 that for any $\mathfrak{p} \notin T$ the representation $\rho_{\mathcal{F}, \mathfrak{p}^\infty}$ is unramified on the dense open subset $\overline{X} \setminus (S \cup S_{\mathfrak{p}}^{\mathcal{F}})$ of \overline{X} .

Define for any $x \in \overline{X} \setminus S$ the polynomial f_x by

$$f_x(z) = \det(y \mathrm{id} - \tau_x | \mathcal{F}_x) |_{y=z^{d_x}} \in A[z].$$

It is monic of degree $r = \mathrm{rank} \mathcal{F}$. By Proposition 3.7, we have, for any $\mathfrak{p} \notin T$ and any $x \in \overline{X} \setminus (S \cup S_{\mathfrak{p}}^{\mathcal{F}})$, that

$$\mathrm{CharPol}_{\rho_{\mathfrak{p}}(\mathrm{Frob}_x)}(z) = f_x(z) \in A[z].$$

This completes the proof that $(\rho_{\mathcal{F}, \mathfrak{p}^\infty})_{\mathfrak{p} \notin T}$ forms a strictly compatible system of F -rational Galois representations with ramification set S , defect set T and divisor $D_{\mathcal{F}}$. \square

Remark 3.11. The proof of Corollary 3.10 shows that the divisor $D_{\mathcal{F}}$, or equivalently the sets $S_{\mathfrak{p}}^{\mathcal{F}}$, $\mathfrak{p} \notin T$, describing ramification on the dense open set where $\underline{\mathcal{F}}_{\eta}$ has a good model, can be obtained in a straightforward manner from the maximal extension of $\underline{\mathcal{F}}_{\eta}$.

DEFINITION 3.12. We say $\underline{\mathcal{F}} \in \mathbf{Crys}^{\text{flat}}(X, A)$ is *Galois abelian* if the members of the compatible system $(\rho_{\underline{\mathcal{F}}, \mathfrak{p}^{\infty}})_{\mathfrak{p}}$ are Galois representations with abelian image for almost all $\mathfrak{p} \in \text{Max}(A)$.

In § 5 we shall deduce the property of being Galois abelian from the existence of a natural Hecke action on the crystal. We end this section by an application of [Böc13, Theorem 2.20] to the present setting. Using the notation of [Böc13], we conclude with the following theorem.

THEOREM 3.13. *Suppose X is a smooth curve and $\underline{\mathcal{F}} \in \mathbf{Crys}^{\text{flat}}(X, A)$ is Galois abelian of generic rank r . Let $D = D_{\mathcal{F}}$. Then there exist a finite extension E of F and Hecke characters $(\chi_i)_{i=1, \dots, r}$ for Σ^D over E such that for all places v of E not above T the v -adic Galois representation $\bigoplus_{i=1}^r \rho_{\chi_i, v}$ of G_K attached to $\bigoplus_{i=1}^r \chi_i$ is isomorphic to the semisimplification $(\rho_{\underline{\mathcal{F}}, \mathfrak{p}^{\infty}} \otimes_{A_{\mathfrak{p}}} E_v)^{\text{ss}}$ for \mathfrak{p} below v . The list of characters $(\chi_i)_{i=1, \dots, r}$ is unique up to permutation.*

4. The endomorphism ring of a uniformizable crystal

The aim of this section is to obtain various results on the endomorphism rings of crystals and τ -sheaves. We shall show that endomorphism rings of generically good τ -sheaves near the generic point are finitely generated over A . The endomorphism ring of a τ -sheaf may, in general, be smaller than that of its associated crystal. If a crystal has the further property that it is uniformizable, cf. Definition 4.11, then its endomorphism ring is a subring of a finite rank matrix algebra over A . The latter appears as the endomorphism ring of the *analytification* of the crystal. Another way to gain information on the endomorphism ring is to consider the étale \mathfrak{p} -adic realizations of the crystal. We end this section with a comparison between the endomorphism ring of the analytification of a crystal and its \mathfrak{p} -adic realization. This will be needed in § 5.

We keep the notation \mathbb{F}_q, X, Y, A introduced at the beginning of § 3.

The following lemma, which is essentially due to Anderson, cf. [And86, Theorem 2], is inspired by analogous considerations for the endomorphism ring of an abelian variety.

LEMMA 4.1. *Suppose X is reduced. Let $\underline{\mathcal{F}}$ and $\underline{\mathcal{G}}$ be generically good τ -sheaves on X such that in addition \mathcal{F} is torsion-free.⁴ Then $\text{Hom}_{\tau}(\underline{\mathcal{F}}, \underline{\mathcal{G}})$ is a finitely generated projective A -module. If X is irreducible, its rank is at most the product of the ranks of \mathcal{F}_{η} and \mathcal{G}_{η} , for η the generic point of X .*

Proof. Because A is a Dedekind domain, we need to show that $\text{Hom}_{\tau}(\underline{\mathcal{F}}, \underline{\mathcal{G}})$ is finitely generated and A -torsion-free. By the torsion-freeness of \mathcal{F} , it suffices to prove the assertion after base change to a generic point η of X , i.e., under the assumption that $X = \text{Spec } K$ for an \mathbb{F}_q -field K . For this it suffices to prove that the natural homomorphism

$$\text{Hom}_{\tau}(\underline{\mathcal{F}}, \underline{\mathcal{G}}) \otimes_{\mathbb{F}_q} K \rightarrow \text{Hom}_{K \otimes_{\mathbb{F}_q} A}(M, N) : (f \otimes \alpha) \mapsto \alpha f \tag{2}$$

with $M = \Gamma(\text{Spec } K \otimes A, \mathcal{F})$ and $N = \Gamma(\text{Spec } K \otimes A, \mathcal{G})$ is injective. The modules M and N are projective and finitely generated over $K \otimes_{\mathbb{F}_q} A$, and so the right-hand side is a projective $K \otimes_{\mathbb{F}_q} A$ -module of finite rank. Moreover, the given map is a homomorphism of $K \otimes_{\mathbb{F}_q} A$ -modules. Thus if the homomorphism is injective then the lemma is proved.

⁴That is, $\mathcal{F} \rightarrow \bigoplus_{\eta} i_{\eta*} \mathcal{F}_{\eta}$ is injective, where the sum is over the generic points of X .

We assume that the homomorphism is not injective and thus that we can find $f_0, \dots, f_m \in \text{Hom}_\tau(\underline{\mathcal{F}}, \underline{\mathcal{G}})$ and $\alpha_0, \dots, \alpha_m \in K$ with $\sum \alpha_i \otimes f_i \neq 0$ and

$$\sum_{i=1}^m \alpha_i f_i = 0 \tag{3}$$

as a homomorphism from M to N . We also assume that m is minimal and that $\alpha_0 = 1$ in $\sum \alpha_i \otimes f_i$. Applying $\tau_{\mathcal{G}}$ to $\sum \alpha_i f_i = 0$ and any section $s \in M$ of \mathcal{F} yields

$$\sum \tau_{\mathcal{G}}(\alpha_i f_i(s)) = \sum \alpha_i^q \tau_{\mathcal{G}}(f_i(s)) = \sum \alpha_i^q f_i(\tau_{\mathcal{F}}(s)) = 0.$$

Since M is torsion-free and $\text{Im}(\tau_{\mathcal{F}}) \subset \mathcal{F}$ generically generates \mathcal{F} we must have $\sum \alpha_i^q f_i = 0$ as a homomorphism from M to N . Subtracting it from (3) yields

$$\sum_{i=2}^m (\alpha^i - \alpha) f_i = 0.$$

By the minimality of m , we deduce that all the α_i satisfy $\alpha_i^q = \alpha_i$ and hence lie in \mathbb{F}_q . But then the f_i are linearly independent over \mathbb{F}_q . This contradicts the minimality hypothesis on m . \square

Remark 4.2. The assertion of the lemma requires both that $\tau_{\mathcal{F}}$ is injective and that \mathcal{F} is torsion-free. If either fails, $\underline{\mathcal{F}}$ may have a non-zero direct summand $\underline{\mathcal{H}}$ on which $\tau_{\mathcal{H}}$ is zero. For such an $\underline{\mathcal{H}}$ one has $\text{End}_\tau(\underline{\mathcal{H}}) = \text{End}_{\mathcal{O}_{X \times C}}(\underline{\mathcal{H}})$, and the right-hand side need not be finitely generated over A .

For flat crystals over a general noetherian X , a finiteness result on Hom_{crys} as an A -module seems conceivable, however only under suitable additional hypotheses, cf. Example 4.5.

Convention 4.3. For the remainder of this article, X will denote a smooth geometrically irreducible curve with constant field \mathbb{F}_q .

PROPOSITION 4.4. *Let $\underline{\mathcal{F}}$ be $\mathbf{Crys}^{\text{flat}}(X, A)$ and write $\underline{\mathcal{F}}$ also for a generically good representative in $\mathbf{Coh}_\tau(X, A)$. Then we have natural homomorphisms of endomorphism rings.*

$$\begin{array}{ccccc} \text{End}_\tau(\underline{\mathcal{F}}) & \longrightarrow & \text{End}_\tau(\underline{\mathcal{F}}^{\text{max}}) & \hookrightarrow & \text{End}_\tau(\underline{\mathcal{F}}_\eta) \\ \downarrow & & \downarrow & & \downarrow \\ \text{End}_{\text{crys}}(\underline{\mathcal{F}}) & \longrightarrow & \text{End}_{\text{crys}}(\underline{\mathcal{F}}^{\text{max}}) & \hookrightarrow & \text{End}_{\text{crys}}(\underline{\mathcal{F}}_\eta) \end{array}$$

All vertical maps as well as the right horizontal ones are injective. If $\underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}^{\text{max}}$ is injective, cf. Remark 2.18, then the same holds for the left horizontal maps.

Proof. We shall only indicate the construction of the horizontal arrow on the lower left, the other assertions being either rather straightforward or similar to but simpler than the argument we give. To see that any endomorphism in $\mathbf{Crys}(X, A)$ of $\underline{\mathcal{F}}$ extends to $\underline{\mathcal{F}}^{\text{max}}$, we consider the following commutative diagram.

$$\begin{array}{ccccc} \underline{\mathcal{F}} & \xleftarrow{\tau^n} & (\sigma^n \times \text{id})^* \underline{\mathcal{F}} & \xrightarrow{\varphi} & \underline{\mathcal{F}} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathcal{F}}^{\text{max}} & \xleftarrow{\tau^n} & (\sigma^n \times \text{id})^* \underline{\mathcal{F}}^{\text{max}} & & \underline{\mathcal{F}}^{\text{max}} \\ & & \downarrow (\sigma^n \times \text{id})^* \iota & & \downarrow \iota \\ & & (\sigma^n \times \text{id})^* \underline{\mathcal{F}}_\eta & \xrightarrow{\varphi_\eta} & \underline{\mathcal{F}}_\eta \end{array}$$

Here the upper horizontal line is the representative of an endomorphism of $\underline{\mathcal{F}}$ as a crystal. Now the point is that the image of $\varphi_\eta \circ (\sigma^n \times \text{id})^* \iota$ is a coherent τ -subsheaf of $\underline{\mathcal{F}}_\eta$, whose generic fiber lies in $\underline{\mathcal{F}}_\eta$. But $\underline{\mathcal{F}}^{\max}$ contains all coherent subsheaves with this property, cf. Remark 2.13, and hence $\varphi_\eta \circ (\sigma^n \times \text{id})^* \iota$ factors via $\underline{\mathcal{F}}^{\max}$ as had to be shown. Alternatively, one can also directly apply the universal property of $\underline{\mathcal{F}}^{\max}$ from Definition 2.12. \square

Example 4.5. We give two examples which explain parts of the difference between $\text{End}_\tau(\underline{\mathcal{F}})$ and $\text{End}_{\text{crys}}(\underline{\mathcal{F}})$. For the first example, let $X = \text{Spec } \mathbb{F}_q[\theta]$ and $A = \mathbb{F}_q[t]$ and let $\underline{\mathcal{C}}$ be the Carlitz- τ -sheaf, i.e., the τ -sheaf corresponding to the pair $(\mathbb{F}_q[\theta, t], (t - \theta)(\sigma \times \text{id}))$ where $(t - \theta)(\sigma \times \text{id})$ maps $f(\theta, t)$ to $(t - \theta)f(\theta^q, t)$. Let $\underline{\mathcal{F}} = \underline{\mathcal{C}} \oplus (\sigma \times \text{id})^* \underline{\mathcal{C}}$. Then the following diagram defines an endomorphism in $\text{End}_{\text{crys}}(\underline{\mathcal{F}})$ which, as we leave to the reader to prove, does not lie in the image of $\text{End}_\tau(\underline{\mathcal{F}})$:

$$\underline{\mathcal{C}} \oplus (\sigma \times \text{id})^* \underline{\mathcal{C}} \xleftarrow{\tau} (\sigma \times \text{id})^* \underline{\mathcal{C}} \oplus (\sigma^2 \times \text{id})^* \underline{\mathcal{C}} \xrightarrow{(x,y) \mapsto (0,x)} \underline{\mathcal{C}} \oplus (\sigma \times \text{id})^* \underline{\mathcal{C}}.$$

In the second example, we let X be arbitrary, take $\mathcal{F} = \mathcal{O}_{X \times C}$ and define τ by adjunction from

$$\mathcal{O}_{X \times C} \xrightarrow{a(\sigma \times \text{id})} \mathcal{O}_{X \times C}$$

for some $a \in A \setminus \{0\}$. Then the homomorphism φ_n in $\text{Hom}_{\text{crys}}(\underline{\mathcal{F}}, \underline{\mathcal{G}})$ defined by the diagram

$$\mathcal{O}_{X \times C} \xleftarrow{\tau^n} (\sigma^n \times \text{id})^* \mathcal{O}_{X \times C} \cong \mathcal{O}_{X \times C} \xrightarrow{\varphi} \underline{\mathcal{G}}$$

for $\varphi \in \text{Hom}(\underline{\mathbb{1}}_{X,A}, \underline{\mathcal{G}})$ is characterized by $a^n \varphi_n = \varphi$. One deduces

$$\text{Hom}_{\text{crys}}(\underline{\mathcal{F}}, \underline{\mathcal{G}}) \cong \text{Hom}_\tau(\underline{\mathcal{F}}, \underline{\mathcal{G}}) \otimes_A A \begin{bmatrix} 1 \\ a \end{bmatrix}.$$

The crystals which we shall ultimately consider will be flat (and in fact have a locally free representing τ -sheaf) and *uniformizable*. The latter condition will have a strong impact on their endomorphism ring. We recall the necessary notions: let $L \subset \mathbb{C}_\infty$ be any complete subfield containing K (where $\text{Spec } K = \eta$) and denote by $L\langle t \rangle$ the Tate-algebra (in the variable t) over L , i.e., the ring

$$L\langle t \rangle = \left\{ \sum_{i \geq 0} a_i t^i \mid a_i \in L, |a_i| \rightarrow 0 \text{ for } i \rightarrow \infty \right\}$$

of convergent power series on the closed unit disc (with coefficients in L). By $\tilde{\sigma}$, we denote the endomorphism of $L\langle t \rangle$ defined by $\tilde{\sigma}(\sum_i a_i t^i) = \sum_i a_i^q t^i$. There is an obvious homomorphism $K[t] = K \otimes_{\mathbb{F}_q} \mathbb{F}_q[t] \hookrightarrow L\langle t \rangle$. It is equivariant for the endomorphism $\sigma \times \text{id}$ on the domain and $\tilde{\sigma}$ on the range.

Since A is of finite type over \mathbb{F}_q and of dimension one, we can find, for instance by the Noether normalization lemma, an element $t \in A$ such that $\mathbb{F}_q[t] \hookrightarrow A$ is finite. One can verify that, for L as above, the ring $A \otimes_{\mathbb{F}_q[t]} L\langle t \rangle$ is independent of the choice of such a t . We also write $\tilde{\sigma}$ for $\text{id}_A \otimes \tilde{\sigma}$.

DEFINITION 4.6. An analytic τ -module on L is a pair (M, τ) consisting of a finitely generated $A \otimes_{\mathbb{F}_q[t]} L\langle t \rangle$ -module and a $\tilde{\sigma}$ -linear endomorphism $\tau : M \rightarrow M$.

The category of analytic τ -modules on L over A is denoted $\mathbf{Coh}_\tau^{\text{an}}(L, A)$. It is A -linear abelian. Similarly, one can define analytic crystals on L over A .

Regarding any τ -sheaf $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(\text{Spec } K, A)$ as a pair (M, τ) consisting of a finitely generated $K \otimes A$ -module and a $(\sigma \times \text{id})$ -linear endomorphism $\tau : M \rightarrow M$, we define $\underline{\mathcal{F}}^{\text{an}} = (M, \tau) \otimes_{K[t]} L\langle t \rangle$. Again this is independent of the choice $\mathbb{F}_q[t] \hookrightarrow A$.

DEFINITION 4.7. We call the pair $\underline{\mathcal{F}}^{\text{an}} = (M, \tau) \otimes_{K[t]} L\langle t \rangle$ the *analytification* of $\underline{\mathcal{F}}$.

The analytification of $\underline{\mathbb{1}}_{\eta, A}$ is thus given by the pair $\underline{\mathbb{1}}_{L, a}^{\text{an}} = (A \otimes_{\mathbb{F}_q[t]} L\langle t \rangle, \tilde{\sigma})$.

PROPOSITION 4.8. *Analytification defines a functor*

$$\mathbf{Coh}_\tau(\text{Spec } K, A) \rightarrow \mathbf{Coh}_\tau^{\text{an}}(L, A) : \underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{\text{an}}$$

and the analogous assertion holds for the corresponding categories of crystals.

In [And86, §§ 2.9 and 2.10] the following properties of $L\langle t \rangle$ are shown:

- (a) the $\mathbb{F}_q[t]$ -module of fixed points $(L\langle t \rangle)^{\tilde{\sigma}}$ is equal to $\mathbb{F}_q[t]$ (from which it follows that $(A \otimes_{\mathbb{F}_q[t]} L\langle t \rangle)^{\tilde{\sigma}} = A$);
- (b) any $F \in L\langle t \rangle$ can be written as $F = uf$ where $u \in L\langle t \rangle$ is a unit and $f \in L[t]$ is monic with all roots of absolute value at most one;
- (c) $L\langle t \rangle$ is a unique factorization domain.

The following lemma is a modification of Lemma 4.1.

LEMMA 4.9. *Suppose for $(M, \tau) \in \mathbf{Coh}_\tau^{\text{an}}(L, A)$ that M is locally free and τ is injective, and define $P = \text{Hom}_{\mathbf{Coh}_\tau^{\text{an}}}(\underline{\mathbb{1}}_{L, A}^{\text{an}}, (M, \tau))$. Then P is a finitely generated projective A -module and the homomorphism*

$$\psi : P \otimes_{\mathbb{F}_q[t]} L\langle t \rangle \rightarrow \text{Hom}_{A \otimes_{\mathbb{F}_q[t]} L\langle t \rangle}(\underline{\mathbb{1}}_{L, A}^{\text{an}}, M) \cong M$$

is injective.

Proof. We first show that ψ is injective. To prove injectivity, we may pass from A to the subring $\mathbb{F}_q[t]$, and so we may assume that $A = \mathbb{F}_q[t]$. We argue by contradiction and let

$$m = \sum_{i=1}^s \varphi_i \otimes m_i \neq 0$$

be an element in the kernel of ψ . We may assume that the number s of summands is minimal among the non-zero elements in $\text{Ker}(\psi)$. Since elements of P take their values in the torsion-free A -module M , the A -module P itself is torsion-free, and hence the submodule generated by $\{\varphi_i\}_{i=1, \dots, s}$ is free over $A = \mathbb{F}_q[t]$. It follows from the minimality of s that the φ_i are linearly independent over $\mathbb{F}_q[t]$. As yet another simplifying hypothesis, we may assume that the elements $m_1, \dots, m_s \in L\langle t \rangle$ have 1 as their greatest common divisor.

The elements m_i can be written as $m_i = u_i f_i$ where u_i is a unit of $L\langle t \rangle$ and f_i is monic in $L[t]$ with roots of absolute value at most one. We may also assume that $u_1 = 1$. We now consider the element

$$m' = (1 \otimes \tilde{\sigma}(f_1))m - (1 \otimes f_1)\tilde{\sigma}(m) = \sum_{i=2}^s \varphi_i \otimes (f_i \tilde{\sigma}(f_1)u_i - f_1 \tilde{\sigma}(f_i u_i)).$$

The element m' lies again in $\text{Ker}(\psi)$, since the kernel is fixed under $\tilde{\sigma}$. From the minimality of s we deduce that

$$f_i \tilde{\sigma}(f_1) u_i = f_1 \tilde{\sigma}(f_i) \tilde{\sigma}(u_i) \quad \text{for } i = 2, \dots, s.$$

Since the f_i are monic, we can separate them from the units and find

$$\frac{f_1}{\tilde{\sigma}(f_1)} = \frac{f_i}{\tilde{\sigma}(f_i)} \in L\langle t \rangle \quad \text{and} \quad u_i = \tilde{\sigma}(u_i) \quad \text{for } i = 2, \dots, s.$$

The latter yields $u_i \in (L\langle t \rangle)^* \cap (L\langle t \rangle)^{\tilde{\sigma}} = A^*$. The former implies that $f_1 / \text{gcd}(f_1, \tilde{\sigma}(f_1))$ must be a divisor of all f_i . Since the greatest common divisor of all f_i is 1, the displayed quotient must be 1, i.e., f_1 must divide $\tilde{\sigma}(f_1)$. Comparing degrees and observing that both are monic, we find $f_1 = \tilde{\sigma}(f_1)$ and then the same for all i . It follows that the coefficients of the f_i lie in $L^{\tilde{\sigma}} = \mathbb{F}_q$, so that $m_i = f_i u_i \in A = \mathbb{F}_q[t]$ for all i . Therefore $m = (\sum_{i=1}^s m_i \varphi_i) \otimes 1$, and hence $\sum_{i=1}^s m_i \varphi_i = 0$. This contradicts the linear independence of the φ_i . The proof of injectivity is thus complete.

To show that P is finitely generated, it suffices to show that P satisfies the ascending chain condition. Now, any ascending chain of A -submodules of P yields, via the injectivity of ψ , an ascending chain of $A \otimes_{\mathbb{F}_q[t]} L\langle t \rangle$ -submodules of M . The latter is noetherian, and hence the same follows for P . □

LEMMA 4.10. For $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau^{\text{an}}(L, A)$, the natural homomorphism

$$\text{Hom}_\tau(\underline{\mathbb{1}}_{L,A}^{\text{an}}, \underline{\mathcal{F}}) \longrightarrow \text{Hom}_{\text{crys}}(\underline{\mathbb{1}}_{L,A}^{\text{an}}, \underline{\mathcal{F}})$$

is bijective, and hence the functor $\underline{\mathcal{F}} \mapsto \text{Hom}_\tau(\underline{\mathbb{1}}_{L,A}^{\text{an}}, \underline{\mathcal{F}})$ factors via $\mathbf{Crys}^{\text{an}}(L, A)$.

Proof. The bijectivity is immediate upon observing that $\tau^n : (\tilde{\sigma}^n)^* \underline{\mathbb{1}}_{L,A}^{\text{an}} \rightarrow \underline{\mathbb{1}}_{L,A}^{\text{an}}$ is an isomorphism for all $n \in \mathbb{N}$. For the second part note that $\text{Hom}_{\text{crys}}(\underline{\mathbb{1}}_{L,A}^{\text{an}}, \underline{\mathcal{F}})$ depends only on the crystal $\underline{\mathcal{F}}$. □

Now let $\underline{\mathcal{F}}$ lie in $\mathbf{Crys}^{\text{flat}}(X, A)$. Extending the analytification functor, we define $\underline{\mathcal{F}}^{\text{an}} := (\underline{\mathcal{F}}_\eta)^{\text{an}}$. On a dense open $U \subset X$, the crystal $\underline{\mathcal{F}}$ has a good representative. Its rank is the rank of the crystal $\underline{\mathcal{F}}$, cf. Definition 2.10. By the previous lemmas, $\text{Hom}_{\mathbf{Crys}^{\text{an}}}(\underline{\mathbb{1}}_{L,A}^{\text{an}}, \underline{\mathcal{F}}^{\text{an}})$ is well defined and is a projective A -module of rank at most the rank of $\underline{\mathcal{F}}$.

DEFINITION 4.11. Let $\underline{\mathcal{F}} \in \mathbf{Crys}^{\text{flat}}(X, A)$ be of rank r . Then $\underline{\mathcal{F}}$ is *uniformizable* for $K \hookrightarrow L$ where $L \subset \mathbb{C}_\infty$ is a complete subfield, if $\text{rank}_A \text{Hom}_{\mathbf{Crys}^{\text{an}}}(\underline{\mathbb{1}}_{L,A}^{\text{an}}, \underline{\mathcal{F}}^{\text{an}}) = r$.

We say that $\underline{\mathcal{F}}$ is *(potentially) uniformizable at* $x \in \overline{X}$ if L is (a finite extension of) the completion of K at x .

PROPOSITION 4.12. Suppose $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X, A)$ is generically good and uniformizable for $K \hookrightarrow L$. Then the homomorphism $\psi : P \otimes_{\mathbb{F}_q[t]} L\langle t \rangle \rightarrow \underline{\mathcal{F}}^{\text{an}}$ from Lemma 4.9 is an isomorphism.

Proof. By Lemma 4.9, ψ is injective and $\text{Im}(\psi)$ is a locally free τ -subsheaf of $\underline{\mathcal{F}}^{\text{an}}$. By uniformizability, they have the same rank. Hence they have the same rank over $L\langle t \rangle$ (for some finite homomorphism $\mathbb{F}_q[t] \hookrightarrow A$). We claim that there exist $a \in \mathbb{F}_q[t] \setminus \{0\}$ such that $a \underline{\mathcal{F}}^{\text{an}} \subset \text{Im}(\psi)$. Then it follows from [And86, § 2.10] that the natural homomorphism

$$(a \underline{\mathcal{F}}^{\text{an}})^\tau \otimes_{\mathbb{F}_q[t]} L\langle t \rangle \longrightarrow a \underline{\mathcal{F}}^{\text{an}}$$

is an isomorphism, from which one easily deduces that ψ is an isomorphism.

To prove the claim, let $\underline{B} = (\mathbf{b}_1, \dots, \mathbf{b}_R)$ be an $\mathbb{F}_q[t]$ -basis of P and hence an $L\langle t \rangle$ -basis of $\text{Im}(\psi)$. Using the elementary divisor theorem, we can also find an $L\langle t \rangle$ -basis $\underline{C} = (\mathbf{c}_1, \dots, \mathbf{c}_R)$

of $\underline{\mathcal{F}}^{\text{an}}$ such that there are $m_i \in L\langle t \rangle$, $i = 1, \dots, R$, such that $\underline{B}' = (m_1 \mathbf{e}_1, \dots, m_R \mathbf{e}_R)$ is an $L\langle t \rangle$ -basis of $\text{Im}(\psi)$. Let $\mathbf{A} \in \text{GL}_R(L\langle t \rangle)$ be the matrix such that $\underline{B}' = \underline{B}\mathbf{A}$. It follows that $\tau_{\text{Im}(\psi)}$ is given by $\mathbf{A}^{-1}\tilde{\sigma}(\mathbf{A}) \in \text{GL}_R(L\langle t \rangle)$. Let \mathbf{D} be the diagonal matrix with entries (m_1, \dots, m_R) , so that $\underline{C} = \underline{B}'\mathbf{D}^{-1}$. Then it follows that $\tau_{\underline{\mathcal{F}}^{\text{an}}}$ is described by the matrix $\mathbf{D}\mathbf{A}^{-1}\tilde{\sigma}(\mathbf{A})\tilde{\sigma}(\mathbf{D})^{-1}$ with respect to \underline{C} . Since it preserves $\underline{\mathcal{F}}^{\text{an}}$, it will lie in $M_R(L\langle t \rangle)$. Taking determinants, we deduce

$$\left(\prod_i m_i\right)\tilde{\sigma}\left(\prod_i m_i\right)^{-1} \in L\langle t \rangle.$$

Since the m_i can be written as a unit u_i times a monic polynomial f_i with all its roots of absolute value at most 1, the expression has the same number of zeros and poles. Since it is not allowed to have any poles, it must be a unit in $L\langle t \rangle$. Separating the u_i from the f_i , we deduce, moreover, that $a = \prod_i f_i$ must be invariant under $\tilde{\sigma}$ and hence lie in $\mathbb{F}_q[t]$. The claim follows for the a so defined. □

PROPOSITION 4.13. *If $\underline{\mathcal{F}} \in \mathbf{Crys}^{\text{flat}}(X, A)$ is uniformizable and P denotes $(\underline{\mathcal{F}}^{\text{an}})^\tau$, then*

$$\text{End}_{\text{crys}}(\underline{\mathcal{F}}_\eta) \hookrightarrow \text{End}_{\text{crys}}(\underline{\mathcal{F}}^{\text{an}}) \cong \text{End}_A(P).$$

Proof. To see the inclusion on the left, let $\underline{\mathcal{F}}$ also denote a generically good representative of $\underline{\mathcal{F}}$, and define M as the projective finitely generated $K \otimes A$ -module $M = \Gamma(\eta \times C, \underline{\mathcal{F}}_\eta)$. Let φ be in $\text{End}_{\text{crys}}(\underline{\mathcal{F}}_\eta)$. Choosing a representing diagram for φ , the underlying sheaf homomorphism yields a $\varphi \in \text{Hom}_{K \otimes A}(K^{\sigma^n} \otimes_K M, M)$ for some $n \in \mathbb{N}$.⁵ Via a choice $\mathbb{F}_q[t] \hookrightarrow A$, the endomorphism φ induces an element in $\text{Hom}_{K[t]}(K^{\sigma^n} \otimes_K M, M)$. Denote by ψ the image of φ in $\text{End}_{\text{crys}}(\underline{\mathcal{F}}^{\text{an}})$. Then, similarly, ψ can be regarded as an element in $\text{Hom}_{L\langle t \rangle}(L\langle t \rangle^{\sigma^n} \otimes_{K[t]} M, L\langle t \rangle \otimes_{K[t]} M)$. Since M is free over $K[t]$ of finite rank, the image of φ in $\text{Hom}_{K[t]}(K^{\sigma^n} \otimes_K M, M)$ may be considered as a square matrix over $K[t]$ and, analogously, the image of ψ as a square matrix over $L\langle t \rangle$, and moreover the first matrix is mapped to the second by applying $K[t] \hookrightarrow L\langle t \rangle$ to its entries. Hence if ψ is the zero endomorphism, then so is φ .

For the isomorphism on the right, observe first that by Lemma 4.10 we have $\text{End}_{\text{crys}}(\underline{\mathbb{1}}_{L,A}^{\text{an}}) = \text{End}_\tau(\underline{\mathbb{1}}_{L,A}^{\text{an}})$, and the latter coincides with the set of τ -invariant elements of $\underline{\mathbb{1}}_{L,A}^{\text{an}}$, i.e. with A . Then from Proposition 4.12 we obtain

$$\text{End}_{\text{crys}}(\underline{\mathcal{F}}^{\text{an}}) = \text{End}_{\text{crys}}(P \otimes_A \underline{\mathbb{1}}_{L,A}^{\text{an}}) = \text{End}_A(P). \quad \square$$

Example 4.14. Let $\underline{\mathcal{F}}$ be a uniformizable crystal and denote by $\underline{\mathcal{F}}$ also a generically good representative. The previous proposition asserts that $\text{End}_\tau(\underline{\mathcal{F}}^{\text{an}}) = \text{End}_{\text{crys}}(\underline{\mathcal{F}}^{\text{an}})$, and so any endomorphism in $\text{End}_{\text{crys}}(\underline{\mathcal{F}}_\eta)$ induces an endomorphism of analytic τ -sheaves. The first example in 4.5 shows that $\text{End}_\tau(\underline{\mathcal{F}}_\eta) \hookrightarrow \text{End}_\tau(\underline{\mathcal{F}}^{\text{an}})$ may be a strict inclusion.

We now explain a parallel but simpler investigation of $\text{End}_{\text{crys}}(\underline{\mathcal{F}})$ based on Galois type properties of $\underline{\mathcal{F}}$. We fix $\mathfrak{p} \in \text{Max}(A)$ and $n \in \mathbb{N}$. The following is an analog of Lemma 4.9, whose proof is immediate from Theorem 3.1.

LEMMA 4.15. *Suppose $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(\eta, A/\mathfrak{p}^n)$ is a locally free τ -sheaf on which τ is injective. Define Q_n as $\text{Hom}_{\mathbf{Coh}_\tau}(\underline{\mathbb{1}}_{\eta, A/\mathfrak{p}^n}, \underline{\mathcal{F}}) = \Gamma(\eta, \underline{\mathcal{F}}_{\mathfrak{p}^n}^{\text{ét}})$. Then Q_n is a free finitely generated A/\mathfrak{p}^n -module and the homomorphism*

$$\psi : Q_n \otimes_{A/\mathfrak{p}^n} \underline{\mathbb{1}}_{\eta, A/\mathfrak{p}^n} \rightarrow \text{Hom}_{K \otimes A/\mathfrak{p}^n}(\underline{\mathbb{1}}_{\eta, A/\mathfrak{p}^n}, \underline{\mathcal{F}}) \cong \underline{\mathcal{F}}$$

⁵ The notation $K^{\sigma^n} \otimes_K$ means that we regard K on the left as a ring over K on the right via σ^n .

is injective. Let $K_{\mathcal{F}, \mathfrak{p}^n} \subset K^{\text{sep}}$ be the fixed field of the kernel of $\rho_{\mathcal{F}, \mathfrak{p}^n}$. Then ψ becomes an isomorphism after base change from η to $\text{Spec } K_{\mathcal{F}, \mathfrak{p}^n}$.

We also have the following analog of Proposition 4.13.

PROPOSITION 4.16. For $\mathcal{F} \in \mathbf{Crys}^{\text{flat}}(\eta, A/\mathfrak{p}^n)$ of generic rank r , $u : \text{Spec } K_{\mathcal{F}, \mathfrak{p}^n} \rightarrow \text{Spec } \eta$ and $Q_n = (u^* \mathcal{F}_{A/\mathfrak{p}^n}^{\text{ét}})^{\tau}$ one has

$$\text{End}_{\text{crys}}(\mathcal{F}) \hookrightarrow \text{End}_{\text{crys}}(u^* \mathcal{F}) \cong \text{End}_{A/\mathfrak{p}^n}(Q_n) \cong M_r(A/\mathfrak{p}^n).$$

Proof. Since u is faithfully flat, the injectivity on the left is straightforward. By the definition of $K_{\mathcal{F}, \mathfrak{p}^n}$ (and Theorem 3.1) we have $Q_n \cong (A/\mathfrak{p}^n)^r$ and by the previous lemma $u^* \mathcal{F} \cong (\mathbb{1}_{\text{Spec } K_{\mathcal{F}, \mathfrak{p}^n}, A/\mathfrak{p}^n})^{\oplus r}$. To complete the proof, it remains to show that $\text{End}_{\text{crys}}(\mathbb{1}_{\eta, A/\mathfrak{p}^n}) \cong A/\mathfrak{p}^n$ (for any η). This is clear, since $\tau : (\sigma \times \text{id})^* \mathbb{1}_{\eta, A/\mathfrak{p}^n} \xrightarrow{\tau} \mathbb{1}_{\eta, A/\mathfrak{p}^n}$ is an isomorphism. \square

THEOREM 4.17. Suppose $\mathcal{F} \in \mathbf{Crys}^{\text{flat}}(X, A)$ is of rank r and uniformizable for $K \hookrightarrow L$. Then for any $\mathfrak{p} \in \text{Max}(A)$ at which \mathcal{F} is generically lisse and any $n \in \mathbb{N}$ there is a naturally defined commutative diagram

$$\begin{array}{ccc} \text{End}_{\text{crys}}(\mathcal{F}_{\eta}) & \hookrightarrow & \text{End}_A(P) \\ \downarrow & & \downarrow \\ \text{End}_{A/\mathfrak{p}^n}(\mathcal{F}_{\eta, \mathfrak{p}^n}^{\text{ét}}) & \hookrightarrow & \text{End}_A(P/\mathfrak{p}^n P) \end{array}$$

where $P = (\mathcal{F}^{\text{an}})^{\tau}$ and the right vertical arrow is reduction modulo \mathfrak{p}^n .

Proof. The ring $A \otimes_{\mathbb{F}_q[t]} L\langle t \rangle$, which appears in the analytification procedure, contains $\mathfrak{p}^n \otimes_{\mathbb{F}_q[t]} L\langle t \rangle$ as an ideal, and the quotient ring is isomorphic to $A/\mathfrak{p}^n \otimes_{\mathbb{F}_q} L$. (There exists a non-zero $g \in \mathbb{F}_q[t]$ which lies in \mathfrak{p}^n , and one verifies that $L\langle t \rangle/(g) \cong L[t]/(g) \cong L \otimes_{\mathbb{F}_q} \mathbb{F}_q[t]/(g)$.) Denoting $\text{Spec } L \rightarrow \text{Spec } K$ by v , one deduces the isomorphism

$$P \otimes_A \mathbb{1}_{\text{Spec } L, A/\mathfrak{p}^n} \cong \mathcal{F}^{\text{an}} \otimes_A A/\mathfrak{p}^n \cong v^*(\mathcal{F}_{\eta} \otimes A/\mathfrak{p}^n) \cong (v^* \mathcal{F})_{\eta} \otimes A/\mathfrak{p}^n. \tag{4}$$

From the definition of $K_{\mathcal{F}, \mathfrak{p}^n}$ we now deduce that it is contained in L . Let u denote $\text{Spec } K_{\mathcal{F}, \mathfrak{p}^n} \rightarrow \text{Spec } K$. Then we obtain the diagram in the theorem by applying End_{crys} to the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_{\eta} & \longrightarrow & \mathcal{F}^{\text{an}} \\ \downarrow & & \downarrow \\ \mathcal{F}_{\eta} \otimes A/\mathfrak{p}^n & \longrightarrow & (u^* \mathcal{F})_{\eta} \otimes A/\mathfrak{p}^n \longrightarrow (v^* \mathcal{F})_{\eta} \otimes A/\mathfrak{p}^n \end{array}$$

and the isomorphism $\text{End}_{A/\mathfrak{p}^n}(\mathcal{F}_{\eta, \mathfrak{p}^n}^{\text{ét}}) \cong \text{End}_{\text{crys}}(\mathcal{F}_{\eta} \otimes A/\mathfrak{p}^n)$ due to Theorem 3.1. \square

5. Hecke crystals

In this section X and C are smooth geometrically connected curves over \mathbb{F}_q with C affine. The function field of X is denoted by K , and $L \subset \mathbb{C}_{\infty}$ denotes a complete subfield containing K .

DEFINITION 5.1. A Hecke crystal on X consists of:

- (a) an A -crystal \mathcal{F} on X which has a locally free representative and is uniformizable for some $K \hookrightarrow L$;

(b) a commutative A -subalgebra $\mathbb{T} \subset \text{End}_{\text{crys}}(\underline{\mathcal{F}})$, called a *Hecke algebra*, generated by elements $T_x, x \in X$, called *Hecke operators*,

such that, denoting by T the finite set of $\mathfrak{p} \in \text{Max}(A)$ for which $\rho_{\underline{\mathcal{F}}, \mathfrak{p}}$ is not lisse, one has for all $\mathfrak{p} \in \text{Max}(A) \setminus T$, all $x \in X \setminus S_{\mathfrak{p}}^{\mathcal{F}}$ (for $S_{\mathfrak{p}}^{\mathcal{F}}$, see Corollary 3.3) and all $n \in \mathbb{N}$,

$$\text{Frob}_x = T_x \in \text{End}_{A/\mathfrak{p}^n}(\underline{\mathcal{F}}_{\mathfrak{p}^n}^{\text{ét}}). \tag{5}$$

Remark 5.2. Note that the divisor $D_{\mathcal{F}}$ on $X \times C$ may have vertical components, i.e., there might be a finite set S of closed points x of X for which (5) is never satisfied for any \mathfrak{p} . One way to bound S is via Corollary 2.19. Namely, choose a zero-dimensional subset $S \subset X$ such that the canonical homomorphism $\underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}^{\text{max}}$ is a nil-isomorphism on $X \setminus S$ and, moreover, $\underline{\mathcal{F}}^{\text{max}}$ is a good representative on all of $X \setminus S$. Then, by its definition, $D_{\mathcal{F}}$ intersected with $(X \setminus S) \times C$ will have no vertical component.

Let $(\underline{\mathcal{F}}, \mathbb{T})$ be a Hecke crystal. Condition (a) implies the following: (i) the crystal $\underline{\mathcal{F}}$ is A -flat; (ii) the A -module $P = (\underline{\mathcal{F}}^{\text{an}})^{\tau}$ is a locally free A -module of rank r equal to the rank of the crystal $\underline{\mathcal{F}}$; and (iii) the ring $\text{End}_{\text{crys}}(\underline{\mathcal{F}}^{\text{an}})$ is an A -subalgebra of $\text{End}_A(P)$, by combining Remark 2.18 and Propositions 4.4 and 4.13. Hence for any T_x its characteristic polynomial CharPol_{T_x} is well defined and monic of degree r with coefficients in A . On the other hand, for $x \in X \setminus (S \cup S_{\mathfrak{p}}^{\mathcal{F}})$ one has the characteristic polynomial $\text{CharPol}_{\rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}(\text{Frob}_x)} \in A_{\mathfrak{p}}[z]$. The two can be compared via Theorem 4.17.

PROPOSITION 5.3. *Let $(\underline{\mathcal{F}}, \mathbb{T})$ be a Hecke crystal. Then for all $\mathfrak{p} \in \text{Max}(A) \setminus T$ and for all $x \in X \setminus (S \cup S_{\mathfrak{p}}^{\mathcal{F}})$ one has*

$$\text{CharPol}_{\rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}(\text{Frob}_x)} = \text{CharPol}_{T_x} \in A[z].$$

Moreover, $\underline{\mathcal{F}}$ is Galois-abelian, cf. Definition 3.12.

Proof. The only assertion which requires proof, in that it is not directly implied by Theorem 4.17, is the final one, i.e., that the $\rho_{\underline{\mathcal{F}}, \mathfrak{p}^n}$ are abelian for all $n \in \mathbb{N}$ and all $\mathfrak{p} \in \text{Max}(A) \setminus T$. The definition of a Hecke crystal implies that the image of \mathbb{T} in $\text{End}_{A/\mathfrak{p}^n}(\underline{\mathcal{F}}_{\mathfrak{p}^n}^{\text{ét}})$ defines an abelian subalgebra. By (5), the endomorphisms $\text{Frob}_x, x \in X \setminus S$, lie in this subalgebra, and hence they commute. By the Čebotarov density theorem, e.g. [Ros02, Theorem 9.13A], any element of the finite Galois group $\text{Gal}(K_{\mathcal{F}, \mathfrak{p}^n})$ is of the form Frob_x for x in a subset of X of positive density. Hence $\rho_{\underline{\mathcal{F}}, \mathfrak{p}^n}$ is abelian. \square

PROPOSITION 5.4. *Let $(\underline{\mathcal{F}}, \mathbb{T})$ be a Hecke crystal. Then there exists $a \in A \setminus \{0\}$ such that over $A' = A[1/a]$ we have $\underline{\mathcal{F}}_i \in \mathbf{Crys}^{\text{flat}}(X, A')$ such that*

$$\underline{\mathcal{F}} \otimes_A A' \cong \bigoplus_i \underline{\mathcal{F}}_i,$$

where the $\underline{\mathcal{F}}_i$ are invariant under \mathbb{T} and $\text{CharPol}_{T_x|_{\underline{\mathcal{F}}_i}} \in A[z]$ is a power of an irreducible polynomial for all $x \in X \setminus S$.

Note that since A is normal and thus integrally closed, any monic irreducible factor in $F[z]$ of the monic polynomial $\text{CharPol}_{T_x} \in A[z]$ will lie in $A[z]$.

Proof. The argument proceeds by induction. As long as the characteristic polynomial of some T_x contains two relatively prime factors, we need to break up $\underline{\mathcal{F}}$ into a direct sum, so that the

summands correspond to the prime power decomposition of CharPol_{T_x} . In each step, we may need to invert another element of $A \setminus \{0\}$. Now, each factor has a generic rank, and the sum of these will be the generic rank of $\underline{\mathcal{F}}$. Hence this inductive procedure will terminate after finitely many steps, completing the proof of the proposition. We now give details on the induction step.

Suppose that $\text{CharPol}_{T_x} = f_1^{\ell_1} \cdot \dots \cdot f_s^{\ell_s} \in A[z]$ for monic pairwise distinct irreducible polynomials $f_i \in A[t]$. We would like to decompose $\underline{\mathcal{F}}$ by the use of idempotents corresponding to the $f_i^{\ell_i}$ as provided by the Chinese remainder theorem over $F[z]$. To clear denominators, we will have to invert some non-zero $a \in A$. More concretely, choose $e_i \in F[z]$ such that e_i is divisible by $\prod_{j \neq i} f_j^{\ell_j}$ and $e_i \equiv 1 \pmod{f_i^{\ell_i}}$. Then

$$\sum_i e_i \equiv 1 \pmod{\prod_j f_j^{\ell_j}}.$$

Let $a \in A$ be non-zero such that $ae_i \in A[z]$ for all i . Then all e_i lie in $A'[z]$ where $A' = A[1/a]$. Define the crystal $\underline{\mathcal{F}}_i$ as $e_i(T_x)(\underline{\mathcal{F}} \otimes_A A')$. Since $e_i e_j = \delta_{ij} e_i$, we find $\underline{\mathcal{F}} \otimes_A A' \cong \bigoplus \underline{\mathcal{F}}_i$ in $\mathbf{Crys}(X, A')$. In particular, the $\underline{\mathcal{F}}_i$ will be flat and also uniformizable (the idempotents are defined over A'). Since the operators $T_{x'}$ commute, we have

$$T_{x'}(\underline{\mathcal{F}}_i) = T_{x'} e_i(T_x)(\underline{\mathcal{F}} \otimes_A A') = e_i(T_x) T_{x'}(\underline{\mathcal{F}} \otimes_A A') \subset e_i(T_x)(\underline{\mathcal{F}} \otimes_A A') = \underline{\mathcal{F}}_i.$$

By its very definition, $f_i^{\ell_i}(T_x)e_i(T_x)$ is a multiple of $\text{CharPol}_{T_x}(T_x)$ and hence zero. This shows in particular that the characteristic polynomial of $T_x|_{\underline{\mathcal{F}}_i}$ is a power of f_i . But the characteristic polynomial of T_x is the product of the characteristic polynomials of its restrictions to the $\underline{\mathcal{F}}_i$, and since the f_i are pairwise relatively prime, we deduce $\text{CharPol}_{T_x|_{\underline{\mathcal{F}}_i}} = f_i^{\ell_i}$. This completes the induction step. \square

Let $(\underline{\mathcal{F}}, \mathbb{T})$ be a Hecke crystal. The action of \mathbb{T} on $\underline{\mathcal{F}}$ induces an A -linear action on the finitely generated projective A -module $P := (\underline{\mathcal{F}}^{\text{an}})^\tau$. Moreover, the decomposition of $\underline{\mathcal{F}} \otimes_A A[1/a] = \bigoplus_i \underline{\mathcal{F}}_i$ from Proposition 5.4 induces a decomposition $P \otimes_A A[1/a] = \bigoplus_i P_i$ of $A[1/a]$ -modules for $P_i := (\underline{\mathcal{F}}_i^{\text{an}})^\tau$. Fix some i and some $x \in X$. By construction of the $\underline{\mathcal{F}}_i$, the characteristic polynomial of T_x on P_i is the power of an irreducible polynomial $f_{x,i} \in A[z]$. By some elementary linear algebra, using the commutativity of \mathbb{T} , one can show for each i the existence of a non-trivial \mathbb{T} -invariant subspace \tilde{P}_i of P such that each T_x has minimal polynomial $f_{x,i}$ for its action on \tilde{P}_i . Conversely one can show that if \tilde{P} is a non-trivial \mathbb{T} -invariant subspace of P such that each T_x has minimal polynomial an irreducible polynomial $g_x \in A[z]$ for its action on \tilde{P} , then there exists some i such that $g_x = f_{x,i}$ for all $x \in X$. The following proposition gives a similar result directly for $\underline{\mathcal{F}}$.

PROPOSITION 5.5. *Let $(\underline{\mathcal{F}}, \mathbb{T})$ be a Hecke crystal, and let A' and $\underline{\mathcal{F}}_i \in \mathbf{Crys}^{\text{flat}}(X, A')$ be as in Proposition 5.4. Then there exist a dense open immersion $j : U \hookrightarrow X$ and subcrystals $\underline{\mathcal{G}}_i \in \mathbf{Crys}^{\text{flat}}(U, A)$ of $j^* \underline{\mathcal{F}}$ such that each $\underline{\mathcal{G}}_i$ is invariant under \mathbb{T} , is annihilated by $\text{CharPol}_{T_x|_{\underline{\mathcal{F}}_i}}$, is uniformizable and satisfies $j^* \underline{\mathcal{F}}_i = \underline{\mathcal{G}}_i \otimes_A A'$.*

Proof. In the following, we also denote by $\underline{\mathcal{F}}$ a locally free τ -sheaf that represents the same-named crystal. Following the proof of Proposition 5.4, there exist idempotents $e_i \in \mathbb{T} \otimes_A A'$ such that $\underline{\mathcal{F}}_i = e_i \underline{\mathcal{F}} \otimes_A A'$ for all i . This implies that $\underline{\mathcal{F}}_i = \text{Ker}(1 - e_i)$ for $1 - e_i \in \text{End}_{\text{crys}}(\underline{\mathcal{F}} \otimes_A A')$. Choose $h_i \in \mathbb{T}$ and $m \in \mathbb{Z}_{\geq 0}$ such that $e_i = a^{-m} h_i$ for all i , so that $\underline{\mathcal{F}}_i = \text{Ker}(a^m - h_i)$. We represent each h_i by a morphism $h_i : (\sigma^{n_i} \times \text{id})^* \underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}$ for some $n_i \geq 0$. The τ -sheaves

$$\underline{\mathcal{G}}_i := \text{Ker}(h_i - a^m \tau^{n_i}) : (\sigma^{n_i} \times \text{id})^* \underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}$$

are torsion-free on the regular scheme $X \times \text{Spec } A$. Hence $\underline{\mathcal{G}}_i$ is free over any point of $X \times \text{Spec } A$ of codimension at most one, and thus there exists $j : U \hookrightarrow X$ dense open such that all $\underline{\mathcal{G}}_i$ are free over $U \times \text{Spec } A$. In particular, $j^* \underline{\mathcal{G}}_i$ is flat as a crystal, and thus has a rank as a crystal. Since $\underline{\mathcal{G}}_i$ is the kernel of some element in \mathbb{T} , the subcrystal $\underline{\mathcal{G}}_i \subset \underline{\mathcal{F}}$ inherits a \mathbb{T} -action from $\underline{\mathcal{F}}$. By construction we have $\underline{\mathcal{G}}_i \otimes_A A' \cong \underline{\mathcal{F}}_i$, and so $\underline{\mathcal{G}}_i$ is annihilated by $\text{CharPol}_{T_x|\underline{\mathcal{F}}_i}$ for all $x \in X$. Moreover, $\underline{\mathcal{G}}_i$ is uniformizable for $K \hookrightarrow L$. From $\underline{\mathcal{F}} \otimes_A A' \cong \bigoplus_i \underline{\mathcal{F}}_i$ we deduce that the $\underline{\mathcal{F}}_i \in \mathbf{Crys}^{\text{flat}}(X, A')$ are uniformizable, i.e., that

$$\text{rank}_{A'} \text{Hom}_{\mathbf{Crys}^{\text{an}}}(\underline{\mathbb{1}}_{L,A}^{\text{an}}, \underline{\mathcal{G}}_i^{\text{an}} \otimes_A A') = \text{rank } \underline{\mathcal{G}}_i = \text{rank } \underline{\mathcal{F}}_i.$$

By multiplying a maximal linear independent subset of $\text{Hom}_{\mathbf{Crys}^{\text{an}}}(\underline{\mathbb{1}}_{L,A}^{\text{an}}, \underline{\mathcal{G}}_i^{\text{an}} \otimes_A A')$ by a suitable power of a , we obtain a linear independent subset of $\text{Hom}_{\mathbf{Crys}^{\text{an}}}(\underline{\mathbb{1}}_{L,A}^{\text{an}}, \underline{\mathcal{G}}_i^{\text{an}})$ of rank \mathcal{G}_i elements, and so $\underline{\mathcal{G}}_i$ is uniformizable. □

Remark 5.6. Following the inductive procedure in the proof of Proposition 5.4, one can give a more explicit description of the crystals $\underline{\mathcal{G}}_i$ constructed in the proof of Proposition 5.5: if $f_{x,i}^{d_{x,i}}$ is the minimal polynomial of T_x acting on $\underline{\mathcal{F}}_i$, then $\underline{\mathcal{G}}_i = \bigcap_x \text{Ker}(f_{x,i}^{d_{x,i}}(T_x))$ where in the latter expression we regard $f_{x,i}^{d_{x,i}}(T_x)$ as an endomorphism of $\underline{\mathcal{F}}$ and thus $\text{Ker}(f_{x,i}^{d_{x,i}}(T_x))$ as a subcrystal of $\underline{\mathcal{G}}$.

Remark 5.7. Let $(\underline{\mathcal{F}}, \mathbb{T})$ be a Hecke crystal. Since the action of \mathbb{T} is faithful on $P = (\underline{\mathcal{F}}^{\text{an}})^{\tau}$, which is finitely generated and projective over A , there exists a finite extension F' of F such that for all $x \in X$ the eigenvalues of T_x lie in the normalization A' of A in E . The minimal choice for F' is the splitting field of all Hecke polynomials over F . Thus for $\underline{\mathcal{G}} := \underline{\mathcal{F}} \otimes_A A'$ the polynomial $\text{CharPol}_{T_x|\underline{\mathcal{G}}}$ splits into linear factors in $F'[z]$ for all $x \in X$, and if we apply Proposition 5.5 to the Hecke crystal $(\underline{\mathcal{G}}, \mathbb{T} \otimes_A A')$ then $\text{CharPol}_{T_x|\underline{\mathcal{F}}_i}$ is a power of a linear polynomial in $F'[z]$ for all i and $x \in X$.

We now prove the following central technical result.

THEOREM 5.8. *Let $(\underline{\mathcal{F}}, \mathbb{T})$ be a Hecke crystal of rank r and $P = (\underline{\mathcal{F}}^{\text{an}})^{\tau}$. Let F'/F be a finite normal extension over which all eigenvalues of all T_x , $x \in X \setminus S$, considered as elements of $\text{End}_A(P)$, are defined. Let $f_1, \dots, f_r \in P \otimes_A F'$ be a system of simultaneous linearly independent eigenvectors for the semisimplified action of $\mathbb{T} \otimes_A F'$ on $P \otimes_A F'$.⁶ Let $\chi_1, \dots, \chi_r : \mathbb{A}_K^* \rightarrow F^{\text{alg}}$ be Hecke characters as in Theorem 3.13 for the strictly compatible abelian system $(\rho_{\underline{\mathcal{F}}, \mathfrak{p}^{\infty}}^{\text{ss}})$, i.e., so that for all places v of F^{alg} above \mathfrak{p} of $\text{Max}(A) \setminus T$, the representations $\rho_{\underline{\mathcal{F}}, \mathfrak{p}^{\infty}}^{\text{ss}}$ and $\bigoplus_{i=1}^r \rho_{\chi_i, \mathfrak{p}^{\infty}}$ are conjugate within the group $\text{GL}_r((F^{\text{alg}})_v)$.*

Then all χ_i can be defined over F' , and after permuting the χ_i , writing ϖ_x for a uniformizer of K_x , we have

$$T_x f_j = \chi_j(1, \dots, 1, \varpi_x, 1, \dots, 1) f_j \quad \forall j = 1, \dots, r \text{ and } x \in X \setminus S.$$

Proof. Let A' be the integral closure of A in F' . Then as in Remark 5.7, all T_x are triagonalizable in $\text{End}_{F'}(P \otimes_A F')$ for $P = (\underline{\mathcal{F}}^{\text{an}})^{\tau}$. Since the assertion of the theorem is not affected by replacing A by A' , we may assume $F = F'$ and $A = A'$ from the start. The uniqueness property of a strictly compatible system does not change under the passage $A \rightarrow A[1/a]$, i.e., if we remove finitely

⁶ That is, we replace the finite dimensional F' vector space $P \otimes_A F'$ by the direct sum over its Jordan–Hölder factors as a module over the finite F' -algebra $\mathbb{T} \otimes_A F'$, where the factors occur according to their multiplicity.

many points from $\text{Max}(A)$. Hence we may apply Proposition 5.4 over $A[1/a]$ to decompose $\underline{\mathcal{F}}$ into isotypical factors for the action of \mathbb{T} . Thus we may assume that $\underline{\mathcal{F}}$ is a single isotypical factor and that all $\text{CharPol}_{T_x|\underline{\mathcal{F}}}$ are powers of a linear polynomial $(z - \lambda_x)$.

Denote by r the rank of $\underline{\mathcal{F}}$ and by $\mathbf{1}_r \in \text{GL}_r(A_{\mathfrak{p}})$ the identity matrix. By Proposition 5.3 we have

$$\rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}^{\text{ss}}(\text{Frob}_x) = \lambda_x \mathbf{1}_r.$$

Now Theorem 3.13 yields a Hecke character χ with values in a finite extension E of F such that $\chi^{\oplus r}$ gives rise to the system $(\rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}^{\text{ss}})$, i.e., such that

$$\rho_{\underline{\mathcal{F}}, \mathfrak{p}^\infty}^{\text{ss}}(\text{Frob}_x) = \rho_{\chi, v}(\text{Frob}_x) \mathbf{1}_r$$

for v a place of E above a place $\mathfrak{p} \in \text{Max}(A) \setminus T$ and for $x \in X \setminus (S_{\mathfrak{p}}^{\mathcal{F}} \cup S)$. The way a Galois representation is attached to a Hecke character implies that

$$\lambda_x = \chi(1, \dots, 1, \varpi_x, 1, \dots, 1) \in F \subset E.$$

It is a *general fact* on Hecke characters that from the above it follows that χ takes its values in F and this completes the proof of the theorem.

In lack of a reference of this *general fact*, we indicate an argument: for $x \in \overline{X}$, let $\text{rec}_x : K_x^* \rightarrow G_{K_x}^{\text{ab}}$ be the reciprocity homomorphism of local class field theory and $\iota_x : G_{K_x}^{\text{ab}} \rightarrow G_K^{\text{ab}}$ the embedding of the local abelianized absolute Galois group at x into the global one. Extending the argument in [Böc13, Lemma 2.18] in an obvious way to the ramified places in S , one obtains $\chi(1, \dots, 1, \alpha_x, 1, \dots, 1) = \rho_{\chi, v}(\iota_x \circ \text{rec}_x(\alpha_x))$ for any $x \in \overline{X}$ and any non-zero $\alpha_x \in K_x$; we assume $v \notin T$. We wish to also show that $\chi(1, \dots, 1, \alpha_{x'}, 1, \dots, 1) \in F^*$ at all places $x' \in S$. If not, then we can find at least one $v \notin T$ such that $\chi(1, \dots, 1, \alpha_{x'}, 1, \dots, 1) \in E_{v'}/F_v$ for some place v' of E above v . In fact, by compactness of G_K , the image will lie in $\mathcal{O}_{E_{v'}}^* \setminus \mathcal{O}_{F_v}^*$. Now, note that $\rho_{\chi, v}$ is unramified outside $S \cup S_v$ and that the set $\{\text{Frob}_x \mid x \notin S \cup S_v\}$ is dense in the maximal quotient of G_K that is unramified outside $S \cup S_v$. By hypothesis, the $\rho_{\chi, v}(\text{Frob}_x)$ take their values in $\mathcal{O}_{F_v}^*$. It follows that $\rho_{\chi, v}(G_{K_{x'}})$ modulo $\mathfrak{m}_{E_{v'}}^c$ is contained in $\mathcal{O}_{F_v}^*/(1 + \mathfrak{m}_{F_v}^c)$ for any $c \geq 0$. This contradicts $\chi(1, \dots, 1, \alpha_{x'}, 1, \dots, 1) \in E_{v'}/F_v$. \square

6. The Hecke crystal for rank two Drinfeld modular forms

In this section we will recall our main example(s) of a Hecke crystal: the crystal of [Böc04] giving rise to the space of rank two Drinfeld (double) cusp forms of fixed weight and level. Applying the main result of the previous section yields the theorem announced in the introduction.

From now on we further restrict the notation, in addition to Conventions 2.6 and 4.3. We denote by X a smooth projective geometrically irreducible curve over \mathbb{F}_q , by ∞ a fixed closed point on X and by C the affine curve $C = X \setminus \{\infty\}$. Then A , with $\text{Spec } A = C$, is a Dedekind domain with finite unit (and class) group. We write F for both $\text{Frac}(A) = \mathbb{F}_q(X)$. The completion of F at the place ∞ will be F_∞ . By \mathfrak{n} we denote a proper non-zero ideal of A , and we define $X_{\mathfrak{n}} = X \setminus (\{\infty\} \cup \text{Spec}(A/\mathfrak{n}))$ and denote by $|X_{\mathfrak{n}}|$ its set of closed points.

Define $\mathcal{K}(\mathfrak{n}) \subset \text{GL}_2(\hat{A})$ as the subgroup of matrices that reduce to the identity modulo \mathfrak{n} . Then $\mathfrak{X}_{\mathcal{K}(\mathfrak{n})}$ is the smooth compactification over F of the moduli space of rank two Drinfeld modules with a full level \mathfrak{n} structure, see [Dri76, Gos80a] or [Böc04]. Moreover, \mathfrak{n} is the conductor of $\mathcal{K}(\mathfrak{n})$. Let $k \geq 2$ be an integer. We define $S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty) = S_{k,0}(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$ and note that, since \mathfrak{n} is non-trivial, one has $S_{k,\ell}(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty) \cong S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$ for all ℓ , cf. [Böc04, Lemma 5.32]. Unlike

in the classical case, the space $S_k^{(2)}(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$ of so-called double cusp forms, i.e., of modular forms vanishing to order 2 at all cusps, is preserved under the Hecke action, cf. [Gos80c].

The following theorem summarizes some of the main results of [Böc04].

THEOREM 6.1. *There exists a Hecke crystal $\underline{\mathcal{S}}_k(\mathfrak{n})$ on $X_{\mathfrak{n}}$ which is uniformizable at ∞ , i.e., for $F \hookrightarrow L = F_\infty$, and has the following property: the projective finitely generated A -module $(\underline{\mathcal{S}}_k(\mathfrak{n})^{\text{an}})^\tau$ when tensored over A with \mathbb{C}_∞ is, as a Hecke module, dual to the space $S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$.*

Proof. By [Böc04, Corollary 10.13], the crystal $\underline{\mathcal{S}}_k(\mathfrak{n})$ is uniformizable. The proof of [Böc04, Proposition 13.4] shows that $\underline{\mathcal{S}}_k(\mathfrak{n})$ has a locally free representative on $X_{\mathfrak{n}} \times C$, and, in particular, $\underline{\mathcal{S}}_k(\mathfrak{n})$ is a flat A -crystal on $X_{\mathfrak{n}}$. Using Corollary 2.19, we choose a good representative $\mathcal{F} \in \mathbf{Coh}_\tau(U, A)$ of $\underline{\mathcal{S}}_k(\mathfrak{n})$ on a dense open subset $U \subset X_{\mathfrak{n}}$. Let $T \subset \text{Spec } A$ be the defect set of those $\mathfrak{p} \in \text{Max}(A)$ such that $\rho_{\underline{\mathcal{S}}_k(\mathfrak{n}), \mathfrak{p}}$ is not lisse, and define for $\mathfrak{p} \notin T$ the set $S_{\mathfrak{p}}^{\mathcal{F}} \subset X_{\mathfrak{n}}$ as the projection onto $X_{\mathfrak{n}}$ of the intersection of $X_{\mathfrak{n}} \times \{\mathfrak{p}\}$ with the support of the cokernel of τ on $\underline{\mathcal{S}}_k(\mathfrak{n}) \otimes_A A/\mathfrak{p}$. We note that if U is taken maximal, then $X_{\mathfrak{n}} \setminus U$ is precisely the intersection of the sets $S_{\mathfrak{p}}^{\mathcal{F}}$ for all $\mathfrak{p} \notin T$.

We now explain relation (5), needed for a Hecke crystal. For $x \in X_{\mathfrak{n}}$ we consider the base change $i_x^* \underline{\mathcal{S}}_k(\mathfrak{n})$ and denote by τ_x the action on it induced by $\tau_{\underline{\mathcal{S}}_k(\mathfrak{n})}$. Then [Böc04, Theorem 13.10] states that the action on $i_x^* \underline{\mathcal{S}}_k(\mathfrak{n})$ induced from T_x on $\underline{\mathcal{S}}_k(\mathfrak{n})$ satisfies

$$T_x = \tau_x^{d_x} : i_x^* \underline{\mathcal{S}}_k(\mathfrak{n}) \rightarrow i_x^* \underline{\mathcal{S}}_k(\mathfrak{n}), \tag{6}$$

where we note that the d_x -fold iterate of τ_x is $(k_x \otimes A)$ -linear. By the proof of [BP09, Theorem 10.6.3], one has $\tau_x^{d_x} = \text{Frob}_x$ under the natural isomorphism $k_x^{\text{sep}} \otimes_{\mathbb{F}_q} \underline{\mathcal{S}}_k(\mathfrak{n})_{\mathfrak{p}^n, \bar{x}}^{\text{ét}} \xrightarrow{\cong} (i_{\bar{x}}^* \underline{\mathcal{S}}_k(\mathfrak{n}) \otimes_A A/\mathfrak{p}^n)_{\text{ss}}$ for $i_{\bar{x}} : \bar{x} \rightarrow X_{\mathfrak{n}}$ a geometric point above x and $(\dots)_{\text{ss}}$ a canonical τ -sheaf representative of (\dots) given by [BP09, Proposition 9.3.4]. By the definition of $S_x^{\mathcal{F}}$, the specialization map $\underline{\mathcal{S}}_k(\mathfrak{n})_{\bar{\eta}, \mathfrak{p}^n}^{\text{ét}} \rightarrow \underline{\mathcal{S}}_k(\mathfrak{n})_{\bar{x}, \mathfrak{p}^n}^{\text{ét}}$ is a G_{F_x} -isomorphism for $\bar{\eta}$ a geometric point above $\eta = \text{Spec } F \in X_{\mathfrak{n}}$. And because $\mathfrak{p} \notin T$, we have $\text{rank } \underline{\mathcal{S}}_k(\mathfrak{n}) = \text{rank}_{A/\mathfrak{p}^n} \underline{\mathcal{S}}_k(\mathfrak{n})_{\eta, \mathfrak{p}^n}^{\text{ét}} (F^{\text{sep}})$. This yields the required compatibility of actions

$$\text{Frob}_x = T_x \in \text{End}_{A/\mathfrak{p}^n}(\underline{\mathcal{S}}_k(\mathfrak{n})_{\mathfrak{p}^n}^{\text{ét}}) \quad \forall n \in \mathbb{N} \tag{7}$$

for $\mathfrak{p} \notin T$ and $x \notin S_{\mathfrak{p}}^{\mathcal{F}}$, which we also refer to as the *Eichler–Shimura relation*. Finally, the Hecke-equivariant isomorphism $((\underline{\mathcal{S}}_k(\mathfrak{n})^{\text{an}})^\tau)^\vee \otimes_A \mathbb{C}_\infty \cong S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$ is [Böc04, Theorems 10.3, 13.2]. \square

Remark 6.2. In Corollaries 8.15 and 8.16 we shall prove: (i) the defect set $T \subset C = \text{Spec } A$ is empty, i.e., that $\rho_{\underline{\mathcal{S}}_k(\mathfrak{n}), \mathfrak{p}}$ is lisse for all $\mathfrak{p} \in \text{Max}(A)$; and (ii) the maximal U possible in the above proof is the set of those $x \in X_{\mathfrak{n}}$ at which $\mathfrak{X}_{\mathfrak{n}}$ has good ordinary reduction.

Remark 6.3. Another main result of [Böc04] asserts that there is also a Hecke crystal $\underline{\mathcal{S}}_k^{(2)}(\mathfrak{n})$ having the same properties as $\underline{\mathcal{S}}_k(\mathfrak{n})$: however, where in the last condition (c) one has a Hecke-isomorphism to the dual of the space of double cusp forms of weight k and level \mathfrak{n} . In addition to the results quoted above, this also needs [Böc04, Theorem 12.3].

Remark 6.4. It should be possible to prove that $\underline{\mathcal{S}}_k(\mathfrak{n})^{\text{max}}$ is a representative of the crystal $\underline{\mathcal{S}}_k(\mathfrak{n})$. However this is not proved in [Böc04]. This would give a natural locally free representative on $X_{\mathfrak{n}}$. We note though that this representative is relevant neither for $(\underline{\mathcal{S}}_k(\mathfrak{n})^{\text{an}})^\tau$ nor for the compatible system $(\rho_{\underline{\mathcal{S}}_k(\mathfrak{n}), \mathfrak{p}^\infty})_{\mathfrak{p}}$, since both depend only on the generic fiber $\underline{\mathcal{S}}_k(\mathfrak{n})_\eta$.

Remark 6.5. Now let $k = 2$ and extend $\mathfrak{X}_{\mathcal{K}(\mathfrak{n})}$ to a smooth projective scheme over $X_{\mathfrak{n}}$ with structure morphism $f_{\mathfrak{n}}$. Let $j_{\mathfrak{n}} : \mathfrak{Y}_{\mathcal{K}(\mathfrak{n})} \hookrightarrow \mathfrak{X}_{\mathcal{K}(\mathfrak{n})}$ denote the open immersion of the moduli space $\mathcal{Y}_{\mathcal{K}(\mathfrak{n})}$ of rank two Drinfeld modules with a full level \mathfrak{n} structure into its compactification. Then by [BP09, Proposition 6.4.10] one has $\underline{\mathcal{S}}_2(\mathfrak{n}) = \underline{\mathcal{G}} \otimes_{\mathbb{F}_p} A$ for $\underline{\mathcal{G}} := R^1 f_{\mathfrak{n}*}(j_{\mathfrak{n}!} \underline{1}_{X_{\mathfrak{n}}, \mathbb{F}_p})$ because the functors pushforward and coefficient change commute. Since $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{\text{ét}}$ commutes with pushforward and extension by zero, we have $\underline{\mathcal{G}}^{\text{ét}} = R^1_{\text{ét}} f_{\mathfrak{n}*} j_{\mathfrak{n}!} \mathbb{F}_p$. Thus for $k = 2$, the Eichler–Shimura relation is

$$\text{Frob}_x = T_x \in \text{End}_{\mathbb{F}_p}(H^1_{\text{ét}}(X/F^{\text{sep}}, j_{\mathfrak{n}!} \mathbb{F}_p)), \tag{8}$$

and one obtains (7) by coefficient change $\otimes_A A/\mathfrak{p}^n$ for all x in a dense open subset $U \subset X$ that is independent of \mathfrak{p} . By Theorem 3.1, one may take as U the subset of those $x \in X_{\mathfrak{n}}$ at which the specialization map $H^1_{\text{ét}}(X/F^{\text{sep}}, j_{\mathfrak{n}!} \mathbb{F}_p) \rightarrow H^1_{\text{ét}}(X/k_x^{\text{sep}}, j_{\mathfrak{n}!} \mathbb{F}_p)$ is an isomorphism of \mathbb{F}_p vector spaces. Similarly $\underline{\mathcal{S}}_2^{(2)}(\mathfrak{n}) = \underline{\mathcal{H}} \otimes_{\mathbb{F}_p} A$ for $\underline{\mathcal{H}} := R^1 f_{\mathfrak{n}*} \underline{1}_{X_{\mathfrak{n}}, \mathbb{F}_p}$, so that $\underline{\mathcal{H}}^{\text{ét}} = R^1 f_{\mathfrak{n}*} \mathbb{F}_p$, and thus (8) holds without $j_{\mathfrak{n}!}$. Moreover, from [Böc04, Theorem 14.8] one easily deduces that as Hecke modules one has⁷

$$H^1(\mathfrak{X}_{\mathfrak{n}}/F^{\text{sep}}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{C}_{\infty} \cong S_2^{(2)}(\mathcal{K}(\mathfrak{n}), \mathbb{C}_{\infty})^{\vee}. \tag{9}$$

Remark 6.6. Due to the known shape of the Hecke algebra for Drinfeld cusp forms on the space of rank r Drinfeld modules (with some level structure), we expect that the analog of Theorem 6.1 holds for any $r \in \mathbb{N}$. The case $r = 1$ is rather easy but unpublished. Recent work [Pin13] of Pink make the cases $r \geq 3$ accessible.

COROLLARY 6.7. *Let f be a cuspidal rank two Drinfeld A -modular Hecke eigenform of weight k and full level \mathfrak{n} . Then there is a unique Hecke character χ_f associated to f , such that for almost all $x \in |X_{\mathfrak{n}}|$ one has*

$$T_x f = \chi_f(1, \dots, 1, \varpi_x, 1, \dots, 1) f.$$

Proof. We apply Theorem 5.8 to the crystal $\underline{\mathcal{S}}_k(\mathfrak{n})$ of the previous theorem. This yields Hecke characters χ_j associated to the f_j defined in Theorem 5.8, such that the inverses of the χ_j have the above property for f_j . Now observe that $A' \subset F'$ both embed into \mathbb{C}_{∞} . Hence the images of the f_j in $(P \otimes_{A'} F') \otimes_{F'} \mathbb{C}_{\infty}$, with $P = (\underline{\mathcal{S}}_k(\mathfrak{n})^{\text{an}})^{\tau}$, form a basis of the dual of $S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_{\infty})$ consisting of eigenvectors of the semisimplified action of the Hecke algebra acting on $S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_{\infty})$. It follows that the eigenvalue system of any Hecke eigenform is equal to the eigenvalue system of some f_j , and the proof of the corollary is complete. \square

7. An example

This section illustrates Corollary 6.7 by an explicit example based on [Böc14, Sec. 10.7]. Let $A = \mathbb{F}_q[T]$ and $\mathfrak{n} = (T)$. To distinguish the coefficient ring A from the base $X = X_{\mathfrak{n}}$, introduced at the beginning of §6, we use the variable θ on the base, so that $X = \text{Spec } \mathbb{F}_q[\theta, \theta^{-1}]$. Note that $X \times \text{Spec } A = \mathbb{F}_q[\theta, \theta^{-1}, T]$. In [Böc14, Proposition 10.29], a filtration of the Hecke crystal $\underline{\mathcal{S}}_k^{(2)}((T))$ is computed.

PROPOSITION 7.1. *Let the notation be as above. Define*

$$F_{k,j} \left(\frac{T}{\theta} \right) = \sum_{\ell=0}^{\min\{j, k-2-j\}} \binom{j}{\ell} \binom{k-2-j}{\ell} \left(\frac{T}{\theta} \right)^{\ell}.$$

⁷ The isomorphism (9) holds for any level \mathcal{K} with target $S_{2,1}^{(2)}(\mathcal{K}, \mathbb{C}_{\infty})^{\vee}$, forms of weight 2 and type 1; cf. [GR96, (6.5)].

Then for $4 \leq k \leq q + 2$ the crystal $\underline{\mathcal{S}}_k^{(2)}((T))$ has a filtration whose associated graded object is isomorphic to

$$\left(\bigoplus_{j=1}^{k-3} \left(\mathcal{O}_{X \times \text{Spec } A}, F_{k,j} \left(\frac{T}{\theta} \right) (\sigma \times \text{id}) \right) \right)^{\oplus q}. \tag{10}$$

There are a number of remarks in order: the expression for $F_{k,j}$ in [Böc04, Proposition 15.3] is $F_{k,j}(T/\theta) = \sum_{\ell=0}^{k-2} \binom{\ell}{j} \binom{j}{k-2-\ell} (T/\theta - 1)^{k-2-\ell}$. The expression we give is from [LM08]. A proof that both these expressions agree can be found in [Böc14, Remark 12.33].

The coefficients of the polynomial $F_{k,j}$ are defined over \mathbb{Z} and are independent of q (and thus also p), subject to the restriction $4 \leq k \leq q + 2$.

The summands occur with multiplicity at least q : this is due to the fact that we work with a full level \mathfrak{n} structure and not a(n enhanced) level $\Gamma_1(\mathfrak{n})$ structure, see the proof of Proposition 8.24. Since (10) is unchanged under $j \mapsto k - 2 - j$, summands with $j \neq k - 2 - j$ occur with multiplicity at least $2q$. But for $q \neq p$ there may be higher multiplicities.

As follows from Theorem 6.1 and Proposition 5.4, each summand corresponds to a Hecke eigenform. Since in general the Hecke action is not semisimple, cf. [LM08], the assignment need not be 1–1. The number of Hecke eigenforms can be much smaller than the number of summands in (10), which is $q(k - 3)$.

Having coefficients in \mathbb{F}_p and constant term one, for suitable $\alpha_{k,j,\ell} \in \mathbb{F}_p^{\text{alg}}$ we obtain

$$F_{k,j} \left(\frac{T}{\theta} \right) = \prod_{\ell=1}^{\min\{j, k-2-j\}} \left(1 - \alpha_{k,j,\ell} \frac{T}{\theta} \right).$$

Now let $\mathbb{A}_{\mathbb{F}_q(\theta)}^{(\infty)}$ denote the finite adeles of $\mathbb{F}_q(\theta)$ with respect to the usual infinite place of $\mathbb{P}_{\mathbb{F}_q}^1$. For \mathfrak{q} a prime ideal of $\mathbb{F}_q[\theta]$, denote by $\mathcal{O}_{\mathfrak{q}}$ the ring of integers of the completion of $\mathbb{F}_q(\theta)$ at \mathfrak{q} and by $\mathcal{O}_{\mathfrak{q}}^1$ those elements of $\mathcal{O}_{\mathfrak{q}}$ reducing to the identity modulo the completion of \mathfrak{q} . Set

$$U_{(\theta)} = \mathcal{O}_{(\theta)}^1 \times \prod_{\mathfrak{q} \neq (\theta)} \mathcal{O}_{\mathfrak{q}}^*.$$

Since $\mathbb{F}_q[\theta]$ is factorial and since the map $\mathbb{F}_q^* = (\mathbb{F}_q[\theta])^* \rightarrow \mathcal{O}_{(\theta)}/\mathcal{O}_{(\theta)}^1$ is an isomorphism, one easily deduces that the diagonal embedding yields isomorphisms

$$(\mathbb{F}_q(\theta))^* \xrightarrow{\cong} \text{GL}_1(\mathbb{A}_{\mathbb{F}_q(\theta)}^{(\infty)})/U_{(\theta)} \xrightarrow{\cong} \mathbb{A}_{\mathbb{F}_q(\theta)}^*/(U_{(\theta)} \times \mathbb{F}_q((\theta^{-1}))). \tag{11}$$

DEFINITION 7.2. Let q be a prime power, $4 \leq k \leq q + 2$ and $1 \leq j \leq k - 3$. For $0 \leq \ell \leq \min\{j, k - 2 - j\}$, define the field embedding

$$\sigma_{k,j,\ell} : \mathbb{F}_q(\theta) \rightarrow \mathbb{F}_q^{\text{alg}}(T) : g(\theta) \mapsto g(\alpha_{k,j,\ell} T).$$

Define the Hecke character

$$\chi_{k,j} : \mathbb{A}_{\mathbb{F}_q(\theta)}^* \rightarrow (\mathbb{F}_q^{\text{alg}}(T))^*$$

by requiring that it is trivial on $U_{(\theta)} \times \mathbb{F}_q((\theta^{-1}))$ and that on $(\mathbb{F}_q(\theta))^*$, embedded diagonally into $\mathbb{A}_{\mathbb{F}_q(\theta)}^*$, it is given by

$$\prod_{\ell=0}^{\min\{j, k-2-j\}} \sigma_{k,j,\ell}.$$

PROPOSITION 7.3. *Suppose $4 \leq k \leq q + 2$ and $1 \leq j \leq k - 3$. The system of Hecke eigenvalues of a Drinfeld modular form corresponding to the crystal*

$$\left(\mathcal{O}_{X \times \text{Spec } A}, F_{k,j} \left(\frac{T}{\theta} \right) (\sigma \times \text{id}) \right)$$

is described by the Hecke character $\chi_{k,j}$ for all prime ideals \mathfrak{q} of $\mathbb{F}_q[\theta]$ different from (θ) .

Proof. Let $f_{k,j}$ be a Hecke eigenform corresponding to the crystal in Proposition 7.1. Let $\mathfrak{q} \neq (\theta)$ be a prime ideal of $\mathbb{F}_q[\theta]$ generated by $g_{\mathfrak{q}}(\theta)$ with $g_{\mathfrak{q}}(0) = 1$, and denote by θ' any root of $g_{\mathfrak{q}}$. Following the derivation of [Böc14, Ex. 10.31] and using (6), one obtains

$$T_{\mathfrak{q}}f = \left(\prod_{\ell=0}^{\min\{j,k-2-j\}} g_{\mathfrak{q}}(\alpha_{k,j,\ell}T) \right) \cdot f$$

by computing the $\deg(\mathfrak{q})$ th iterate of $F_{k,j}(T/\theta')(\sigma \times \text{id})$.

Let us now evaluate the Hecke character $\chi_{k,j}$ at the idele $(1, \dots, 1, \varpi_{\mathfrak{q}}, 1, \dots, 1)$. Under the isomorphism (11) this idele is the image of $g_{\mathfrak{q}} \in (\mathbb{F}_q(\theta))^*$. By the definition of $\chi_{k,j}$, its value at $g_{\mathfrak{q}}$ is

$$\prod_{\ell=0}^{\min\{j,k-2-j\}} \sigma_{k,j,\ell}(g_{\mathfrak{q}}) = \prod_{\ell=0}^{\min\{j,k-2-j\}} g_{\mathfrak{q}}(\alpha_{k,j,\ell}T).$$

This completes the proof of the proposition. □

Remark 7.4. Proposition 7.3 has two interesting features not explained by Corollary 6.7: the conductor of each $\chi_{k,j}$ is equal to the level \mathfrak{n} of the corresponding Drinfeld cusp form and the character $\chi_{k,j}$ computes the Hecke eigenvalues at all primes not dividing \mathfrak{n} . Corollary 8.25 in § 8 will give a theoretical explanation. It asserts that for any cuspidal Hecke eigenform of level dividing a prime of degree at most one and of any weight, the associated Hecke character is unramified outside the level.

8. Ramification

In the remainder of this article we investigate the ramification of \mathfrak{p} -adic Galois representations attached to a cuspidal Drinfeld modular Hecke eigenform f at places x that divide neither the level \mathfrak{n} of f nor the characteristic of the prime \mathfrak{p} . In weight 2, we shall link this to the geometry of the underlying Drinfeld modular curve: We show that if f is ramified at a place x outside \mathfrak{n} , then the p -rank of a corresponding Drinfeld modular curve must strictly decrease under reduction at x . Moreover, in this case, the corresponding Hecke eigenvalue $a_x(f)$ will vanish. For the latter we exploit the relation between function field automorphic forms and weight 2 Drinfeld cusp forms, first noticed in [GR96]. In weight 2 the main results are Proposition 8.7 and Theorem 8.8. In higher weight cases, we shall use congruences. There the main results are Propositions 8.18 and 8.22.

We often restrict our attention to doubly cuspidal Hecke eigenforms. The reason is [Böc04, Corollary 14.7], which states that for any weight $k \geq 2$ one has a surjection $S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_{\infty}) \rightarrow S_k^{(2)}(\mathcal{K}(\mathfrak{n}), \mathbb{C}_{\infty})$ of Hecke modules whose kernel admits a basis of cuspidal Hecke eigenforms g whose associated Galois representations all factor via the ray class field of F of conductor $g\infty$. This implies that for any such Hecke eigenform g the corresponding Hecke character χ_g is unramified outside $\mathfrak{n}\infty$.

We continue with the notation from § 6. We write X for a smooth projective geometrically irreducible curve over \mathbb{F}_q , fix a point $\infty \in |X|$ and set $C = X \setminus \{\infty\} = \text{Spec } A$. We write F for both $\text{Frac}(A) = \mathbb{F}_q(X)$. By $\mathfrak{n} \subset A$ we denote a proper non-zero ideal, and $X_{\mathfrak{n}}$ is $X \setminus (\{\infty\} \cup \text{Spec}(A/\mathfrak{n}))$.

We begin with the following result on p -ranks and ramification that we regard as a complement to the well-known criterion of Néron–Ogg–Shafarevich (NOS). It will be proved in Appendix A and is a simple consequence of [deJ98].

PROPOSITION 8.1 (Theorem A.1). *Let X be a smooth projective curve over a finite field of characteristic p and with function field K . Let J be an ordinary abelian variety over K with good reduction at a place x of X . Then for the representation of G_K on the group $J[p^\infty](K^{\text{sep}})$ of p -power torsion points of J over K^{sep} the following are equivalent:*

- (a) *the representation is unramified at x ;*
- (b) *the abelian variety J has ordinary reduction at x .*

A curve is ordinary if its Jacobian has this property. Therefore the following result is relevant.

THEOREM 8.2 (Gekeler). *For any congruence subgroup $\Gamma \subset \text{GL}_2(A)$, the curve \mathfrak{X}_Γ over its field of definition F_Γ is ordinary, and for any open subgroup $\mathcal{K} \subset \text{GL}_2(\hat{A})$, the curve $\mathfrak{X}_\mathcal{K}$ over $F_\mathcal{K}$ is ordinary.*

The proof is a consequence of the ramification calculations of Drinfeld modular curves in [Gek86, § 5] combined with [Nak87, Theorem 2(i)] and $\mathfrak{X}_\mathcal{K}(\mathbb{C}_\infty) \setminus \mathfrak{C}_\mathcal{K} = \bigsqcup_i \Gamma_i \setminus \Omega$, cf. page 2007; see also [Pin00, Theorem 1.2]. One also has the following well-known result whose proof, in lack of a suitable reference, we indicate.

PROPOSITION 8.3. *Let X be a smooth curve over a finite field of characteristic p with function field K and let \mathcal{C} be a smooth projective curve over X . Suppose that \mathcal{C} is ordinary over the generic point η of X . Then for all but finitely many $x \in X$, the curve \mathcal{C}_x is ordinary.*

Proof. Ordinarity of a smooth projective curve C over a field k of characteristic $p > 0$ can be measured in terms of its p -rank: Denote by φ the endomorphism of $H^1(C, \mathcal{O}_C)$ induced from the absolute Frobenius endomorphism $\sigma_C : \mathcal{O}_C \rightarrow \mathcal{O}_C, a \mapsto a^p$. This is a p -linear endomorphism on the finite-dimensional k vector space $H^1(C, \mathcal{O}_C)$; the rank of the matrix representing φ with respect to any k -basis is independent of the choice of basis and called the p -rank of C . If k is separably closed, the p -rank is equal to the \mathbb{F}_p -dimension of the subspace of $H^1(C, \mathcal{O}_C)$ of fixed points under φ . If C is geometrically irreducible, the p -rank of C is the \mathbb{F}_p -dimension of the module of p -torsion points of the Jacobian of C . A curve is ordinary precisely if its p -rank is equal to $\dim_k H^1(C, \mathcal{O}_C)$. If C is geometrically irreducible, the latter dimension is equal to the genus of C .

Now if $f : \mathcal{C} \rightarrow X$ denotes the structure morphism, then $\mathcal{M} = R^1 f_* \mathcal{O}_\mathcal{C}$ is a locally free \mathcal{O}_X -module whose rank is equal to $\dim_k H^1(C, \mathcal{O}_C)$. The absolute Frobenius σ_C on \mathcal{C} induces an endomorphism $\varphi_\mathcal{M} : \mathcal{M} \rightarrow \mathcal{M}$, and one easily verifies that $\varphi_\mathcal{M}$ is σ_X -linear, i.e., for any sections c of \mathcal{M} and α of \mathcal{O}_X (on the same open U of X), one has $\varphi(\alpha c) = \alpha^p \varphi(c) = \sigma_X(\alpha) \varphi(c)$. By passing from X to an open subset, which suffices for the proof of the assertion, we may assume that \mathcal{M} is a free \mathcal{O}_X -module. Choosing an \mathcal{O}_X -basis of \mathcal{M} , the endomorphism φ can thus be described by a square matrix $\beta_\mathcal{M}$, the Hasse–Witt matrix, whose entries are global sections of \mathcal{O}_X . The fact that \mathcal{C}_η is ordinary means that $\beta_\mathcal{M}$ as a matrix over K has full rank. Now the rank of $\beta_\mathcal{M}$ under reduction can go down at most at finitely many places of X , namely at those where $\det \beta_\mathcal{M}$ vanishes. Hence φ_x has full rank for all but finitely many $x \in X$. But then the p -rank of \mathcal{C}_x is full for all such x . This concludes the proof. □

Remark 8.4. By using minors of the matrix $\beta_{\mathcal{M}}$ in the above proof, one can strengthen Proposition 8.3 as follows: the p -rank of a curve over a function field K remains constant under reduction for all but finitely many places of K (at which it can only decrease).

After some further preparation we will be ready to discuss the ramification properties of Galois representations and Hecke characters attached to weight 2 cuspidal Drinfeld Hecke eigenforms. For this, let $\mathfrak{X}_{\mathcal{K}(n)}$ be the compactified modular curve as in §6, and abbreviate $\mathfrak{X}_n = \mathfrak{X}_{\mathcal{K}(n)}$. The curve \mathfrak{X}_n is smooth projective over F but rarely geometrically connected. Denoting by $\text{Jac}(\mathfrak{X}_n)$ the Jacobian of \mathfrak{X}_n , one has the well-known Galois equivariant isomorphism

$$H_{\text{ét}}^1(\mathfrak{X}_n/F^{\text{sep}}, \mathbb{F}_p) \cong \text{Jac}(\mathfrak{X}_n)[p](F^{\text{sep}})^{\vee}.$$

On the left-hand side one has a Hecke action via correspondences of \mathfrak{X}_n . These induce endomorphisms on $\text{Jac}(\mathfrak{X}_n)$ and thus a Hecke action on the right-hand side. Because of the geometric nature of the two Hecke actions, the displayed isomorphism is Hecke equivariant. We combine this isomorphism with the observations from Remark 6.5 to obtain a Hecke equivariant isomorphism

$$\text{Jac}(\mathfrak{X}_n)[p](F^{\text{sep}}) \otimes_{\mathbb{F}_p} \mathbb{C}_{\infty} \cong S_2^{(2)}(\mathcal{K}(n), \mathbb{C}_{\infty}).$$

We define the Hecke algebra $\mathbb{T}_{\mathbb{Z},n}$ as the \mathbb{Z} -subalgebra of $\text{End}(\text{Jac}(\mathfrak{X}_n))$ generated by the standard Hecke endomorphisms T_x , for $x \in |X_n|$, and let $\mathbb{T}_{\mathbb{F}_p,n}^{(2)}$ be the analogous subalgebra of $\text{End}_{\mathbb{F}_p}(\text{Jac}(\mathfrak{X}_n)[p](F^{\text{sep}}))$. Then $\mathbb{T}_{\mathbb{F}_p,n}^{(2)} \otimes_{\mathbb{F}_p} \mathbb{C}_{\infty}$ is the Hecke algebra of $S_2^{(2)}(\mathcal{K}(n), \mathbb{C}_{\infty})$. It has been observed by Gekeler and Reversat in [GR96, (6.5)] that reduction defines a surjective ring homomorphism

$$\mathbb{T}_{\mathbb{Z},n} \twoheadrightarrow \mathbb{T}_{\mathbb{F}_p,n}^{(2)}.$$

We choose a finite field $\mathbb{F} \supset \mathbb{F}_p$ over which the action of $\mathbb{T}_{\mathbb{F}_p,n}^{(2)}$ on $\text{Jac}(\mathfrak{X}_n)[p](F^{\text{sep}})$ becomes triangularizable. We fix an embedding $\iota : \mathbb{F} \hookrightarrow F^{\text{sep}} \hookrightarrow \mathbb{C}_{\infty}$, and we write $\mathbb{T}_{\mathbb{F},n}^{(2)}$ for $\mathbb{T}_{\mathbb{F}_p,n}^{(2)} \otimes_{\mathbb{F}_p} \mathbb{F}$.

Now let f be a Hecke eigenform in $S_2^{(2)}(\mathcal{K}(n), \mathbb{C}_{\infty})$ with Hecke eigenvalue system $(a_x(f))_x$, where the x range over $|X_n|$. Via ι , this defines a homomorphism $\bar{t}_f : \mathbb{T}_{\mathbb{F},n}^{(2)} \rightarrow \mathbb{C}_{\infty}$. By \bar{V}_f we denote an irreducible subspace of $\text{Jac}(\mathfrak{X}_n)[p](F^{\text{sep}}) \otimes_{\mathbb{F}_p} \mathbb{F}$ on which $\mathbb{T}_{\mathbb{F},n}^{(2)}$ also acts by \bar{t}_f . By the definition of \mathbb{F} , we have $\dim_{\mathbb{F}} \bar{V}_f = 1$. Because of the Eichler–Shimura relation from Remark 6.5, there is a well-defined commutative action $\bar{\rho}_f : G_F \rightarrow \text{Aut}_{\mathbb{F}}(\bar{V}_f)$ and, moreover, we have the following proposition.

PROPOSITION 8.5. *Let f be a Hecke eigenform in $S_2^{(2)}(\mathcal{K}(n), \mathbb{C}_{\infty})$. Then for $\bar{\rho}_f$ defined as above, we have*

$$\bar{\rho}_f(\text{Frob}_x) \stackrel{(8)}{=} a_x(f) \stackrel{\text{Corollary 6.7}}{=} \chi_f(1, \dots, 1, \varpi_x, 1, \dots, 1)$$

for all but finitely many $x \in X_n$.

The existence of $\bar{\rho}_f$ can also be obtained from [GR96] by Gekeler and Reversat or from Appendix B.

Remark 8.6. Let $F' \subset F^{\text{sep}}$ be the field generated by F and \mathbb{F} . Clearly, the above χ_f is defined over F' . Moreover, one can easily show that the compatible system $(\rho_{\chi_f, \mathfrak{P}^{\infty}})_{\mathfrak{P}}$ where \mathfrak{P} ranges over all places of F' can be obtained from $\bar{\rho}_f$ by $\rho_{\chi_f, \mathfrak{P}^{\infty}} := \bar{\rho}_f \otimes_{\mathbb{F}} F'_{\mathfrak{P}}$ where $F'_{\mathfrak{P}}$ denotes the completion of F' at \mathfrak{P} .

We write $\mathrm{Ta}_p \mathrm{Jac}(\mathfrak{X}_n)$ for the p -adic Tate module $\varprojlim_n \mathrm{Jac}(\mathfrak{X}_n)[p^n](F^{\mathrm{sep}})$. Its mod p reduction is $\mathrm{Jac}(\mathfrak{X}_n)[p](F^{\mathrm{sep}})$. By $X_n^{\mathrm{ord}} \subset X_n$ we denote the dense open subscheme of points where \mathfrak{X}_n has good ordinary reduction, cf. Proposition 8.3. Then Proposition 8.1 applied to the present situation gives the following result.

PROPOSITION 8.7. *The G_F -representation $\mathrm{Ta}_p \mathrm{Jac}(\mathfrak{X}_n)$ is unramified at all places $x \in |X_n^{\mathrm{ord}}|$. It is ramified at all places in $X_n \setminus X_n^{\mathrm{ord}}$. In particular, for any doubly cuspidal Hecke eigenform f of weight 2 and minimal level $\mathcal{K}(n)$, the Hecke character χ_f is unramified at all $x \in |X_n^{\mathrm{ord}}|$.*

We note that, by Proposition 8.3 (and its proof), the finite list of good reduction non-ordinary primes of \mathfrak{X}_n can in principle be found by analyzing the Hasse–Witt matrix for \mathfrak{X}_n .

Our next aim is a refinement of Proposition 8.7. For this we define $\mathbb{T}_{\mathbb{Z},n}^{\mathrm{ord}} \subset \mathrm{End}(\mathrm{Jac}(\mathfrak{X}_n))$ as the subring of $\mathbb{T}_{\mathbb{Z},n}$ generated by the T_x for $x \in X_n^{\mathrm{ord}}$. We write $t_f : \mathbb{T}_{\mathbb{Z},n} \rightarrow \mathbb{F}_f$ for the composite of $\mathbb{T}_{\mathbb{Z},n} \rightarrow \mathbb{T}_{\mathbb{F}_p,n}^{(2)}$ with the homomorphism $\bar{t}_f : \mathbb{T}_{\mathbb{F}_p,n}^{(2)} \rightarrow \mathbb{F}_f$. Let \mathfrak{m}_f and $\mathfrak{m}_f^{\mathrm{ord}}$ be the kernels of t_f and of the restriction of t_f to $\mathbb{T}_{\mathbb{Z},n}^{\mathrm{ord}}$. Then $\mathfrak{m}_f^?$ is a maximal ideal of $\mathbb{T}_{\mathbb{Z},n}^?$ for $? \in \{\emptyset, \mathrm{ord}\}$ that contains the prime p . For the completions at p we write $\widehat{\mathbb{T}}_{\mathbb{Z},n}^?$ and $\widehat{\mathfrak{m}}_f^?$, respectively. The completions $\widehat{\mathbb{T}}_{\mathbb{Z},n}^?$ act on the p -divisible group $\mathrm{Jac}(\mathfrak{X}_n)(p)$.

By $\mathfrak{P}_1, \dots, \mathfrak{P}_s$ we denote the minimal primes of $\widehat{\mathbb{T}}_{\mathbb{Z},n}^{\mathrm{ord}}$ contained in $\widehat{\mathfrak{m}}_f^{\mathrm{ord}}$. Then each $\widehat{\mathbb{T}}_{\mathbb{Z},n}^{\mathrm{ord}}/\mathfrak{P}_i$ is an integral domain of finite rank over \mathbb{Z}_p , and thus its fraction field is one of the p -adic completions of the ring of fractions of $\widehat{\mathbb{T}}_{\mathbb{Z},n}^{\mathrm{ord}}$. By strong multiplicity one for automorphic forms for GL_2 , which allows one to disregard Hecke operators at finitely many places, each \mathfrak{P}_i defines a cuspidal automorphic form Φ_i together with a place \mathfrak{p}_i above p of the field of definition E_i of Φ_i ; one can think of E_i as the fraction field of $\widehat{\mathbb{T}}_{\mathbb{Z},n}^{\mathrm{ord}}/\mathfrak{P}_i$ and of \mathfrak{p}_i as the maximal ideal of the normalization of $\widehat{\mathbb{T}}_{\mathbb{Z},n}^{\mathrm{ord}}/\mathfrak{P}_i$ in E_i . By \mathfrak{n}_i we denote the conductor of Φ_i , so that Φ_i is a new form of level \mathfrak{n}_i dividing \mathfrak{n} . It is not excluded that $\Phi_i = \Phi_j$ for different i, j ; however, then we have $\mathfrak{p}_i \neq \mathfrak{p}_j$.

We now apply the results of Appendix B, where we formulate and prove some general results on the representation of G_F on p -adic Tate modules and their mod p reductions. There, for each Φ_i we define a simple abelian variety A_{Φ_i} in the \mathfrak{n}_i -new part of the Jacobian for level $\mathcal{K}_1(\mathfrak{n}_i)$, a field of endomorphisms E_{Φ_i} acting on A_{Φ_i} with $[E_{\Phi_i} : \mathbb{Q}] = \dim A_{\Phi_i}$ and, for each place \mathfrak{p}_i of E_{Φ_i} above p , a representation $\rho_{\Phi_i, \mathfrak{p}_i} : G_F \rightarrow \mathrm{GL}_1((E_{\Phi_i})_{\mathfrak{p}_i})$ unramified outside \mathfrak{n}_i and with $(E_{\Phi_i})_{\mathfrak{p}_i}$ the \mathfrak{p}_i -adic completion of E_{Φ_i} . In fact, one has $E_{\Phi_i} = E_i$. The $\rho_{\Phi_i, \mathfrak{p}_i}$ describe the action of G_F on the \mathfrak{p}_i -component of $\mathrm{Ta}_p A_{\Phi_i}$. They are characterized by $\rho_{\Phi_i, \mathfrak{p}_i}(\mathrm{Frob}_x) = \alpha_{i,x}$ for all $x \in X_{\mathfrak{n}_i}^{\mathrm{ord}}$, where $\alpha_{i,x}$ is the root of slope zero of $g_x(z) = z^2 - T_x z + S_x q_x$ in $(E_{\Phi_i})_{\mathfrak{p}_i}$. Let $A_{\Phi_i}(p)_{\mathfrak{p}_i}$ be the \mathfrak{p}_i -component of the p -divisible group of A_{Φ_i} over the base $X_{\mathfrak{n}_i}$.

Write \mathcal{O}_i for the ring of integers of $(E_i)_{\mathfrak{p}_i}$ and \mathfrak{p}_i also for the maximal ideal of \mathcal{O}_i . Then one has a monomorphism $\widehat{\mathbb{T}}_{\mathbb{Z},n}^{\mathrm{ord}}/\widehat{\mathfrak{m}}_f^{\mathrm{ord}} \hookrightarrow \mathcal{O}_i/\mathfrak{p}_i$ that sends $T_x \pmod{\widehat{\mathfrak{m}}_f^{\mathrm{ord}}}$ to $\alpha_x \pmod{\mathfrak{p}_i}$ for all $x \in |X_n^{\mathrm{ord}}|$. By $X_{\Phi_i, \mathfrak{p}_i}^{\mathrm{ord}}$ we denote the set of $x \in X_{\mathfrak{n}_i}$ at which $A_{\Phi_i/k_x}(p)_{\mathfrak{p}_i}$ has non-trivial étale part. By Corollary B.4, $|X_{\Phi_i, \mathfrak{p}_i}^{\mathrm{ord}}|$ is the set of $x \in |X_{\mathfrak{n}_i}|$ where T_x acts by a unit of \mathcal{O}_i on Φ_i . Note that $X_{\Phi_i}^{\mathrm{ord}} \supseteq X_n^{\mathrm{ord}}$ for all i .

THEOREM 8.8. *Let $\bar{\rho}_f : G_F \rightarrow \mathrm{GL}_1(\mathbb{T}_{\mathbb{Z},n}^{\mathrm{ord}}/\mathfrak{m}_f^{\mathrm{ord}})$ be the Galois representation corresponding to \bar{V}_f from Proposition 8.5. Then the following hold.*

- (a) *The representation $\bar{\rho}_f$ is unramified for $x \in \bigcup_i X_{\Phi_i, \mathfrak{p}_i}^{\mathrm{ord}}$. For $x \in X_{\Phi_i, \mathfrak{p}_i}^{\mathrm{ord}}$ one has*

$$\chi_f((1, \dots, 1, \varpi_x, 1, \dots, 1)) = \bar{\rho}_f(\mathrm{Frob}_x) \equiv T_x \equiv \alpha_{i,x} = \rho_{\Phi_i, \mathfrak{p}_i}(\mathrm{Frob}_x) \pmod{\mathfrak{p}_i}$$

under $\mathbb{T}_{\mathbb{Z},n}^{\mathrm{ord}}/\mathfrak{m}_f^{\mathrm{ord}} \hookrightarrow \mathcal{O}_i/\mathfrak{p}_i$.

- (b) If $\bar{\rho}_f$ is ramified at $x \in X_{n_i}$, then $T_x \equiv 0 \pmod{\mathfrak{p}_i}$ for its action on Φ_i and $x \notin X_{\Phi_i, \mathfrak{p}_i}^{\text{ord}}$.
- (c) There exists an $i \in \{1, \dots, s\}$ such that for all $x \in |X_n|$ the mod \mathfrak{p}_i reduction of the eigenvalue of T_x on Φ_i is equal to the eigenvalue of T_x acting on f . In particular:
 - (i) if $a_x(f) \neq 0$, then the étale part of $A_{\Phi_i/F}(p)_{\mathfrak{m}_f}$ extends to a p -divisible group on $\text{Spec } \mathcal{O}_{X,x}$;
 - (ii) if $a_x(f) = 0$, then $A_{\Phi_i/k_x}(p)_{\mathfrak{m}_f}$ has trivial étale part, and so $x \notin X_{\Phi_i, \mathfrak{p}_i}^{\text{ord}}$.
- (d) At all places $x \notin X_n$ where some Φ_i is semistable, the representation $\bar{\rho}_f$ is unramified.

Proof. By the choice of \mathfrak{P}_i we have for all $x \in |X_n|$ where \mathfrak{X}_n has ordinary reduction that the value of T_x in $\widehat{\mathbb{T}}_{\mathbb{Z},n}^{\text{ord}}/\mathfrak{P}_i$ after reduction modulo $\widehat{\mathfrak{m}}_f^{\text{ord}}$ agree with the value of T_x in $\widehat{\mathbb{T}}_{\mathbb{Z},n}^{\text{ord}}/\widehat{\mathfrak{m}}_f^{\text{ord}}$. On the one hand we have $\bar{\rho}_f(\text{Frob}_x) = T_x \pmod{\mathfrak{m}_f^{\text{ord}}}$ for almost all such x . On the other we have from Theorem B.14(d) that $\rho_{\Phi_i, \mathfrak{p}_i}(\text{Frob}_x) = \alpha_{i,x} \equiv T_x \pmod{\mathfrak{p}_i}$ for all $x \in X_{\Phi_i, \mathfrak{p}_i}^{\text{ord}}$.⁸ Part (a) now follows from the Čebotarev density theorem. The proof of (b) is simple: if $\bar{\rho}_f$ is ramified at x , then by (a) so is $\rho_{\Phi_i, \mathfrak{p}_i}$ for all $i = 1, \dots, s$. By Theorem B.14(f) this implies (b). Part (d) is a direct consequence of Proposition B.6, where we also recall the definition of semistable.

To prove (c), let Φ be a cuspidal automorphic new form corresponding to a minimal prime \mathfrak{P} under \mathfrak{m}_f in $\widehat{\mathbb{T}}_{\mathbb{Z},n}$, and let \mathfrak{p} be the corresponding place of $\text{Frac}(\widehat{\mathbb{T}}_{\mathbb{Z},n}/\mathfrak{P})$. By multiplicity one, (Φ, \mathfrak{p}) must coincide with one of the (Φ_i, \mathfrak{p}_i) . Because of the congruence by which we chose (Φ, \mathfrak{p}) , the first assertion of (c) follows. Parts (i) and (ii) are then implied by Theorem B.14(b), (f). \square

Remark 8.9. It follows from Theorem 8.8(a) that from the Hecke action on $S_2^{(2)}(\mathcal{K}(\mathbf{n}_i), \mathbb{C}_\infty)$, $i = 1, \dots, s$, one can compute $\chi_f((1, \dots, 1, \varpi_x, 1, \dots, 1))$ at all places $x \in \bigcup_i X_{\Phi_i, \mathfrak{p}_i}^{\text{ord}}$.

Remark 8.10. The ramification properties of $\rho_{\Phi_i, \mathfrak{p}_i}$ at places where A_{Φ_i} has semistable reduction are completely described by Corollary B.5 and Proposition B.6. Jointly with T. Centeleghe we plan to work out the ramification for the further reduction types of A_{Φ_i} . There seem to be plausible conjectures to what happens. They suggest that one should be able to describe the ramification $\bar{\rho}_f$ in terms of that of the $\rho_{\Phi_i, \mathfrak{p}_i}$ at all places where A_{Φ_i} has either potentially semistable but not potentially good reduction, or where it has potentially good ordinary reduction. For instance, if Φ is a ramified principal series at x such that one character is unramified with image of Frobenius a p -adic unit, then we expect for such x that $\rho_{\Phi, \mathfrak{p}}$, and hence $\bar{\rho}_f$, is unramified at x , and yet x is in the support of the conductor of Φ .

In the remaining cases, i.e., at places x where the reduction of all A_{Φ_i} is potentially good non-ordinary, we have currently no approach to understand the restriction $\bar{\rho}_f|_{G_{F_x}}$ or its ramification.

Example 8.11. The following example shows that $a_x(f) = 0$ does not imply that $\bar{\rho}_f$ is ramified at x : Let E be an elliptic curve over F with split multiplicative reduction at ∞ , and assume that E has good supersingular reduction at a place x of F . Let F'/F be a Galois extension such that the $G_{F'}$ -representation on $E[p](F^{\text{sep}})$ is unramified at the places above x . Denote by E' the base change $E \times_F F'$. At any place ∞' of F' above ∞ the curve E' has split multiplicative reduction. By [GR96], which goes back to Drinfeld, there exists a cuspidal automorphic Hecke eigenform Φ' over F' that is Steinberg at ∞' and such that $\text{Ta}_\ell E' = \text{Ta}_\ell \Phi'$

⁸ We caution the reader that the Hecke algebra $\mathbb{T}_{\mathbb{Z},n}^{\text{ord}}$ used here and $\mathbb{T}_{\mathbb{Z},n}$ used in Appendix B are different for $? \in \{\mathbb{Z}, \mathbb{F}_p\}$. For $? = \mathbb{Z}$ this does not matter, since the automorphic multiplicity one, crucial to our applications, requires only the Hecke operators T_x , $x \in X_n^{\text{ord}}$ and not the T_x , $x \in X_n \setminus X_n^{\text{ord}}$, nor the additional $q_x S_x$ of $\mathbb{T}_{\mathbb{Z},n}$. For $? = \mathbb{F}_p$ we have $q_x S_x = 0$, and results for the possibly larger algebra $\mathbb{T}_{\mathbb{F}_p, n}$ still hold for $\mathbb{T}_{\mathbb{F}_p, n}^{\text{ord}}$ used here; see also Remark B.11.

for all primes ℓ including p . Let f denote the doubly cuspidal weight 2 Drinfeld Hecke eigenform such that $\bar{\rho}_f$ agrees with the mod p reduction of $\mathrm{Ta}_p \Phi'$, i.e., with the $G_{F'}$ -representation on $E'[p](F^{\mathrm{sep}})$. Because E' is supersingular at x , at all places x' of F' above x we have $a_{x'}(f) = 0$ and yet, by construction of F' , the representation $\bar{\rho}_f$ is unramified at x' .

The above example uses that F' is not the minimal field of definition of f (or Φ or E). We do not know whether, over such a minimal field, $\bar{\rho}_f$ has to be ramified if $a_x(f) = 0$.

Example 8.12. We claim that there exist doubly cuspidal weight 2 Hecke eigenforms f_1, f_2 with $\bar{\rho}_{f_1} = \bar{\rho}_{f_2}$ for which there exists $x \in |X_n|$ with $T_x f_1 = 0$ and $T_x f_2 \neq 0$. In particular, the Φ_i in Theorem 8.8(b) may depend on f and not just on $\bar{\rho}_f$.

To show the above claim, we follow the method employed in the previous example. We let E_1 and E_2 be any elliptic curves over F that are split multiplicative at ∞ and such that, at some place x of F , both curves E_i have good reduction: however, E_1 has ordinary and E_2 supersingular reduction. Now choose F' finite separable over F such that $G_{F'}$ acts trivially on $E_i[p](F^{\mathrm{sep}})$ for both i . Arguing as in Example 8.11, the claim follows.

We now turn to the case of arbitrary weight $k \geq 2$, and we fix an \mathfrak{n} that is a proper non-zero ideal of A . Let $\mathcal{K} \subset \mathrm{GL}_2(\hat{A})$ be a compact open subgroup that is *neat*, i.e., such that for all $g \in \mathrm{GL}_2(\mathbb{A}_F)$ the torsion in $\mathrm{GL}_2(F) \cap g\mathcal{K}g^{-1}$ is p -torsion.⁹ By $\mathfrak{Y}_{\mathcal{K}}$ we denote the moduli space for rank two Drinfeld modules with a level \mathcal{K} structure and by $j_{\mathcal{K}} : \mathfrak{Y}_{\mathcal{K}} \rightarrow \mathfrak{X}_{\mathcal{K}}$ the open immersion into the smooth compactification of $\mathfrak{Y}_{\mathcal{K}}$. Abbreviate $\mathfrak{Y}_{\mathfrak{n}} = \mathfrak{Y}_{\mathcal{K}(\mathfrak{n})}$ and $j_{\mathfrak{n}} = j_{\mathcal{K}(\mathfrak{n})}$. We regard $\mathfrak{X}_{\mathfrak{n}}$ and $\mathfrak{Y}_{\mathfrak{n}}$ either as smooth schemes over $X_{\mathfrak{n}}$ or as curves over F . Let $\mathbb{F}_{\mathfrak{p}^n}$ be the étale sheaf of \mathfrak{p}^n -torsion points of the universal Drinfeld module on $\mathfrak{Y}_{\mathfrak{n}}$. By [Böc04, Corollary 7.2], the dual $\mathbb{F}_{\mathfrak{n}}^{\vee} := \mathrm{Hom}_{A/\mathfrak{p}^n}(\mathbb{F}_{\mathfrak{n}}, \mathfrak{p}^{-n}\Omega_{A/\mathbb{F}_q}/\Omega_{A/\mathbb{F}_q})$ is isomorphic to $(\underline{\mathcal{M}}_k(\mathfrak{n}) \otimes_A A/\mathfrak{p}^n)^{\acute{e}t}$, where we recall from [Böc04, Definition 10.1] that $\underline{\mathcal{M}}_k(\mathfrak{n})$ is the A -motive attached to the universal Drinfeld module on $\mathfrak{Y}_{\mathfrak{n}}$. We define $\mathbb{G}_{\mathfrak{p}^n} = j_{\mathfrak{n}!}\mathbb{F}_{\mathfrak{p}^n}^{\vee}$. Since $\underline{\mathcal{S}}_k(\mathfrak{n})$ is defined as the pushforward under $\mathfrak{X}_{\mathfrak{n}}$ to $X_{\mathfrak{n}}$ of $j_{\mathfrak{n}!}\mathrm{Sym}^{k-2}\underline{\mathcal{M}}_k(\mathfrak{n})$, we have as G_F -representations:

$$H_{\acute{e}t}^1(\mathfrak{X}_{\mathfrak{n}/F^{\mathrm{sep}}}, \mathrm{Sym}^{k-2}\mathbb{G}_{\mathfrak{p}^n}) \cong (\underline{\mathcal{S}}_k(\mathfrak{n}) \otimes_A A/\mathfrak{p}^n)^{\acute{e}t}(F^{\mathrm{sep}}) \cong \rho_{\underline{\mathcal{S}}_k(\mathfrak{n}), \mathfrak{p}^n}. \tag{12}$$

Thus $H_{\acute{e}t}^1(\mathfrak{X}_{\mathfrak{n}/F^{\mathrm{sep}}}, \mathrm{Sym}^{k-2}\mathbb{G}_{\mathfrak{p}^n})_{n \geq 1}$ is an inverse system. Note also that one has a short exact sequence

$$0 \rightarrow H_{\acute{e}t}^1(\mathfrak{X}_{\mathfrak{n}/F^{\mathrm{sep}}}, \mathrm{Sym}^{k-2}\mathbb{G}_{\mathfrak{p}}) \rightarrow H_{\acute{e}t}^1(\mathfrak{X}_{\mathfrak{n}/F^{\mathrm{sep}}}, \mathrm{Sym}^{k-2}\mathbb{G}_{\mathfrak{p}^n}) \rightarrow H_{\acute{e}t}^1(\mathfrak{X}_{\mathfrak{n}/F^{\mathrm{sep}}}, \mathrm{Sym}^{k-2}\mathbb{G}_{\mathfrak{p}^{n-1}}) \rightarrow 0 \tag{13}$$

for all n . It is part of a long exact cohomology sequence in which the H^0 and H^i , $i \geq 2$, terms vanish: for H^0 this holds because $\mathrm{Sym}^{k-2}\mathbb{G}_{\mathfrak{p}^n}$ is the extension by zero from an affine to a projective scheme; for H^i , $i \geq 2$, this uses that $\mathfrak{X}_{\mathfrak{n}}$ is a curve in characteristic p , and that the coefficients $j_{\mathfrak{n}!}\mathrm{Sym}^{k-2}\mathbb{G}_{\mathfrak{p}^n}$ are in characteristic p , as well.

LEMMA 8.13. *Let $d_{k,\mathfrak{n}}$ be the dimension of $S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_{\infty})$.*

- (a) *For any n, \mathfrak{p}, k , the A/\mathfrak{p}^n -module $H_{\acute{e}t}^1(\mathfrak{X}_{\mathfrak{n}/F^{\mathrm{sep}}}, \mathrm{Sym}^{k-2}\mathbb{G}_{\mathfrak{p}^n})$ is free of rank $d_{k,\mathfrak{n}}$.*
- (b) *For $\mathfrak{p}|\mathfrak{n}$, any n, k , and $x \in |X_{\mathfrak{n}}^{\mathrm{ord}}|$, the A/\mathfrak{p}^n -module $H_{\acute{e}t}^1(\mathfrak{X}_{\mathfrak{n}/k_x^{\mathrm{sep}}}, \mathrm{Sym}^{k-2}\mathbb{G}_{\mathfrak{p}^n})$ is free of rank $d_{k,\mathfrak{n}}$.*

Proof. To prove (a), denote by π the Galois cover $\mathfrak{X}_{\mathfrak{n}'} \rightarrow \mathfrak{X}_{\mathfrak{n}}$ over F^{sep} for $\mathfrak{n}' := \mathrm{lcm}(\mathfrak{n}, \mathfrak{p})$. By [Gek86, § 5], all non-trivial inertia subgroups of π at closed points of $\mathfrak{X}_{\mathfrak{n}}$, i.e., at the cusps

⁹ This generality is convenient for Proposition 8.24, Corollary 8.25 and Example 8.26.

of \mathfrak{X}_n , are p -groups. Since clearly the Galois action on $\pi^* \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}^n}$ is of p -power order, this implies (i) and (ii) of:

- (i) the action of $G_{\mathbb{K}}$ on $\pi^* \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}^n}$, for \mathbb{K} the function field of $\mathfrak{X}_{n'}$, is via a p -group;
- (ii) the action of inertia of a closed point of \mathfrak{X}_n on $\text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}^n}$ is via a p -group;
- (iii) the curve $\mathfrak{X}_{n'}/F^{\text{sep}}$ is ordinary.

Property (iii) is implied by Theorem 8.2. Now by [Pin00, Theorem 0.2], which is an analog of the Grothendieck–Ogg–Shafarevich formula, and by [Pin00, Proposition 5.6(a)], which computes some local terms, it follows that the Euler–Poincaré characteristic of the étale cohomology of $\text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}^n}$, considered as a vector space over \mathbb{F} , is given by

$$(1 - g_{\mathfrak{X}_n} - h_{\mathfrak{X}_n}) \dim_{\mathbb{F}}(\text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}^n}(\mathbb{K}^{\text{sep}})) = -\dim_{\mathbb{F}} A/\mathfrak{p}^n \cdot d_{k,n},$$

where $g_{\mathfrak{X}_n}$ is the genus of \mathfrak{X}_n and $h_{\mathfrak{X}_n}$ is the number of cusps of $\mathfrak{X}_n/F^{\text{sep}}$. Arguing as for (13), we see that $H^i(\mathfrak{X}_n/F^{\text{sep}}, \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}^n})$ is possibly non-zero only for $i = 1$. From this, (a) follows.

For (b), observe that the analogs of (i)–(iii) hold for k_x instead of F : conditions (i) and (ii) are clear since \mathfrak{p} divides \mathfrak{n} , so that the Galois cover $\mathfrak{X}_{\mathfrak{n}\mathfrak{p}^{n-1}} \rightarrow \mathfrak{X}_n$ trivializing $\text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}^n}$ has degree a power of p ; (iii) holds because x lies in $|X_n^{\text{ord}}|$. One now argues as in (a) to obtain (b). \square

For the following result, we regard $j_n : \mathfrak{Y}_n \hookrightarrow \mathfrak{X}_n$ as a compactification over X_n^{ord} , and we regard the étale sheaf $\mathbf{F}_{\mathfrak{p}^n}$ of \mathfrak{p}^n -torsion points of the universal Drinfeld module as a sheaf on \mathfrak{Y}_n . Then the extension by zero $\mathbf{G}_{\mathfrak{p}^n} = j_{n!} \mathbf{F}_{\mathfrak{p}^n}^{\vee}$ is an étale sheaf on \mathfrak{X}_n . By $\bar{f}_n : \mathfrak{X}_n \rightarrow X_n^{\text{ord}}$ we denote the structure morphism.

COROLLARY 8.14. *If $\mathfrak{p}|\mathfrak{n}$, then the étale sheaves $R_{\text{ét}}^1 \bar{f}_{n*} \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}^n}$ of A/\mathfrak{p}^n -modules are lisse on X_n^{ord} .*

Proof. Using the specialization from η to $x \in |X_n^{\text{ord}}|$ for the sequence (13) as given to the same sequence with F^{sep} replaced by k_x^{sep} , it suffices to prove the result for $n = 1$. Next, by the sheaf analog of (12) we have

$$R_{\text{ét}}^1 \bar{f}_{n*} \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}} \cong (\underline{\mathcal{S}}_k(\mathfrak{n}) \otimes_A A/\mathfrak{p})^{\text{ét}}$$

as étale sheaves on X_n^{ord} . Since $\underline{\mathcal{S}}_k(\mathfrak{n})$ has a locally free representative $\underline{\mathcal{F}}$ on $X_n \times C$, so does $\underline{\mathcal{S}}_k(\mathfrak{n}) \otimes_A A/\mathfrak{p}$ on $X \times \text{Spec } A/\mathfrak{p}$. Thus \mathcal{F} is a vector bundle over the curve $X_n^{\text{ord}} \times \text{Spec } A/\mathfrak{p}$. Replacing the representative by $\text{Im}(\tau^n)$ for $n \gg 0$, which over the curve is again locally free, we can assume that τ is generically injective on \mathcal{F} . Because A/\mathfrak{p} is finite it follows that in fact τ is generically an isomorphism, and thus, by Lemma 8.13(a), the rank of \mathcal{F} over $X_n \times \text{Spec } A/\mathfrak{p}$ is $d_{k,n}$. Consider now $i_x^* \underline{\mathcal{F}}$ for $x \in X_n^{\text{ord}}$. Because of Theorem 3.1 and Lemma 8.13(b), the induced τ_x must be an isomorphism. Again from Theorem 3.1, we deduce that $\underline{\mathcal{F}}^{\text{ét}}$ is lisse over X_n^{ord} , and this concludes the proof of the corollary. \square

COROLLARY 8.15. *The compatible system $(\rho_{\underline{\mathcal{S}}_k(\mathfrak{n}), \mathfrak{p}^\infty})_{\mathfrak{p}}$ has empty defect set $T \subset \text{Max}(A)$.*

Proof. Let \mathfrak{p} be in $\text{Max}(A)$ and let $\mathfrak{n}' := \text{lcm}(\mathfrak{p}, \mathfrak{n})$. Then by (12) and by Corollary 8.14, the G_F -representation $\rho_{\underline{\mathcal{S}}_k(\mathfrak{n}'), \mathfrak{p}^\infty}$ is lisse on $X_{\mathfrak{n}'}^{\text{ord}}$. Since for any \mathfrak{p} , the crystal $\underline{\mathcal{S}}_k(\mathfrak{n})$ is a subcrystal of $\underline{\mathcal{S}}_k(\mathfrak{n}')$, the assertion follows. \square

COROLLARY 8.16. *The ramification set of $(\rho_{\underline{\mathcal{S}}_k(\mathfrak{n}), \mathfrak{p}^\infty})_{\mathfrak{p}}$ is contained in $\text{Spec } A \setminus X_n^{\text{ord}}$. In particular, for any Hecke eigenform $f \in S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$, the conductor of χ_f is supported on $\text{Spec } A \setminus X_n^{\text{ord}}$.*

Proof. Because of Corollary 6.7, its proof and Theorem 5.8, which link the system $(\rho_{\underline{S}_k(\mathfrak{n}, \mathfrak{p}^\infty)})_{\mathfrak{p}}$ to the direct sum of Hecke characters χ_f , with f traversing all Hecke eigensystems for level \mathfrak{n} and weight k , it suffices to show that the conductors of all χ_f have support outside $X_{\mathfrak{n}}^{\text{ord}}$. Suppose x is in the support of the conductor of some χ_f . Then $(\rho_{\underline{S}_k(\mathfrak{n}, \mathfrak{p}^\infty)})_{\mathfrak{p}}$ is ramified at x for all \mathfrak{p} . However, by Corollary 8.14, the representation $\rho_{\underline{S}_k(\mathfrak{n}, \mathfrak{p}^\infty)}$ is unramified at all $x \in X_{\mathfrak{n}}^{\text{ord}}$ for any \mathfrak{p} dividing \mathfrak{n} . \square

To study the compatible system of Galois representations attached to a single Hecke eigenform, we introduce some further notation. We continue with $k \geq 2$ and a non-trivial proper ideal $\mathfrak{n} \subset A$. Denote by $\mathbb{T}_k(\mathfrak{n}, A)$ the A -subalgebra of $\text{End}_{\text{crys}}(\underline{S}_k(\mathfrak{n}))$ generated by the Hecke operators $T_x, x \in |X_{\mathfrak{n}}|$. As explained above Proposition 5.3, the algebra $\mathbb{T}_k(\mathfrak{n}, A)$ is a subalgebra of $\text{End}_A(P)$ for a finitely generated projective A -module $P = (\underline{S}_k(\mathfrak{n})^{\text{an}})^{\tau}$.

DEFINITION 8.17. An extension F' of F inside \mathbb{C}_∞ is said to *split* $\mathbb{T}_k(\mathfrak{n}, A)$ if the action of T_x on $P \otimes_A F'$ is triagonalizable for all $x \in |X_{\mathfrak{n}}|$.

In the following, we denote by F' a finite extension of F that splits $\mathbb{T}_k(\mathfrak{n}, A)$ and by A' the normalization of A in F' . We note that F' splits $\mathbb{T}_k(\tilde{\mathfrak{n}}, A)$ for all divisors $\tilde{\mathfrak{n}}$ of \mathfrak{n} . For any homomorphism $A \rightarrow \tilde{A}$ we define $\mathbb{T}_k(\mathfrak{n}, \tilde{A}) := \mathbb{T}_k(\mathfrak{n}, A) \otimes_A \tilde{A}$. If $A \subset \tilde{A} \subset \mathbb{C}_\infty$, then $\mathbb{T}_k(\mathfrak{n}, \tilde{A})$ is isomorphic to the \tilde{A} -subalgebra of $\text{End}_{\text{crys}}(\underline{S}_k(\mathfrak{n}))$ generated by the Hecke operators $T_x, x \in |X_{\mathfrak{n}}|$.

Let f be a Hecke eigenform in $S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$. By $a_x(f)$ we denote the eigenvalue of f under T_x for $x \in |X_{\mathfrak{n}}|$. We describe two concrete ways to attach the strictly compatible system $(\rho_{f, \mathfrak{P}^\infty})_{\mathfrak{P}}$. One way is to consider the subcrystal $\underline{\mathcal{F}}_f$ of $\underline{S}_k(\mathfrak{n}) \otimes_A A'$ defined as in Proposition 5.5 as the intersection of the kernels of all $\mu_{f,x}(T_x)^{d_{k,\mathfrak{n}}}$, where $\mu_{f,x}$ is the linear polynomial $z - a_x(f) \in A'[z]$ and x ranges over $|X_{\mathfrak{n}}|$; see also Remark 5.6. The crystal $\underline{\mathcal{F}}_f$ is a Hecke crystal on a dense open subscheme U_f of $X_{\mathfrak{n}}$. Denoting by r the rank of the crystal $\underline{\mathcal{F}}_f$, the system $\rho_{f, \mathfrak{P}^\infty}$ is uniquely defined by

$$\rho_{f, \mathfrak{P}^\infty}^{\oplus r} = \rho_{\underline{\mathcal{F}}_f, \mathfrak{P}^\infty}^{\text{ss}}, \tag{14}$$

where \mathfrak{P} ranges over $\text{Max}(A')$ and where the superscript *ss* means that we pass to the semisimplification of $\rho_{\underline{\mathcal{F}}_f, \mathfrak{P}^\infty} \otimes_{A_{\mathfrak{P}}} F'_{\mathfrak{P}}$. A second alternative construction runs as follows: for $\mathfrak{P} \in \text{Max}(A')$ above $\mathfrak{p} \in \text{Max}(A)$, abbreviate $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty}) := H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{p}^\infty} \otimes_{A_{\mathfrak{p}}} A'_{\mathfrak{P}})$ and define

$$H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty}) = \varprojlim H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty}),$$

so that by (12) the G_F -representation $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty})$ is isomorphic to $\rho_{\underline{S}_k(\mathfrak{n}, \mathfrak{P}^\infty)} \otimes_{A_{\mathfrak{p}}} A'_{\mathfrak{P}}$. In particular, the action of T_x on $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty}) \otimes_{A'} F'$ is diagonalizable for all $x \in X_{\mathfrak{n}}$, and by Theorem 4.17 the latter module has the same eigenvalue systems as $P \otimes_A F'_{\mathfrak{P}}$. We define $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty})^f$ as the intersection

$$\bigcap_{x \in |X_{\mathfrak{n}}|} \text{Ker}(\mu_{f,x}(T_x)^{d_{k,\mathfrak{n}}} : H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty}) \rightarrow H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty})). \tag{15}$$

Because of $\rho_{\underline{S}_k(\mathfrak{n}, \mathfrak{p}^\infty)} \otimes_{A_{\mathfrak{p}}} F'_{\mathfrak{P}} \cong \bigoplus_f \rho_{\underline{\mathcal{F}}_f, \mathfrak{P}^\infty} \otimes_{A'_{\mathfrak{P}}} F'_{\mathfrak{P}}$, where the sum runs over all Hecke eigensystems of eigenforms $f \in S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$, and which follows from Propositions 5.4 and 5.5 and Corollary 8.15, we see that the G_F -representation $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty})^f \otimes_{A'_{\mathfrak{P}}} F'_{\mathfrak{P}}$ is isomorphic to $\rho_{\underline{\mathcal{F}}_f, \mathfrak{P}^\infty} \otimes_{A'_{\mathfrak{P}}} F'_{\mathfrak{P}}$. With r from (14) it follows that

$$\rho_{f, \mathfrak{P}^\infty}^{\oplus r} = (H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbb{G}_{\mathfrak{P}^\infty})^f)^{\text{ss}}.$$

PROPOSITION 8.18. *Let $f \in S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$ be a Hecke eigenform.*

- (a) *The defect set of $(\rho_{f, \mathfrak{P}^\infty})_{\mathfrak{P}}$ in $\text{Max}(A')$ is empty, its ramification set contained in $\text{Max}(A) \setminus X_{\mathfrak{n}}^{\text{ord}}$.*
- (b) *Let $\mathfrak{P} \in \text{Max}(A')$ be above $\mathfrak{p} \in \text{Max}(A)$, $\mathfrak{n}' := \text{lcm}(\mathfrak{n}, \mathfrak{p})$. Then $\rho_{f, \mathfrak{P}^\infty}$ is unramified at $x \in |X_{\mathfrak{n}'}^{\text{ord}}|$ and $\rho_{f, \mathfrak{P}^\infty}(\text{Frob}_x) = a_x(f)$.*

Proof. Part (a) follows from (14) and Corollaries 8.15 and 8.16. For (b) observe that the Eichler-Shimura relation (7) as proved in Theorem 6.1 yields

$$T_x = \rho_{\mathcal{S}_k(\mathfrak{n}), \mathfrak{P}^\infty} \tag{16}$$

as actions on $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{P}^\infty})$. Because of Corollary 8.15 this holds for all $\mathfrak{P} \in \text{Max}(A')$. Now Corollary 8.16 implies that $S_{\mathfrak{P}}^{\mathcal{F}}$ in the proof of Theorem 6.1 is disjoint from $|X_{\mathfrak{n}'}^{\text{ord}}|$, and so for a given \mathfrak{P} equality (16) holds for all $x \in |X_{\mathfrak{n}'}^{\text{ord}}|$. Specializing (16) to the generalized f -eigenspace of $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{P}^\infty})$ and semisimplifying completes the proof of (b). \square

Let $f \in S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$ be a Hecke eigenform and fix $\mathfrak{P} \in \text{Max}(A')$. Define \mathfrak{m} as the kernel of the homomorphism $\mathbb{T}_k(\mathfrak{n}, A') \rightarrow A'_{\mathfrak{P}}/_{\mathfrak{P}}, T_y \mapsto a_y(f) \pmod{\mathfrak{P}}$. For any $\mathbb{T}_k(\mathfrak{n}, A')$ -module M , we denote by $M_{\mathfrak{m}}$ the localization of M at \mathfrak{m} , and we note that localization is exact.

The following result describes congruences between weight k and weight 2 forms. Results similar to part (a) are well known for classical modular forms.

LEMMA 8.19. *Set $\mathfrak{p} := A \cap \mathfrak{P} \in \text{Max}(A)$ and $\mathfrak{n}' := \text{lcm}(\mathfrak{n}, \mathfrak{p})$. Choose a field extension $F' \supset F$ that splits $\mathbb{T}_k(\mathfrak{n}', A)$.*

- (a) *There exists a Hecke eigenform $g \in S_2(\mathfrak{n}', \mathbb{C}_\infty)$ such that*

$$a_y(f) \equiv a_y(g) \pmod{\mathfrak{P}} \quad \text{for all } y \in |X_{\mathfrak{n}'}|.$$

- (b) *If $\rho_{f, \mathfrak{P}^\infty}$ is ramified at some $x \in |X_{\mathfrak{n}'}|$, then $a_x(f) \equiv 0 \pmod{\mathfrak{P}}$.*
- (c) *If $a_x(f) \equiv 0 \pmod{\mathfrak{P}}$, then any g as in (a) is doubly cuspidal.*

Proof. For (a) we have to show that the ideal \mathfrak{m}' of $\mathbb{T}_2(\mathfrak{n}', A')$ generated by \mathfrak{P} and $\{T_y - a_y(f) \mid y \in |X_{\mathfrak{n}'}|\}$ is a proper ideal. Then \mathfrak{m}' is maximal, as a kernel of a homomorphism to $A'/_{\mathfrak{P}}$; the localization $S_2(\mathfrak{n}', \mathbb{C}_\infty)_{\mathfrak{m}'}$ is non-zero and Hecke stable; and any Hecke eigenform $g \in S_2(\mathfrak{n}', \mathbb{C}_\infty)$ which is non-zero after localization at \mathfrak{m} would work for (a).

To see that \mathfrak{m}' is proper, it suffices to show that $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/F^{\text{sep}}}, \text{Sym}^0 \mathbf{G}_{\mathfrak{P}})_{\mathfrak{m}'}$ is non-zero. Since \mathfrak{p} divides \mathfrak{n}' , the sheaf $\text{Sym}^{k-2} \mathbf{F}_{\mathfrak{p}}$ is constant and isomorphic to $A/\mathfrak{p}^{\oplus(k-1)}$. We deduce the isomorphism $(\text{Sym}^0 \mathbf{G}_{\mathfrak{p}})^{\oplus(k-1)} \cong \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{p}}$ and hence the isomorphism of modules over the free commutative A -algebra $A[T_x \mid x \in |X_{\mathfrak{n}}|]$,

$$H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/F^{\text{sep}}}, \text{Sym}^0 \mathbf{G}_{\mathfrak{P}})^{\oplus(k-1)} \cong H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{P}}).$$

This implies $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/F^{\text{sep}}}, \text{Sym}^0 \mathbf{G}_{\mathfrak{P}})_{\mathfrak{m}'}^{\oplus(k-1)} \cong H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{P}})_{\mathfrak{m}}$ as $A'/_{\mathfrak{P}}$ -modules and thus completes (a).

We now prove (b). Let x be in $|X_{\mathfrak{n}'}|$. Arguing as in Lemma 8.13(b), we deduce from the exactness of localization that the $A'/_{\mathfrak{P}^n}$ -rank of $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/k_x^{\text{sep}}}, \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{P}^n})_{\mathfrak{m}}$ is independent of n . If this rank was equal to the $A'/_{\mathfrak{P}^n}$ -rank of $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/F^{\text{sep}}}, \text{Sym}^{k-2} \mathbf{G}_{\mathfrak{P}^n})_{\mathfrak{m}}$, then using $\underline{\mathcal{F}}_f$ and

following the proof of Corollary 8.14, one deduces that $\rho_{f,\mathfrak{P}^\infty}$ is unramified at x , which is ruled out by the hypothesis of (b). The congruence argument employed in (a) now yields

$$\dim_{A'/\mathfrak{P}} H_{\text{ét}}^1(\mathfrak{X}_{n'}/F^{\text{sep}}, \text{Sym}^0 \mathbf{G}_{\mathfrak{P}})_{\mathfrak{m}'} > \dim_{A'/\mathfrak{P}} H_{\text{ét}}^1(\mathfrak{X}_{n'}/k_x^{\text{sep}}, \text{Sym}^0 \mathbf{G}_{\mathfrak{P}})_{\mathfrak{m}'}$$

We claim that $T_x \in \mathfrak{m}'$, so that $0 = a_x(g) \equiv a_x(f) \equiv 0 \pmod{\mathfrak{P}}$.

Denoting by $i_n : \mathcal{C}_{\mathcal{K}(\mathfrak{n})}/F \rightarrow \mathfrak{X}_n$ the closed immersion of the cusps, one has the short exact sequence $0 \rightarrow j_{n!} j_n^* A'/\mathfrak{P} \rightarrow A'/\mathfrak{P} \rightarrow i_{n*} i_n^* A'/\mathfrak{P} \rightarrow 0$ on \mathfrak{X}_n , whose left term is isomorphic to $\text{Sym}^0 \mathbf{G}_{\mathfrak{P}}$. The associated long exact sequence of étale cohomology together with the specialization homomorphism to x give the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A'/\mathfrak{P} & \longrightarrow & H_{\text{ét}}^0(\mathcal{C}_{\mathcal{K}(\mathfrak{n})}/F^{\text{sep}}, A'/\mathfrak{P}) & \longrightarrow & H_{\text{ét}}^1(\mathfrak{X}_n/F^{\text{sep}}, j_{n!} A'/\mathfrak{P}) & \longrightarrow & H_{\text{ét}}^1(\mathfrak{X}_n/F^{\text{sep}}, A'/\mathfrak{P}) & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'/\mathfrak{P} & \longrightarrow & H_{\text{ét}}^0(\mathcal{C}_{\mathcal{K}(\mathfrak{n})}/k_x^{\text{sep}}, A'/\mathfrak{P}) & \longrightarrow & H_{\text{ét}}^1(\mathfrak{X}_n/k_x^{\text{sep}}, j_{n!} A'/\mathfrak{P}) & \longrightarrow & H_{\text{ét}}^1(\mathfrak{X}_n/k_x^{\text{sep}}, A'/\mathfrak{P}) & \longrightarrow & 0 \end{array}$$

The isomorphism on the far left is clear. The isomorphism on the second term is shown as follows: The compactification \mathfrak{X}_n of \mathfrak{Y}_n is obtained by gluing in formal Drinfeld–Tate curves at the cusps. The special fibers of the latter, i.e., the components of $\mathcal{C}_{\mathcal{K}(\mathfrak{n})}/F$, are isomorphic to $\text{Spec } F(\mathfrak{n})$, where $F(\mathfrak{n})$ is the Drinfeld cyclotomic extensions of F of conductor \mathfrak{n} , see [Böc04, §§ 2.2–2.4]. It follows that the action of G_F on $H_{\text{ét}}^0(\mathcal{C}_{\mathcal{K}(\mathfrak{n})}, \mathbb{F}_p)$ is unramified outside \mathfrak{n} . The above diagram remains exact after localization at \mathfrak{m}' . It follows that the specialization $H_{\text{ét}}^1(\mathfrak{X}_n/F^{\text{sep}}, A'/\mathfrak{P})_{\mathfrak{m}'} \rightarrow H_{\text{ét}}^1(\mathfrak{X}_n/k_x^{\text{sep}}, A'/\mathfrak{P})_{\mathfrak{m}'}$ is not an isomorphism. From the alternative described in Theorem 8.8(c), we deduce that $T_x \in \mathfrak{m}'$.

Finally, to show (c), we argue by contradiction and assume that some g from (a) is not doubly cuspidal. Then its Galois representation belongs to $H_{\text{ét}}^0(\mathcal{C}_{\mathcal{K}(\mathfrak{n})}/F^{\text{sep}}, A'/\mathfrak{P})$, and so we know that the specialization homomorphism to $H_{\text{ét}}^0(\mathcal{C}_{\mathcal{K}(\mathfrak{n})}/k_x^{\text{sep}}, A'/\mathfrak{P})$ is an isomorphism. As explained in the proof of Theorem 6.1, this implies that for the actions on $H_{\text{ét}}^0(\mathcal{C}_{\mathcal{K}(\mathfrak{n})}/F^{\text{sep}}, A'/\mathfrak{P})$ we have $T_x = \text{Frob}_x$. As explained in the previous paragraph, the Galois action on this module is unramified on X_n and via a finite order character. Because x is in $|X_{n'}|$, it follows that Frob_x , and hence also T_x , act via some invertible matrix over A'/\mathfrak{P} . In particular, $a_x(g)$ must be a unit, which is a contradiction. \square

As a corollary to the proof just given, we also record the following.

COROLLARY 8.20. *Let f be an eigenform in $S_2(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty) \setminus S_2^{(2)}(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$. Then for all $x \in X_n$ we have $a_x(f) = \bar{\rho}_f(\text{Frob}_x)$.*

Remark 8.21. With slightly more work but similar methods, one can show that for $k \geq 3$ and a Hecke eigenform $f \in S_k(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty) \setminus S_k^{(2)}(\mathcal{K}(\mathfrak{n}), \mathbb{C}_\infty)$, one has for all $\mathfrak{P} \in \text{Max}(A')$ and $\mathfrak{p} = \mathfrak{P} \cap A$ the equality $a_x(f) = \rho_{f,\mathfrak{P}^\infty}(\text{Frob}_x)$ for all $x \in X_{\mathfrak{np}}$. Recall, from the second paragraph of the introduction to the present section, that it follows from [Böc04, Corollary 14.7] that χ_f is unramified outside \mathfrak{n} .

PROPOSITION 8.22. *Set $\mathfrak{p} := A \cap \mathfrak{P} \in \text{Max}(A)$ and $\mathfrak{n}' := \text{lcm}(\mathfrak{n}, \mathfrak{p})$. Choose a field extension $F' \supset F$ that splits $\mathbb{T}_k(\mathfrak{n}', A)$. Let $\mathfrak{m}' \subset \mathbb{T}_2(\mathfrak{n}', A')$ be as above, and let x be in $|X_{n'}|$. Then:*

- (a) *if $a_x(f) \not\equiv 0 \pmod{\mathfrak{P}}$, then $\rho_{f,\mathfrak{P}^\infty}$ is unramified at x and $\rho_{f,\mathfrak{P}^\infty}(\text{Frob}_x) = a_x(f)$;*
- (b) *if $a_x(f) \equiv 0 \pmod{\mathfrak{P}}$, then $\text{Jac}(\mathfrak{X}_{n'}/k_x)(p)_{\mathfrak{m}'}$ has trivial étale part and $\text{Jac}(\mathfrak{X}_{n'}/k_x)$ is non-ordinary.*

Proof. Suppose first that $a_x(f) \not\equiv 0 \pmod{\mathfrak{P}}$. Then by Lemma 8.19(b), $\rho_{f,\mathfrak{P}^\infty}$ is unramified at x . Moreover, Lemma 8.19(a) provides a weight 2 doubly cuspidal eigenform g of level \mathfrak{n}' such that $a_x(g) \equiv a_x(f) \pmod{\mathfrak{P}}$ for all $x \in |X_{\mathfrak{n}'}|$. Let $\mathfrak{m} \subset \mathbb{T}_k(\mathfrak{n}, A')$ be as defined above Lemma 8.19. Because of Theorem 8.8(c), the specialization map $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/F^{\text{sep}}}, \text{Sym}^0 \mathfrak{G}_{\mathfrak{P}})_{\mathfrak{m}'} \rightarrow H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/k_x^{\text{sep}}}, \text{Sym}^0 \mathfrak{G}_{\mathfrak{P}})_{\mathfrak{m}'}$ is an isomorphism. Arguing as in the proof of Lemma 8.19(b), we deduce that the specialization map $H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/F^{\text{sep}}}, \text{Sym}^0 \mathfrak{G}_{\mathfrak{P}^\infty})_{\mathfrak{m}} \rightarrow H_{\text{ét}}^1(\mathfrak{X}_{\mathfrak{n}'/k_x^{\text{sep}}}, \text{Sym}^0 \mathfrak{G}_{\mathfrak{P}^\infty})_{\mathfrak{m}}$ is an isomorphism. Passing to the f -component defined in (15), it remains an isomorphism. Therefore the Eichler–Shimura relation (7) applies to give $\rho_{f,\mathfrak{P}^\infty}(\text{Frob}_x) = a_x(f)$. This completes (a).

Suppose now that $a_x(f) \equiv 0 \pmod{\mathfrak{P}}$, and let g be as in Lemma 8.19(a). By Lemma 8.19(c), we deduce that g is doubly cuspidal. Hence we can apply Theorem 8.8 to g , and by its part (c)(i) the proof of (b) is immediate. \square

The following is an immediate consequence of Proposition 8.22.

COROLLARY 8.23.

- (a) Let x be in $|X_{\mathfrak{n}}^{\text{ord}}|$. Then $a_x(f) \neq 0$, and for $\mathfrak{P} \in \text{Max}(A'[1/a_x(f)])$ the representation $\rho_{f,\mathfrak{P}^\infty}$ is unramified at x and satisfies the Eichler–Shimura relation $a_x(f) = \rho_{f,\mathfrak{P}^\infty}(\text{Frob}_x)$.
- (b) Let \mathfrak{P} be in $\text{Max}(A')$ and $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\rho_{f,\mathfrak{P}^\infty}$ is unramified at all $x \in |X_{\mathfrak{np}}^{\text{ord}}|$.

The next result is useful for the two examples given below.

PROPOSITION 8.24. Let $\mathcal{K}' \subset \mathcal{K} \subset \text{GL}_2(\widehat{A})$ be open subgroups such that \mathcal{K}' is normal in \mathcal{K} with p -group quotient. Then any cuspidal weight k Hecke eigensystem of level \mathcal{K}' occurs as a weight k cuspidal Hecke eigensystem of level \mathcal{K} .

Proof. Consider the diagram

$$\begin{array}{ccc} \mathfrak{Y}_{\mathcal{K}'} & \xrightarrow{j_{\mathcal{K}'}} & \mathfrak{X}_{\mathcal{K}'} \\ \downarrow \pi_{\mathfrak{Y}} & & \downarrow \pi_{\mathfrak{X}} \\ \mathfrak{Y}_{\mathcal{K}} & \xrightarrow{j_{\mathcal{K}}} & \mathfrak{X}_{\mathcal{K}} \end{array}$$

where $\mathfrak{Y}_{\mathcal{K}}$ is the modular curve for Drinfeld modules with a level \mathcal{K} structure, $\mathfrak{X}_{\mathcal{K}}$ is its smooth compactification, $j_{\mathcal{K}}$ is the corresponding open immersion, and similarly for \mathcal{K}' in place of \mathcal{K} ; and, moreover, $\pi_{\mathfrak{Y}}$ and $\pi_{\mathfrak{X}}$ are the respective covering morphisms. Note that $\pi_{\mathfrak{Y}}$ is a Galois cover with Galois group \mathcal{K}/\mathcal{K}' . However, there might be ramification at the boundary of $\mathfrak{C}_{\mathcal{K}'} \rightarrow \mathfrak{C}_{\mathcal{K}}$.

The weight k Hecke eigensystems of level \mathcal{K}' occur in $H_{\text{ét}}^1(\mathfrak{X}_{\mathcal{K}'/F^{\text{sep}}}, j_{\mathcal{K}'}! \text{Sym}^{k-2} \mathbb{F}_{\mathfrak{p}^\infty}^\vee)$. By the spectral sequence of composition of $\pi_{\mathfrak{X}}$ with the structure morphism of $\mathfrak{X}_{\mathcal{K}}$, we obtain, using that $\pi_{\mathfrak{Y}}$ is finite étale, the isomorphism

$$H_{\text{ét}}^1(\mathfrak{X}_{\mathcal{K}'/F^{\text{sep}}}, j_{\mathcal{K}'}! \text{Sym}^{k-2} \mathbb{F}_{\mathfrak{p}^\infty}^\vee) \cong H_{\text{ét}}^1(\mathfrak{X}_{\mathcal{K}/F^{\text{sep}}}, j_{\mathcal{K}'}!(\pi_{\mathfrak{Y}*} \text{Sym}^{k-2} \mathbb{F}_{\mathfrak{p}^\infty}^\vee)).$$

The isomorphism is compatible with Hecke correspondences away from the level of \mathcal{K}' .

Next, observe that $\text{Sym}^{k-2} \mathbb{F}_{\mathfrak{p}^\infty}^\vee$ over $\mathfrak{Y}_{\mathcal{K}'}$ is $\pi_{\mathfrak{Y}}^* \text{Sym}^{k-2} \mathbb{F}_{\mathfrak{p}^\infty}^\vee$, so that by the projection formula we have

$$\pi_{\mathfrak{Y}*} \text{Sym}^{k-2} \mathbb{F}_{\mathfrak{p}^\infty}^\vee \cong \text{Sym}^{k-2} \mathbb{F}_{\mathfrak{p}^\infty}^\vee \otimes_{\mathbb{F}_p} \pi_{\mathfrak{Y}*} \mathbb{F}_p.$$

Furthermore, $\pi_{\mathfrak{Y}*} \mathbb{F}_p \cong \text{Ind}_{\mathcal{K}'}^{\mathcal{K}} \mathbb{F}_p \cong \mathbb{F}_p[\mathcal{K}/\mathcal{K}']$ as a representation of the fundamental group of $\mathfrak{Y}_{\mathcal{K}}$. Because \mathcal{K}' is normal in \mathcal{K} with p -group quotient, it follows that the semisimplification

of $\mathbb{F}_p[\mathcal{K}/\mathcal{K}']$ is isomorphic to \mathbb{F}_p^d for $d = [\mathcal{K} : \mathcal{K}']$. Arguing as for (13), the H^0 and H^2 terms vanish for étale cohomology with \mathbb{F}_p -coefficients for a proper extension by zero on a curve. Hence

$$\begin{aligned} H_{\text{ét}}^1(\mathfrak{X}_{\mathcal{K}'/F^{\text{sep}}}, j_{\mathcal{K}'}! \text{Sym}^{k-2} \mathbb{F}_{p^\infty}^\vee)^{\text{ss}} &\cong H_{\text{ét}}^1(\mathfrak{X}_{\mathcal{K}'/F^{\text{sep}}}, j_{\mathcal{K}'}! \text{Sym}^{k-2} \mathbb{F}_{p^\infty}^\vee)^{\text{ss}} \\ &\cong H_{\text{ét}}^1(\mathfrak{X}_{\mathcal{K}/F^{\text{sep}}}, j_{\mathcal{K}}! \text{Sym}^{k-2} \mathbb{F}_{p^\infty}^\vee)^{\text{ss}, \oplus d} \end{aligned}$$

as Hecke modules, and this completes the proof. □

We define $\mathcal{K}'_1(\mathfrak{n})$ with $\mathcal{K}(\mathfrak{n}) \subset \mathcal{K}'_1(\mathfrak{n}) \subset \text{GL}_2(\widehat{A})$ as the set of those matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that satisfy $a - 1 \equiv c \equiv d - 1 \equiv 0 \pmod{\mathfrak{n}}$.

COROLLARY 8.25. *Let $A = \mathbb{F}_q[T]$ and suppose that \mathfrak{n} is a prime of degree one of A . Then for any cuspidal Drinfeld Hecke eigenform f of level dividing \mathfrak{n} and any weight, the Hecke character χ_f is unramified outside the level of f , and $a_x(f) = \chi_f(1, \dots, 1, \varpi_x, 1, \dots, 1)$ for all x outside the level of f .*

Proof. Suppose first that $\mathfrak{n} = (T)$. By Proposition 8.24, any f as in the theorem has the same eigensystem as some $g \in S_k(\mathcal{K}'_1(\mathfrak{n}), \mathbb{C}_\infty)$. Now in this case, the curve $\mathfrak{X}_{\mathcal{K}'_1(\mathfrak{n})}$ over $X_{\mathfrak{n}} = \text{Spec } A[1/T]$ is simply $\mathbb{P}^1_{X_{\mathfrak{n}}}$, e.g. [Böc14, Proposition 10.10(b)]. Now, trivially, the curve \mathbb{P}^1 has everywhere ordinary reduction. Then by Proposition 8.18,¹⁰ the conductor of $\chi_f = \chi_g$ divides (T) for g of level $\mathcal{K}'_1(T)$. By acting on the primes of degree one of \mathbb{P}^1 via an automorphism of the form $T \mapsto T + a$ for a suitable $a \in \mathbb{F}_q$, one sees that the argument just given applies to all primes \mathfrak{n} of degree one. So suppose finally that f has level one. Then f also occurs in levels (T) and $(T + 1)$. By what we have just proved, the conductor of χ_f divides T and $T + 1$. But then the conductor must be one, as asserted. The last part is immediate from Corollary 8.23. □

Example 8.26. We now apply Proposition 8.22 to the examples in § 7. For $j = 1$ and $4 \leq k \leq q + 2$ we have $F_{k,1} = 1 + (k - 3)(T/\theta)$. Then for the corresponding form f , the Hecke eigenvalue of $T_{(h)}$ for $h \in \mathbb{F}_q[\theta]$ with constant term 1 is given by (top of page 2032)

$$h((k - 3)T).$$

Suppose now further that $5 \leq k \leq p + 2$, so that $k - 3$ lies in $\mathbb{F}_p^* \setminus \{1\}$, and that $h(\theta)$ is linear. Then the linear polynomials $h(T)$ and $h((k - 3)T)$ are distinct and both have constant term 1, and so they have no common roots. In particular, if we choose $\mathfrak{p} = (h((k - 3)T))$, and $x = (h)$, then $a_x(f) \equiv 0 \pmod{\mathfrak{p}}$. Thus the Jacobian of $\mathfrak{X}_{(th((k-3)t))}$ has a factor of non-ordinary reduction at the prime $h(\theta)$. Since h was arbitrary, it follows that for any linear polynomial $h \in \mathbb{F}_p[T]$ with constant term 1, the Drinfeld modular curve $\mathfrak{X}_{(th(t))}$ has non-ordinary reduction at the prime $h((k - 3)^{-1}\theta)$. Now note that $k \in \{5, \dots, p - 2\}$ was arbitrary. We thus deduce that $\mathfrak{X}_{(th(t))}$ has non-ordinary reduction at all primes $h(\alpha\theta)$ for $\alpha \in \mathbb{F}_p^* \setminus \{1\}$.

By Proposition 8.24 below, we can descend from a full level t structure to a $\Gamma'_1(t)$ -type level structure. The advantage of this is that numerical computations with the latter type level structure have a lower complexity. To be more explicit, set $\Gamma'_1(th(t)) := \text{GL}_2(\mathbb{F}_q[t]) \cap \mathcal{K}'_1(th(t))$. Now by Proposition 8.24, the Drinfeld modular curve $\mathfrak{X}_{\Gamma'_1(th(t))}$ of level $\Gamma'_1(th(t))$ over $\mathbb{F}_q(\theta)$ has non-ordinary reduction for all primes of the form $h(\alpha\theta)$, $\alpha \in \mathbb{F}_p^* \setminus \{1\}$. In fact, for each $\alpha \in \mathbb{F}_p^* \setminus \{1\}$ there is a weight 2 cuspidal Hecke eigenform g_α of level $\Gamma'_1(th(t))$ whose Hecke eigenvalue at $h(\alpha T)$ is zero. The last result was verified independently by R. Butenuth who

¹⁰ Strictly speaking, we have only proved Proposition 8.18 for levels $\mathcal{K}(\mathfrak{n})$. However, the arguments all go through for neat levels \mathcal{K} . We stuck to $\mathcal{K}(\mathfrak{n})$ because we did not wish to get into fields of definitions of the curves $\mathfrak{X}_{\mathcal{K}}$.

computed numerically the Hecke operators on weight 2 forms for $\Gamma_1'(th(t))$. We leave it up to the reader to draw further conclusions for primes \mathfrak{p} and x of degree larger than one.

ACKNOWLEDGEMENTS

My sincere thanks go to Centeleghe, Goss, Khare, Lau, Papikian, Petrov and Thakur for their interest and conversations related to this work. I thank Chang for his careful reading and many helpful comments on a previous version of the present work. For an exchange regarding the appendix, I thank de Jong, and for the numerical verification of the result in Example 8.26, I thank Butenuth. We acknowledge the use of the computer algebra systems MAGMA and SAGE in his computations. I also thank the referee for a careful reading of the manuscript and many helpful suggestions.

Appendix A. Ramification and the p -rank of abelian varieties under reduction

The present appendix is motivated by the results in §8 on the ramification of Galois representations associated to Drinfeld modular forms. Its main result is a simple consequence of [deJ98] by de Jong and it is certainly well known to the experts. At the time of writing this we could not find an argument in the literature. De Jong pointed out that results of this kind are described in the language of F -crystals independently in the recent work [Yan11, Proposition 4.6]. Due to the simplicity of the argument and the complementary nature of our results in §8, we give a complete treatment.

Let us recall the theorem of Néron–Ogg–Shafarevich, cf. [ST68]. It links the reduction behavior of an abelian variety A over a local field K to ramification properties of the action of the absolute Galois group of K on the ℓ -adic Tate module of A where ℓ is any prime different from the residue characteristic p of K : the variety has good reduction if and only if the Galois representation is unramified. If $p > 0$ a similar much deeper theorem holds if instead of the Tate module at p one considers the p -divisible group attached to A . This is due to Tate for K of characteristic zero, [Tat67], and de Jong for K of characteristic p , [deJ98].

In this appendix we shall derive a similar type Galois criterion in a related setting: let K be a local field of characteristic p with ring of integers \mathcal{O} and residue field k . Let \mathcal{A} over \mathcal{O} be an abelian scheme of dimension g with generic fiber A over K and special fiber \mathcal{A}_k , so that in particular A has good reduction. It is well known that the p -adic Tate module $\mathrm{Ta}_p A$ of A is free over \mathbb{Z}_p and that its rank $\mathrm{rank}_p A = \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Ta}_p A$ satisfies $0 \leq \mathrm{rank}_p A \leq g$. One calls $\mathrm{rank}_p A$ the p -rank of A .¹¹ Moreover, $\mathrm{Ta}_p A$ carries a natural continuous \mathbb{Z}_p -linear action of G_K which provides one with a Galois representation:

$$\rho_{A,p} : G_K \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(\mathrm{Ta}_p A) \cong \mathrm{GL}_{\mathrm{rank}_p A}(\mathbb{Z}_p).$$

Considering the étale quotient of the p^n -torsion scheme of \mathcal{A} , whose reduction to k agrees with the étale quotient of the p^n -torsion scheme of \mathcal{A}_k , one sees that any p^n -torsion point of \mathcal{A}_k has a unique lift to a p^n -torsion point of A . In particular,

$$\mathrm{rank}_p \mathcal{A}_k \leq \mathrm{rank}_p A.$$

Our main result relates the reduction behavior of the p -rank to the ramification of $\rho_{A,p}$.

THEOREM A.1. *The p -rank of A is constant under reduction if and only if $\rho_{A,p}$ is unramified.*

¹¹ The formula $\mathrm{rank}_p A = \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Ta}_p A$ is used over any field K of characteristic $p > 0$ to define the p -rank of A .

The proof given below is a straightforward consequence of the main theorem of [deJ98].

Remark A.2. As the p -rank of an abelian variety is unchanged under finite extension of its base field, Theorem A.1 has the following consequence: the representation $\rho_{A,p}$ is unramified if and only if it is potentially unramified. It might be interesting to explore possible relations between the decrease of the p -rank under reduction and the rank of the inertial action.

As a corollary to Theorem A.1 we obtain a Galois criterion for the constancy of the p -rank of curves under reduction: let \mathcal{C} over \mathcal{O} be a smooth projective geometrically irreducible curve with generic fiber C and special fiber \mathcal{C}_k . As in the proof of Proposition 8.3, one defines the p -rank of C as

$$\text{rank}_p C = \dim_{\mathbb{F}_p} H^1_{\text{ét}}(C_{K^{\text{sep}}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^1(C_{K^{\text{sep}}}, \mathcal{O}_C)^\varphi \leq \dim_K H^1(C, \mathcal{O}_C).$$

Clearly the \mathbb{Q}_p -vector space $H^1_{\text{ét}}(C_{K^{\text{sep}}}, \mathbb{Q}_p)$ comes naturally equipped with an action of G_K .

COROLLARY A.3. *The action of G_K on $H^1_{\text{ét}}(C_{K^{\text{sep}}}, \mathbb{Q}_p)$ is unramified if and only if $\text{rank}_p C = \text{rank}_p \mathcal{C}_k$.*

Proof. This is straightforward from Theorem A.1: The Jacobian $\text{Jac}_{\mathcal{C}}$ of \mathcal{C} is an abelian scheme over \mathcal{O} whose generic fiber is the Jacobian Jac_C of C and whose special fiber is the Jacobian of \mathcal{C}_k . Now the Tate module of $\text{Jac}_{\mathcal{C}}$ tensored with \mathbb{Q}_p over \mathbb{Z}_p is dual to $H^1_{\text{ét}}(C_{K^{\text{sep}}}, \mathbb{Q}_p)$ as a module for G_K . Moreover, the rank of $H^1_{\text{ét}}(C_{K^{\text{sep}}}, \mathbb{Q}_p)$ over \mathbb{Q}_p is equal to that of $H^1_{\text{ét}}(C_{K^{\text{sep}}}, \mathbb{F}_p)$ over \mathbb{F}_p . Thus we have on the one hand $\text{rank}_p C = \text{rank}_p \text{Jac}_C$ and on the other that the G_K -action on $H^1_{\text{ét}}(C_{K^{\text{sep}}}, \mathbb{Q}_p)$ is unramified if and only if this action of $\rho_{\text{Jac}_C,p}$ is unramified. \square

To prove Theorem A.1, we shall in fact prove a result on p -divisible groups. Recall that a p -divisible group \mathcal{G} over \mathcal{O} of height $h = \text{height } \mathcal{G}$ is an inductive system $(\mathcal{G}_n, \iota_n)_{n \geq 0}$ where:

- (a) \mathcal{G}_n is a finite flat group scheme over \mathcal{O} of rank p^{hn} ;
- (b) for each $n \geq 0$, the following sequence is exact:

$$0 \longrightarrow \mathcal{G}_n \xrightarrow{\iota_n} \mathcal{G}_{n+1} \xrightarrow{p^n} \mathcal{G}_{n+1}.$$

One example of a p -divisible group is the inductive system $\mathcal{A}[p^n]$ of the p^n -torsion subschemes of the abelian scheme \mathcal{A} over \mathcal{O} . We denote it by $\mathcal{A}(p)$. Any p -divisible group has a maximal étale quotient $\mathcal{G}_{\text{ét}}$. It is formed by the sequence of maximal étale quotients $\mathcal{G}_{n,\text{ét}}$ of \mathcal{G}_n over \mathcal{O} . If \mathcal{G}_K denotes the generic fiber of \mathcal{G} , then its étale quotient $\mathcal{G}_{K,\text{ét}}$ surjects onto the generic fiber $(\mathcal{G}_{\text{ét}})_K$, but this need not be an isomorphism.

To any p -divisible group over K one attaches its p -adic Tate modules $\text{Ta}_p \mathcal{G}_K = \varprojlim_n \mathcal{G}_n(K^{\text{alg}})$. It carries a natural action of G_K . The assignment $\mathcal{G}_K \mapsto \text{Ta}_p \mathcal{G}_K$ defines a functor from p -divisible groups over K and p -adic Galois representations of G_K , which restricts to an equivalence of categories when restricted to étale p -divisible groups over K . Moreover, the equivalence restricts to an equivalence between p -divisible étale groups \mathcal{G} over \mathcal{O} and unramified p -adic Galois representations.

Since the p -rank of A is the same as the height of $A(p)_{\text{ét}}$, Theorem A.1 is now an immediate consequence of the following more general result on p -divisible groups, which will also be useful in Appendix B.

THEOREM A.4. *Suppose \mathcal{G} is a p -divisible group over \mathcal{O} . Then $\text{Ta}_p \mathcal{G}$ is an unramified representation of G_K if and only if $\text{height } \mathcal{G}_{K,\text{ét}} = \text{height } \mathcal{G}_{\text{ét}}$.*

Remark A.5. The height is invariant under base change. Thus if $\text{height } \mathcal{G}_{K,\text{ét}} > \text{height } \mathcal{G}_{\text{ét}}$, then $\text{Ta}_p \mathcal{G}$ restricted to $G_{K'}$ for any finite extension field K' of K remains ramified, analogous to Remark A.2.

Our essential tool to prove Theorem A.4 is the following result of de Jong adapted to our situation.

THEOREM A.6 [deJ98, Corollary 1.2]. *Let \mathcal{G} and \mathcal{H} be p -divisible groups over \mathcal{O} . Then the natural homomorphism*

$$\text{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{H}) \longrightarrow \text{Hom}_K(\mathcal{G}_K, \mathcal{H}_K)$$

is bijective. In particular, any homomorphism that is an isomorphism over K is an isomorphism over \mathcal{O} .

Proof of Theorem A.4. By a suitable base change, we may assume for the proof that the residue field k of \mathcal{O} is algebraically closed. Then G_k is trivial and hence a representation of G_K is unramified if and only if G_K acts trivially.

One direction of the theorem is now immediate: Suppose that $\text{height } \mathcal{G}_{K,\text{ét}} = \text{height } \mathcal{G}_{\text{ét}}$ and consider for any $n \geq 1$ the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{G}_{n,K,\text{ét}}(K^{\text{sep}}) & \xrightarrow{\cong} & \mathcal{G}_{n,\text{ét}}(K^{\text{sep}}) & \longleftarrow & \mathcal{G}_{n,\text{ét}}(K) \\ & & & \searrow \cong & \downarrow \cong \\ & & & & \mathcal{G}_{n,\text{ét}}(k) \end{array}$$

The left horizontal map is an isomorphism by hypothesis. The map from the center to the lower right is an isomorphism because $\mathcal{G}_{n,\text{ét}}$ is finite étale and k is algebraically closed. The right vertical map is the composition of the isomorphism $\mathcal{G}_{n,\text{ét}}(\mathcal{O}) \rightarrow \mathcal{G}_{n,\text{ét}}(k)$, which holds since $\mathcal{G}_{n,\text{ét}} \rightarrow \text{Spec } \mathcal{O}$ is étale, and the inverse of the isomorphism $\mathcal{G}_{n,\text{ét}}(\mathcal{O}) \rightarrow \mathcal{G}_{n,\text{ét}}(K)$, that uses that $\mathcal{G}_{n,\text{ét}}$ is proper. It follows that the right horizontal map is an isomorphism and hence that G_K acts trivially on $\mathcal{G}_{K,n}(K^{\text{sep}})$.

Suppose now that the action of G_K on $\text{Ta}_p A$ is trivial. Then one can associate to $\text{Ta}_p A$ the following trivial p -divisible group: denote by \mathcal{T}_n the constant group scheme over \mathcal{O} with generic fiber

$$\mathcal{G}_n(K) \cong (\mathbb{Z}/(p^n))^{\text{height } \mathcal{G}_{K,\text{ét}}}.$$

With respect to the canonical inclusion $\mathcal{T}_n(K) \hookrightarrow \mathcal{T}_{n+1}(K)$ these group schemes form a p -divisible group $\mathcal{T} = (\mathcal{T}_n, \iota_n)$ which is obviously étale over \mathcal{O} . Note that \mathcal{T}_K is the étale quotient of \mathcal{G}_K by the equivalence of Galois representations and étale p -divisible groups over K .

Now Theorem A.6 of de Jong yields a homomorphism $\mathcal{G} \rightarrow \mathcal{T}$ of p -divisible groups over \mathcal{O} , which over the generic fiber arises from the isomorphism $\mathcal{G}_{K,\text{ét}} \rightarrow \mathcal{T}_K$. Since \mathcal{T} is étale, we obtain an induced homomorphism $\mathcal{G}_{\text{ét}} \rightarrow \mathcal{T}$ which over the generic fiber yields

$$\mathcal{G}_{K,\text{ét}} \longrightarrow (\mathcal{G}_{\text{ét}})_K \longrightarrow \mathcal{T}_K.$$

As the composite is an isomorphism and the left-hand map is surjective, all maps are isomorphisms, and thus $\text{height } \mathcal{G}_{K,\text{ét}} = \text{height } \mathcal{G}_{\text{ét}}$.

Appendix B. Basic results on p -adic Galois representations and their reductions for automorphic forms over characteristic p function fields

Gebhard Böckle and Tommaso Centeleghe

This second appendix is concerned with results on p -adic Galois representations and their reductions attached to cuspidal automorphic forms over characteristic p function fields for which we could not find an adequate reference. Many results are inspired by related results on Galois representations over number fields. However there are significant differences, for instance that the representations we deal with are abelian.

We assume some familiarity with the theory of automorphic forms and representations. Details can be found in the following references. A good presentation of the general theory is [BJ79, §5]. In particular, it gives the correct framework of cuspidal automorphic forms and automorphic representations. Much narrower, but very close to the present setting is [vdPR97, §2]. The reference [Pia79] presents strong multiplicity one for cuspidal automorphic representations for GL_n . The tensor product theorem can be found in [Fla79] and the local theory for GL_2 is nicely presented in [BH06].

In this appendix, the basic notation is as in §6. We let k be the finite field \mathbb{F}_q . By X we denote a smooth projective geometrically irreducible curve over k on which we fix a closed point ∞ . By F we denote the function field $k(X)$ of X . We set $C = X \setminus \{\infty\}$ and let A be the coordinate ring of the affine curve C . We choose a non-zero proper ideal \mathfrak{n} of A . As before, by $X_{\mathfrak{n}}$ we denote $X \setminus (\{\infty\} \cup \text{Spec}(A/\mathfrak{n}))$; it is a dense open subscheme in X . The residue field of X at x is k_x and its order is q_x . By \mathcal{O}_x we denote the completion of the local ring $\mathcal{O}_{X,x}$ and by \mathfrak{M}_x the maximal ideal of $\mathcal{O}_{X,x}$.

By $\mathcal{K}_1(\mathfrak{n})$ we denote the compact open subgroup of $GL_2(\hat{A})$ of matrices whose reduction modulo \mathfrak{n} is the subgroup

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \mid b \in A/\mathfrak{n}, c \in (A/\mathfrak{n})^* \right\} \subset GL_2(A/\mathfrak{n}),$$

and we write $\mathfrak{X}_{1,\mathfrak{n}}$ for the corresponding modular curve $\mathfrak{X}_{\mathcal{K}_1(\mathfrak{n})}$ over F . The curve $\mathfrak{X}_{1,\mathfrak{n}}$ is in general only a coarse moduli scheme; by [Gek80, (3.4.17)] it is fine if either $\deg \infty$ is even or if \mathfrak{n} contains a prime divisor of odd degree, because under these conditions the corresponding congruence subgroups admit no elliptic fixed points. Arguing as in [KM85, ch. 9, Paragraph 9.4.3], one also finds that $\mathfrak{X}_{1,\mathfrak{n}}$ is defined and geometrically irreducible over F .

Because some arguments we use follow [Wil86], who works in the classical case of Hilbert modular forms with a $\Gamma_1(N)$ structure, we chose to work with $\mathfrak{X}_{1,\mathfrak{n}}$. This setup has the advantage over $\mathfrak{X}_{\mathfrak{n}}$ that the corresponding space of newforms satisfies the multiplicity one property with respect to the Hecke algebra generated by the Hecke operators T_x and S_x (diamond operators) where x ranges over all places $x \in |X_{\mathfrak{n}}|$; for multiplicity one, the operators T_x with x in a cofinite subset of $|X_{\mathfrak{n}}|$ are in fact sufficient. Moreover, one has the following uniqueness result.

LEMMA B.1. *Let Π be an irreducible cuspidal automorphic representation for $GL_2(\mathbb{A}_F)$. Then there exists a unique non-zero ideal \mathfrak{n} of A that is maximal under inclusion, such that $\Pi^{\mathcal{K}_1(\mathfrak{n})}$ is non-zero. Moreover, if Π is special at ∞ , then $\dim_{\mathbb{C}} \Pi^{\mathcal{K}_1(\mathfrak{n}) \times \mathcal{K}_{\infty}} = 1$ for \mathcal{K}_{∞} the Iwahori subgroup at ∞ . In particular, there exists a unique cuspidal automorphic eigenform Φ of level $\mathcal{K}_1(\mathfrak{n})$ that spans Π .*

Proof. This follows as in [Cas73, Theorems 1, 3], except that no discussion of Archimedean places is needed. Write $\Pi = \bigotimes_x \Pi_x$ in its tensor product representation with local factor Π_x at each place x of F . Define $\mathcal{K}_{1,x,c}$ as the set of matrices in $\mathrm{GL}_2(\mathcal{O}_{X,x})$ whose reduction modulo \mathfrak{M}_x^c is of the form $\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$ with $b \in \mathcal{O}_{X,x}/\mathfrak{M}_x^c$ and $d \in (\mathcal{O}_{X,x}/\mathfrak{M}_x^c)^*$. By Theorem 1 of [Cas73], for each place x there exists a unique integer $c_x \geq 0$ such that the fixed point set $\Pi_x^{\mathcal{K}_{1,x,c_x}}$ is a one-dimensional complex vector space. Moreover, by the global theory of automorphic forms almost all c_x are zero. Define \mathfrak{n} as the product $\prod_{x \neq \infty} \mathfrak{M}_x^{c_x}$, so that one has $\mathcal{K}_1(\mathfrak{n}) = \prod_{x \neq \infty} \mathcal{K}_{1,x,c_x}$. Then the subspace of vectors of Π fixed under $\mathcal{K}_1(\mathfrak{n}) \times \mathcal{K}_\infty$ is one-dimensional. \square

In the following we consider the abelian variety

$$A_{\mathfrak{n}} := J_{1,\mathfrak{n}}^{\mathrm{new}} := \mathrm{Jac}(\mathfrak{X}_{1,\mathfrak{n}})^{\mathfrak{n}\text{-new}}$$

over F , which is the quotient of $\mathrm{Jac}(\mathfrak{X}_{1,\mathfrak{n}})$ modulo the subgroups of the form $\mathrm{Jac}(\mathfrak{X}_{1,\mathfrak{m}})$ for proper divisors \mathfrak{m} of \mathfrak{n} , mapping to $\mathrm{Jac}(\mathfrak{X}_{1,\mathfrak{n}})$ via the usual degeneracy maps. By $\mathcal{A}_{\mathfrak{n}}$ we denote the Néron model of $A_{\mathfrak{n}}$ over X , and by $\mathcal{A}_{\mathfrak{n},\mathcal{O}_x}$ and $\mathcal{A}_{\mathfrak{n},x}$, its base change to \mathcal{O}_x and k_x respectively. We denote by

$$E_{\mathfrak{n}} := \mathbb{T}_{1,\mathfrak{n}}^{\mathrm{new}} \hookrightarrow \mathrm{End}_F^0(J_{1,\mathfrak{n}}^{\mathrm{new}})$$

the \mathbb{Q} -subalgebra spanned by the Hecke operators T_x and S_x where x ranges over all places $x \in |X_{\mathfrak{n}}|$. Using the Néron model, one obtains for any place $x \in |X_{\mathfrak{n}}|$ an algebra-homomorphism $\mathrm{End}_F(A_{\mathfrak{n}}) \hookrightarrow \mathrm{End}_{k_x}(\mathcal{A}_{\mathfrak{n},x})$ which is injective, so we may identify $\mathrm{End}_F(A_{\mathfrak{n}})$ with a subring of $\mathrm{End}_{k_x}(\mathcal{A}_{\mathfrak{n},x})$, and in particular we regard $E_{\mathfrak{n}}$ as subalgebra of $\mathrm{End}_{k_x}^0(\mathcal{A}_{\mathfrak{n},x})$. The following properties will be important.

(a) The abelian variety $A_{\mathfrak{n}}$ is ordinary over F by Theorem 8.2 and has good reduction at all places $x \in |X_{\mathfrak{n}}|$, because $\mathfrak{X}_{1,\mathfrak{n}}$ is a smooth projective curve over $X_{\mathfrak{n}}$.

(b) $E_{\mathfrak{n}}$ is an étale \mathbb{Q} -algebra, i.e., a product of finite field extensions of \mathbb{Q} , by the multiplicity one property mentioned above.

(c) For each indecomposable idempotent ε of $E_{\mathfrak{n}}$ (so that $\varepsilon E_{\mathfrak{n}}$ is a field), we have $\dim(\varepsilon A_{\mathfrak{n}}) = [\varepsilon E_{\mathfrak{n}} : \mathbb{Q}]$ by the theory of newforms.

(d) At all $x \in X \setminus X_{\mathfrak{n}}$, each factor $\varepsilon A_{\mathfrak{n}}$ as in (c) has bad reduction; the base change to $X_{\mathfrak{n}}$ of $\mathcal{A}_{\mathfrak{n}}$ is an abelian scheme.

(e) For each $x \in |X_{\mathfrak{n}}|$, denote by $\pi_x, V_x \in \mathrm{End}_{k_x}(\mathcal{A}_{\mathfrak{n},x})$ the q_x -Frobenius endomorphism and q_x -Verschiebung on $\mathcal{A}_{\mathfrak{n},x}$, so that $T_x = \pi_x + S_x V_x$ in $\mathrm{End}_{k_x}(\mathcal{A}_{\mathfrak{n},x})$ by [Dri76, § 11, Theorem 2].

For each place x of $|X_{\mathfrak{n}}|$ we define $g_x(z) \in E_{\mathfrak{n}}[z]$ as the polynomial

$$g_x(z) = z^2 - T_x z + S_x q_x. \tag{B1}$$

We note that T_x and S_x lie in the maximal order of $E_{\mathfrak{n}}$.

For a (rational) prime $\ell \neq p$, we denote by $\mathrm{Ta}_{\ell}(A_{\mathfrak{n}})$ the ℓ -adic Tate module; it is a representation of G_F on a free \mathbb{Z}_{ℓ} -module of rank $2 \dim A_{\mathfrak{n}}$. We set $V_{\ell}(A_{\mathfrak{n}}) := \mathrm{Ta}_{\ell}(A_{\mathfrak{n}}) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}$. Correspondingly we define $E_{\mathfrak{n},\ell} := E \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$. For every prime λ of $E_{\mathfrak{n}}$ above ℓ , we define $V_{\lambda}(A_{\mathfrak{n}}) := V_{\ell}(A_{\mathfrak{n}}) \otimes_{E_{\mathfrak{n},\ell}} (E_{\mathfrak{n}})_{\lambda}$ where $(E_{\mathfrak{n}})_{\lambda}$ denotes the completion at λ . The notation V_{ℓ} and V_{λ} is used analogously for the pair $(\varepsilon A_{\mathfrak{n}}, \varepsilon E_{\mathfrak{n}})$.

Lacking a suitable reference, we give a proof of the following result which describes properties well known for abelian varieties attached to the weight 2 elliptic modular form by Shimura.

PROPOSITION B.2. Denote by A the quotient abelian variety εA_n of A_n and set $E := \varepsilon E_n$. Then:

- (a) let ℓ be a prime different from p and λ a place of E above ℓ . Then $V_\lambda(A)$ is an absolutely irreducible two-dimensional representation of G_F over E_λ ;
- (b) the embedding $E \hookrightarrow \text{End}_F^0(A)$ is an isomorphism and the abelian variety A is F -simple.

Proof. For (a) we first recall the following well-known argument that $V_\lambda(A)$ is a free E_λ -module of rank two: By replacing A by an isogenous abelian variety one can assume that the ring of integers \mathcal{O}_E of E embeds into $\text{End}_F(A)$. Since the class group of \mathcal{O}_E is finite, there exists m such that λ^m is a principal ideal, say (a) for some $a \in \mathcal{O}_E \setminus \{0\}$. Now multiplication by a induces an endomorphism a_A of A that is an isogeny of degree $\text{Norm}_{E/\mathbb{Q}}(a)^2 = \text{Norm}_{E/\mathbb{Q}}(\lambda)^{2m}$. A key insight to see this is that degree and Norm for non-zero elements in \mathcal{O}_E are multiplicative maps to \mathbb{Z} that agree on \mathbb{Z} up to a power and hence can only differ on \mathcal{O}_E by that power; see [Mum70, § 19]. Since ℓ is different from p , the kernel of a_A is an étale ℓ -primary torsion scheme of order $|\mathcal{O}_E/\lambda|^{2m}$. One deduces that the limit $\varprojlim_n \text{Ker}(a_A^n)$ must be free over the completion of \mathcal{O}_E at λ of rank two, which implies that $V_\lambda(A)$ is free of rank two over E_λ .

The proof of the absolute irreducibility is now similar to that of [Rib77, Theorem 2.3], which goes at least back to Deligne. Suppose that the semisimplification of $V_\lambda(A)$ over some extension E'_λ of E_λ is the direct sum of two one-dimensional representations, say given by characters χ_i , $i = 1, 2$, of G_F . Because A has total toric reduction at ∞ , one of the characters must be trivial when restricted to the decomposition group at ∞ ; the other restriction must be the cyclotomic character. At places not dividing n , the representations are unramified; at those dividing n , they are finitely ramified. We may thus assume that χ_1 is of finite order and χ_2 is the product of the cyclotomic character by a character χ'_2 of finite order. For a place x not above $n\infty$, we deduce that the trace of Frob_x acting on $V_\lambda(A)$ is $\chi_1(\text{Frob}_x) + q_x \chi'_2(\text{Frob}_x)$. From [Dri76, § 11, Theorem 2] it is known that at places x not above $n\infty$, the complex absolute value of the trace satisfies the Ramanujan–Petersson bound $2\sqrt{q_x}$, and this gives a contradiction.

For the proof of (b), we follow [Rib92, Theorem 3.3] but use the Tate conjecture for function fields over finite fields, which is a theorem due to Zarhin [Zar75]. It asserts that

$$\text{End}_F^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \text{End}_{\mathbb{Q}_\ell[G_F]}(V_\ell(A)).$$

The ring $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ embeds into the left-hand side. Decomposing it and $V_\ell(A)$ according to the primes λ above ℓ we obtain the canonical embedding $E_\lambda \hookrightarrow \text{End}_{\mathbb{Q}_\ell[G_F]}(V_\lambda(A))$, which is an isomorphism, by (a). It follows that the embedding $E \hookrightarrow \text{End}_F^0(A)$ becomes an isomorphism after tensoring with \mathbb{Q}_ℓ over \mathbb{Q} , and by faithful flatness of this operation we deduce $E = \text{End}_F^0(A)$. Now the Poincaré complete reducibility theorem for abelian varieties shows that A is simple. \square

We recall the standard short exact sequence of p -divisible groups over F :

$$0 \longrightarrow A_n(p)_{\text{loc}} \longrightarrow A_n(p) \longrightarrow A_n(p)_{\text{ét}} \longrightarrow 0, \tag{B2}$$

where the subscripts *loc* and *ét* denote, as usual, the local and étale parts. If we wish to work in the isogeny category of p -divisible groups we add a superscript 0 to the notation.

The action of E_n on A_n (up to isogeny) induces an action of the completion $E_{n,p}$ of E_n at p on the p -divisible group $A_n^0(p)$. The \mathbb{Q}_p -algebra $E_{n,p}$ is a product of fields $(E_n)_\mathfrak{p}$ obtained as the completion of E_n at its places \mathfrak{p} above p . Using for instance the corresponding idempotents, one can decompose $A_n^0(p)$ as a direct product over p -divisible groups $(A_n^0(p))_\mathfrak{p}$ which carry an action of $(E_n)_\mathfrak{p}$. By an argument as in the proof of Proposition B.2(a), one shows that the height of

$(A_n^0(p))_{\mathfrak{p}}$ is a multiple of the degree $[(E_n)_{\mathfrak{p}} : \mathbb{Q}_p]$; we define the *height of $(A_n^0(p))_{\mathfrak{p}}$ over $(E_n)_{\mathfrak{p}}$* as the positive integer

$$\text{height}_{(E_n)_{\mathfrak{p}}}(A_n^0(p))_{\mathfrak{p}} := \frac{\text{height}(A_n^0(p))_{\mathfrak{p}}}{[(E_n)_{\mathfrak{p}} : \mathbb{Q}_p]}.$$

A similar notation of height over E applies to any finite field extension E of \mathbb{Q}_p and any p -divisible group \mathcal{G} over an irreducible scheme S such that E embeds into $\text{End}^0(\mathcal{G})$. Passing to local and étale parts of a p -divisible group is functorial in homomorphism. Therefore the action of $E_{n,p}$ on $A_n^0(p)$ induces actions of $E_{n,p}$ on $(A_n^0(p))_?$ for $? \in \{\text{loc}, \text{ét}\}$. For the same reason, the passage to local or étale parts commutes with passing from $A_n^0(p)$ to $(A_n^0(p))_{\mathfrak{p}}$.

LEMMA B.3. *Let x be a place of $|X_n|$ and \mathfrak{p} a prime ideal of E_n above p . Let $v_{\mathfrak{p}}$ be the valuation on $(E_n)_{\mathfrak{p}}$ such that $v_{\mathfrak{p}}(q_x) = 1$. Let $\varepsilon \in E_n$ denote an indecomposable idempotent. Then the following hold.*

- (a) *The polynomial $g_x(z)$ from (B1) annihilates π_x .*
- (b) *If $T_x^2 \neq 4S_x q_x$ in εE_n , then the subalgebra $\varepsilon E_n[\pi_x] \subset \text{End}_{k_x}^0(\varepsilon \mathcal{A}_{n,x})$ is free over εE_n of rank two, and $(\mathcal{A}_{n,\mathcal{O}_x}^0(p))_{\mathfrak{p}}$ has height 2 over $(E_n)_{\mathfrak{p}}$.*
- (c) *Localizing (B2) in the isogeny category at \mathfrak{p} , one obtains the short exact sequence*

$$0 \longrightarrow (A_n^0(p)_{\text{loc}})_{\mathfrak{p}} \longrightarrow (A_n^0(p))_{\mathfrak{p}} \longrightarrow (A_n^0(p)_{\text{ét}})_{\mathfrak{p}} \longrightarrow 0, \tag{B3}$$

in which the outer terms have height 1 over $(E_n)_{\mathfrak{p}}$.

- (d) *If $v_{\mathfrak{p}}(T_x) = 0$, the sequence (B3) extends to a local-étale sequence over \mathcal{O}_x :*

$$0 \longrightarrow (\mathcal{A}_{n,\mathcal{O}_x}^0(p)_{\text{loc}})_{\mathfrak{p}} \longrightarrow (\mathcal{A}_{n,\mathcal{O}_x}^0(p))_{\mathfrak{p}} \longrightarrow (\mathcal{A}_{n,\mathcal{O}_x}^0(p)_{\text{ét}})_{\mathfrak{p}} \longrightarrow 0.$$

- (e) *Whenever $v_{\mathfrak{p}}(T_x) < 1/2$, the slope filtration sequence (slopes are indicated by superscripts)*

$$0 \longrightarrow (\mathcal{A}_{n,x}^0(p))_{\mathfrak{p}}^{1-v_{\mathfrak{p}}(T_x)} \longrightarrow (\mathcal{A}_{n,x}^0(p))_{\mathfrak{p}} \longrightarrow (\mathcal{A}_{n,x}^0(p))_{\mathfrak{p}}^{v_{\mathfrak{p}}(T_x)} \longrightarrow 0,$$

is split, with outer terms of height 1 over $(E_n)_{\mathfrak{p}}$.

- (f) *If $v_{\mathfrak{p}}(T_x) \geq 1/2$, then $(\mathcal{A}_{n,x}^0(p))_{\mathfrak{p}}$ has constant slope $1/2$.*

Proof. Since all assertions are assertions in the respective isogeny categories, we may choose an indecomposable idempotent ε and give the proof after replacing $(A_n, E_n, \mathcal{A}_n)$ by $(A := \varepsilon A_n, E := \varepsilon E_n, \mathcal{A} := \varepsilon \mathcal{A}_n)$ so that E is a field and A is a simple abelian variety with $\dim A = [E : \mathbb{Q}]$ and Néron model \mathcal{A} . We use π_x, g_x etc. for this situation as well. This simplification simply means that we focus on the Galois orbit of a single automorphic eigenform Φ of level $\mathcal{K}_1(\mathfrak{n})$ that is new at \mathfrak{n} , and on the corresponding factor A of A_n and coefficient field E .

By the easy part of the Tate conjecture one has a monomorphism

$$\text{End}_{k_x}^0(\mathcal{A}_x) \hookrightarrow \text{Aut}_{\text{Gal}(\overline{k_x}/k_x)}(V_{\ell}(\mathcal{A}_x))$$

for any $\ell \neq p$. It is well known, e.g. [Dri76, § 11, Theorem 2, part 2], that g_x is the characteristic polynomial of $\text{Frob}_x \in \text{Gal}(\overline{k_x}/k_x)$ and that π_x maps to Frob_x under the above inclusion. Hence (a) is proved.

The condition in (b) is that the two eigenvalues of Frob_x acting on $V_{\ell}(A)$ are distinct. Since they both occur in the characteristic polynomial, the extension $E \hookrightarrow E[\pi_x]$ is proper. Now since π_x satisfies a polynomial of degree two, we have either that $E[\pi_x]$ is a field extension of E of

degree two or that $E[\pi_x] \cong E \times E$ with E being embedded diagonally. This proves the assertion on $E \rightarrow E[\pi_x]$. From this it follows that $\mathcal{A}_{\mathcal{O}_x}^0(p)$ is of height 1 over $E[\pi_x] \otimes_{\mathbb{Q}} \mathbb{Q}_p$. After localization we obtain that $(\mathcal{A}_{\mathcal{O}_x}^0(p))_{\mathfrak{p}}$ is of height 1 over $E[\pi_x]_{\mathfrak{p}}$, and this proves the remaining part of (b).

Consider now a pair (x, \mathfrak{p}) with $x \in |X_n|$ and \mathfrak{p} such that $v_{\mathfrak{p}}(T_x) < 1/2$. Since $v_{\mathfrak{p}}(S_x q_x) = 1$ we have $T_x^2 \neq 4S_x q_x$, and so (b) applies. By (b) and the p -part of the Tate conjecture over k_x , proven by Tate, see [CCO14, Appendix 1], the minimal polynomial of π_x is g_x . Part (e) now follows from the Dieudonné classification of p -divisible groups over perfect fields. In fact, since k_x is perfect, the sequence in (e) is split. Let us also note that, independently of (b), if $v_{\mathfrak{p}}(T_x) \geq 1/2$, then the possible roots of π_x must have slope $1/2$, and this proves (f).

If we further assume $v_{\mathfrak{p}}(T_x) = 0$, then the Newton polygons of Frobenius on the special and on the generic point of $(\mathcal{A}_{\mathcal{O}_x}^0(p))_{\mathfrak{p}}$ agree. The existence of the filtration as asserted in (d) is therefore implied by [Kat79, (2.6)], see also [Zin01, Theorem 7]. Finally we note that by Theorem 8.2 and Proposition 8.3, the set of points $x \in |X_n|$ at which A is ordinary is dense open. Thus by passing to the generic fiber of such a point, (c) follows from (d). \square

Because of (c) of the above lemma, for every prime \mathfrak{p} of E_n above p , the étale part of $(A_n^0(p))_{\mathfrak{p}}$ defines a one-dimensional Galois representation of G_F on $((\text{Ta}_{\mathfrak{p}} A_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{\mathfrak{p}}$. We shall refer to it by

$$\rho_{\Phi, \mathfrak{p}} : G_F \rightarrow \text{GL}_1((E_n)_{\mathfrak{p}}),$$

where Φ is an automorphic form of level $\mathcal{K}_1(\mathfrak{n})$ that is new at \mathfrak{n} corresponding to an indecomposable idempotent ε of E_n such that \mathfrak{p} is a prime of εE_n above p ; note that given \mathfrak{p} there is a unique such E and hence a unique Galois orbit of Φ . Our notation emphasizes the relation to automorphic forms. The idempotent ε is also denoted by ε_{Φ} if we wish to stress the relation to Φ , and then we write A_{Φ} for $\varepsilon_{\Phi} A_n$, E_{Φ} for εE_n etc.

COROLLARY B.4. *Let Φ be an automorphic form of level $\mathcal{K}_1(\mathfrak{n})$ that is new at \mathfrak{n} . Let \mathfrak{p} be a prime above p of E_{Φ} . Then $\rho_{\Phi, \mathfrak{p}}$ is ramified at $x \in |X_n|$ if and only if $v_{\mathfrak{p}}(T_x) > 0$, i.e., if and only if A_{Φ} has non-ordinary reduction at x .*

Proof. Let x be in $|X_n|$. By Theorem A.4 applied to $(A_n^0(p))_{\mathfrak{p}}$, the representation $\rho_{\Phi, \mathfrak{p}}$ is ramified at x if and only if the étale part of $(\mathcal{A}_{\mathfrak{n}, \mathcal{O}_x}^0(p))_{\mathfrak{p}}$ is of height 1 over $(E_n)_{\mathfrak{p}}$. Going through the possibilities in Lemma B.3, the corollary follows. \square

We also give the following intrinsic characterization of $\rho_{\Phi, \mathfrak{p}}$.

COROLLARY B.5. *Let Φ be a cuspidal automorphic eigenform of level $\mathcal{K}_1(\mathfrak{n})$ that is new at \mathfrak{n} . Let \mathfrak{p} be a prime above p of E_{Φ} . Denote by X_{Φ}^{ord} the set of places $x \in |X_n|$ at which A_{Φ} has ordinary reduction and by $\alpha_{x, \mathfrak{p}} \in (E_{\Phi})_{\mathfrak{p}}$ the unique root of g_x of slope zero at $x \in X_{\Phi}^{\text{ord}}$. Then*

$$\forall x \in X_{\Phi}^{\text{ord}} : \rho_{\Phi, \mathfrak{p}}(\text{Frob}_x) = \alpha_{x, \mathfrak{p}}, \tag{B4}$$

and $\rho_{\Phi, \mathfrak{p}}$ is uniquely characterized by (B4). Moreover, $\rho_{\Phi, \mathfrak{p}}$ is infinitely ramified when restricted to the decomposition group at any $x \in |X_n| \setminus X_{\Phi}^{\text{ord}}$.

In fact, condition (B4) for any subset of X_{Φ}^{ord} of density one already characterizes $\rho_{\Phi, \mathfrak{p}}$.

Proof. The first assertion follows from Lemma B.3(d): The restriction of $\rho_{\Phi, \mathfrak{p}}$ to a decomposition group G_{F_x} for $x \in X_{\Phi}^{\text{ord}}$ is the Galois representation of G_{F_x} attached to $(\mathcal{A}_{\mathfrak{n}, \mathcal{O}_x}^0(p)_{\text{ét}})_{\mathfrak{p}}$ that is unramified at x . It is determined by the image of Frob_x which in turn is equal to the action of the geometric Frobenius on the special fiber $(\mathcal{A}_{\mathfrak{n}, x}^0(p)_{\text{ét}})_{\mathfrak{p}}$. This is the étale quotient of $(\mathcal{A}_{\mathfrak{n}, x}^0(p))_{\mathfrak{p}}$ on

which Frob_x has characteristic polynomial g_x . From here the first assertion is straightforward. The second follows from the Čebotarov density theorem. The last assertion is implied by Theorem A.4 and Remark A.5. \square

We say that Φ is semistable but non-good at x if its local representation at x is an unramified twist of a Steinberg representation, say by a character $\chi_{\Phi,x}$, or equivalently if the associated Weil–Deligne representation has unramified underlying Weil representation, and that the monodromy N is non-zero. At such a place, the conductor exponent of Φ is 1. For the Hecke operator U_x defined as for instance in [Wil86, (1.3.1)], its eigenvalue on Φ is equal to $\chi_{\Phi,x}(\text{Frob}_x)$. Moreover, the following proposition describes $\rho_{\Phi,\mathfrak{p}}$ at x in a simple way.

PROPOSITION B.6. *Let Φ be an automorphic form of level $\mathcal{K}_1(\mathfrak{n})$ that is new at \mathfrak{n} . Let x be a place of $\text{Spec } A \setminus |X_{\mathfrak{n}}|$ at which Φ is semistable. Then for any prime \mathfrak{p} above p of E_{Φ} the representation $\rho_{\Phi,\mathfrak{p}}$ is isomorphic to $\chi_{\Phi,x}$ when restricted to G_{F_x} , and in particular it is unramified at x .*

Proof. If Φ is an unramified Steinberg representation at x , then, by the local Langlands correspondence for the ℓ -adic representations of A_{Φ} , one deduces that A_{Φ} has semistable reduction but not good reduction at x . Since the action of E_{Φ} induces a faithful action on the reduction, the reduction must be totally toric because $[E_{\Phi} : \mathbb{Q}] = \dim A_{\Phi}$. Note that this behavior also occurs for the place ∞ .

By the generalization of the Tate curve construction due to Mumford and Raynaud to abelian varieties, e.g. [BL91], there exists a π -adic uniformization of \mathcal{A}_x , i.e., a short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow T(\overline{F_x}) \longrightarrow A_{\Phi}(\overline{F_x}) \longrightarrow 0$$

such that:

- (a) T is a torus over F_x with $\dim T = \dim A_{\Phi}$ that is split over an unramified extension of F_x ;
- (b) there exists an E_{Φ} -order \mathcal{O} such that T carries an action $\mathcal{O} \rightarrow \text{End}(T)$ of \mathcal{O} , Λ is a torsion-free \mathcal{O} -module of generic rank one and $\Lambda \rightarrow T(\overline{F_x})$ is \mathcal{O} equivariant;
- (c) there exists an \mathcal{O} -linear positive definite Riemann form, i.e., an \mathcal{O} -homomorphism $\sigma : \Lambda \rightarrow X^*(T)$, the character group of T , such that (i) for all $\lambda, \lambda' \in \Lambda$: $\sigma(\lambda')(\lambda) = \sigma(\lambda)(\lambda')$ and (ii) the symmetric bilinear form $\langle \lambda, \lambda' \rangle := -\log |\sigma(\lambda)(\lambda')|$ on Λ is positive definite;
- (d) the sequence is G_{F_x} -equivariant.

Using the ℓ -adic representations, one sees that the action of G_{F_x} on Λ is given by $\chi_{\Phi,x}$. Now the p^n -torsion of A_{Φ} is clearly $\Lambda^{1/p^n} / \Lambda \subset T(\overline{F_x}) / \Lambda$, and the proposition follows. \square

Next, we present some results on the (semisimplification of the) mod p reduction of the representation on $\text{Ta}_p A_{\mathfrak{n}}$. The main complication here is that we have to do this integrally over \mathbb{Z}_p and not after tensoring $\text{Ta}_p A_{\mathfrak{n}}$ with \mathbb{Q}_p . This leads to the usual difficulties when studying congruences. For this we define the orders $\Lambda \subset \tilde{\Lambda} \subset \text{End}_F(A_{\mathfrak{n}})$, where Λ is the \mathbb{Z} -span in $\text{End}_F(A_{\mathfrak{n}})$ of the Hecke-operators T_x $x \in |X_{\mathfrak{n}}|$, and $\tilde{\Lambda}$ is the \mathbb{Z} -span of T_x and $q_x S_x$, $x \in |X_{\mathfrak{n}}|$. By $\mathcal{O}_{\mathfrak{n}}$ we denote the maximal order of $E_{\mathfrak{n}}$ which we consider inside $\text{End}_F^0(A_{\mathfrak{n}})$. One has $\mathcal{O}_{\mathfrak{n}} \supset \tilde{\Lambda}$ because $\text{End}_F(A_{\mathfrak{n}})$ is finitely generated over \mathbb{Z} . Before we go on, we need the following result.

LEMMA B.7. *The subrings $\Lambda \subset \tilde{\Lambda}$ are orders of $E_{\mathfrak{n}}$.*

Proof. The result is proved as in [Rib77, Corollary 3.1] by Ribet and rests on results from [DS74] by Deligne and Serre. We indicate the argument: it suffices to show that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = E_{\mathfrak{n}}$ under the given inclusion $\Lambda \subset E_{\mathfrak{n}}$. To prove this, we will show that for any (Galois orbit of an) eigenform

Φ , the ring E_Φ is generated by the images of the Hecke operators T_x , where x ranges over a dense open subset U of X_n .

Let ℓ be a prime different from p , and consider the ℓ -adic representation $\rho_{\Phi,\ell}$ of G_F on $V_\ell(A_\Phi)$. We denote by $\rho_{\Phi,\ell}^{ss}$ its semisimplification. Since the representation $\rho_{\Phi,\ell}$ is unramified at the places x in U and since they are dense, the elements Frob_x , $x \in U$, form a dense subgroup in the image of $G_F \rightarrow \text{Aut}(V_\ell(A_\Phi))$. Hence $\rho_{\Phi,\ell}^{ss}$ is determined uniquely by the traces $\rho_{\Phi,\ell}(\text{Frob}_x)$, $x \in U$; cf. [DS74, Lemma 3.2]. Let E'_Φ be the subfield of E_Φ generated over \mathbb{Q} by the eigenvalues of T_x acting on Φ for all $x \in U$. Let σ be an automorphism of \mathbb{C} that fixes E'_Φ , and let Φ^σ denote the corresponding eigenform. Then $\rho_{\Phi,\ell}^{ss}$ and $\rho_{\Phi^\sigma,\ell}^{ss}$ are isomorphic because we have $\text{Tr } \rho_{\Phi,\ell}(\text{Frob}_x) = \text{Tr } \rho_{\Phi^\sigma,\ell}(\text{Frob}_x)$ for all $x \in U$. But then $\det \rho_{\Phi,\ell}(\text{Frob}_x) = \det \rho_{\Phi^\sigma,\ell}(\text{Frob}_x) = \det \rho_{\Phi,\ell}(\text{Frob}_x)^\sigma$ for all $x \in |X_n|$, and so the eigenvalues of Φ under S_x must lie in E'_Φ , which completes the argument. \square

As we shall see shortly, mod p congruences are measured by the maximal ideals \mathfrak{m} of Λ that contain p . Any such \mathfrak{m} may be contained in several maximal ideals \mathfrak{p} of \mathcal{O}_n , which are the prime ideals we have considered so far. We define $\Lambda_{\mathfrak{m}}$ to be the completion of Λ at its maximal ideal \mathfrak{m} . For any maximal ideal $\tilde{\mathfrak{m}}$ of $\tilde{\Lambda}$ above \mathfrak{m} we define $\tilde{\Lambda}_{\tilde{\mathfrak{m}}}$ analogously. We note that congruences can occur in two ways: First, it may happen that for a fixed form Φ , two prime ideals \mathfrak{p}_1 and \mathfrak{p}_2 of E_Φ above p may lie above \mathfrak{m} . Second, for two different forms Φ_1 and Φ_2 there could be ideals \mathfrak{p}_i of E_{Φ_i} above p that both contain \mathfrak{m} . We also note that the residue field Λ/\mathfrak{m} could be properly contained in the residue field $\mathcal{O}_n/\mathfrak{p}$ for any \mathfrak{p} above \mathfrak{m} .

LEMMA B.8. *The map $\tilde{\mathfrak{m}} \mapsto \tilde{\mathfrak{m}} \cap \Lambda$ is a bijection between the maximal ideals of $\tilde{\Lambda}$ above p and the maximal ideals of Λ above p . Moreover, one has $\Lambda/\mathfrak{m} \cong \tilde{\Lambda}/\tilde{\mathfrak{m}}$ if \mathfrak{m} and $\tilde{\mathfrak{m}}$ correspond under this bijection.*

Proof. It is a standard argument to see that the map is surjective: The maximal ideals of $\tilde{\Lambda}$ and Λ are in bijection with the indecomposable idempotents of the p -adic completions of the two rings, denoted by $\tilde{\Lambda}_p$ and Λ_p , respectively. By the previous lemma, $\Lambda \rightarrow \tilde{\Lambda}$ is an inclusion of free \mathbb{Z} -modules of the same rank and hence $\Lambda_p \rightarrow \tilde{\Lambda}_p$ is an inclusion of free \mathbb{Z}_p -modules of the same rank. Therefore, distinct idempotents of Λ_p have to map to distinct idempotents of $\tilde{\Lambda}_p$, and this gives the surjectivity of the map in the lemma.

The main point of the proof of injectivity is the claim that under any homomorphism $\tilde{\Lambda}$ to a field of characteristic p , the elements $q_x S_x$ map to zero. This is so because S_x has finite order, say d_x , in E_n , so that $(q_x S_x)^{d_x} = q_x^{d_x} \cdot 1$, and the latter element becomes zero under maps to characteristic p . But if the target is a field, any nilpotent element will be zero, and so the claim follows. We now show how to deduce the injectivity from the claim.

Let $\tilde{\mathfrak{m}} \subset \tilde{\Lambda}$ be a maximal ideal containing p that contracts to $\mathfrak{m} \subset \Lambda$. By the previous paragraph, $\tilde{\mathfrak{m}}$ contains the elements $q_x S_x$ ($\in \text{End}_F(A)$), and hence $\tilde{\Lambda}/\tilde{\mathfrak{m}}$ is generated over \mathbb{F}_p by the images of the T_x , $x \in |X_n|$. It follows that the composite $\Lambda \rightarrow \tilde{\Lambda} \rightarrow \tilde{\Lambda}/\tilde{\mathfrak{m}}$ is surjective. By definition, the kernel of this composite is \mathfrak{m} , and it follows that the induced map $\Lambda/\mathfrak{m} \rightarrow \tilde{\Lambda}/\tilde{\mathfrak{m}}$ is a surjection between finite fields and hence an isomorphism. We also see that $\tilde{\mathfrak{m}}$ is the unique ideal of $\tilde{\Lambda}$ that is generated by \mathfrak{m} and by $\{q_x S_x \mid x \in |X_n|\}$, and this shows the surjectivity. \square

We now fix a maximal ideal \mathfrak{m} of Λ above p and denote by $\tilde{\mathfrak{m}}$ the corresponding maximal ideal of $\tilde{\Lambda}$. We denote by $P_{\mathfrak{m}}$ the set of prime ideals \mathfrak{p} of \mathcal{O}_n that contain \mathfrak{m} . Then $\tilde{\Lambda}_{\tilde{\mathfrak{m}}} \hookrightarrow \prod_{\mathfrak{p} \in P_{\mathfrak{m}}} (E_n)_{\mathfrak{p}}$ factors via

$$\tilde{\Lambda}_{\tilde{\mathfrak{m}}} \hookrightarrow \prod_{\mathfrak{p} \in P_{\mathfrak{m}}} (\mathcal{O}_n)_{\mathfrak{p}}. \tag{B5}$$

We also denote by $\Phi_{\mathfrak{p}}$ the automorphic form (up to Galois conjugacy) such that $(E_{\Phi})_{\mathfrak{p}} \neq 0$ so that we have the representation

$$\prod_{\mathfrak{p} \in P_m} \rho_{\Phi_{\mathfrak{p}}, \mathfrak{p}} : G_F \rightarrow \prod_{\mathfrak{p} \in P_m} \mathrm{GL}_1((\mathcal{O}_n)_{\mathfrak{p}}).$$

COROLLARY B.9. (a) *The image of $\prod_{\mathfrak{p} \in P_m} \rho_{\Phi_{\mathfrak{p}}, \mathfrak{p}}$ is contained in $\mathrm{GL}_1(\tilde{\Lambda}_{\tilde{\mathfrak{m}}})$ under the inclusion (B5), and so this representation factors via a unique representation*

$$\rho_{\tilde{\mathfrak{m}}} :: G_F \rightarrow \mathrm{GL}_1(\tilde{\Lambda}_{\tilde{\mathfrak{m}}}).$$

(b) *The reduction $\rho_{\tilde{\mathfrak{m}}} \pmod{\tilde{\mathfrak{m}}}$ defines a representation*

$$\bar{\rho}_m :: G_F \rightarrow \mathrm{GL}_1(\Lambda/\mathfrak{m}) \cong \mathrm{GL}_1(\tilde{\Lambda}/\tilde{\mathfrak{m}}),$$

whose image spans the finite field Λ/\mathfrak{m} over \mathbb{F}_p .

Proof. Denote by U an open dense subset of X_n that consists of ordinary places only. By the Čebotarov density theorem, together with Theorem 8.2 and Proposition 8.3, the Frobenius automorphisms Frob_x , $x \in U$, are dense in the maximal quotient of G_F unramified outside U . It follows that the images of these Frob_x are dense in the closed subset $\prod_{\mathfrak{p} \in P_m} \rho_{\Phi_{\mathfrak{p}}, \mathfrak{p}}(G_F)$ of $\mathrm{GL}_1(\prod_{\mathfrak{p} \in P_m} (\mathcal{O}_n)_{\mathfrak{p}})$. Hence $\prod_{\mathfrak{p} \in P_m} \rho_{\Phi_{\mathfrak{p}}, \mathfrak{p}}$ factors via the smallest subring of $\prod_{\mathfrak{p} \in P_m} (\mathcal{O}_n)_{\mathfrak{p}}$ over \mathbb{Z}_p that contains $\prod_{\mathfrak{p} \in P_m} \rho_{\Phi_{\mathfrak{p}}, \mathfrak{p}}(\mathrm{Frob}_x)$ for all such x , and this choice of subring is optimal.

Denote by α_x and β_x , for x of good ordinary reduction, the roots of g_x in the completion $E_{n,p}$ at p (which is a product of p -adic fields). We make choices so that $v_{\mathfrak{p}}(\alpha_x) = 0$ and $v_{\mathfrak{p}}(\beta_x) = 1$ for all primes \mathfrak{p} of E_n above p ; note that α_x and β_x can be found in $E_{n,p}$ since for an ordinary place x the polynomial g_x splits in $E_{n,p}$. It is then clear that α_x and β_x lie in $(\mathcal{O}_n)_{\mathfrak{p}}$. We observe that by this choice we have $\alpha_x \equiv T_x \pmod{q_x}$ in $(\mathcal{O}_n)_{\mathfrak{p}}$. Using the Newton method for $g_x(z) = z^2 + T_x z + q_x S_x$ with initial value $z_0 = T_x$, we see that α_x lies in fact in $\mathbb{Z}_p[T_x, q_x S_x]$. Hence, after completion at $\tilde{\mathfrak{m}}$, we find that α_x projected to $\prod_{\mathfrak{p} \in P_m} (\mathcal{O}_n)_{\mathfrak{p}}$ lies in $\tilde{\Lambda}_{\tilde{\mathfrak{m}}}$. Now this projection of α_x is equal to $\prod_{\mathfrak{p} \in P_m} \rho_{\Phi_{\mathfrak{p}}, \mathfrak{p}}(\mathrm{Frob}_x)$, and this completes the proof of (a). Part (b) is now immediate from Lemma B.8. \square

Remark B.10. We do not know whether in fact $\tilde{\Lambda}_{\tilde{\mathfrak{m}}}$ is the smallest subring of $\prod_{\mathfrak{p} \in P_m} (\mathcal{O}_n)_{\mathfrak{p}}$ over \mathbb{Z}_p over which $\prod_{\mathfrak{p} \in P_m} \rho_{\Phi_{\mathfrak{p}}, \mathfrak{p}}$ is defined.

Remark B.11. Let $U \subset X_n$ be dense open. The proof of Corollary B.9 shows that $\tilde{\Lambda}_{\tilde{\mathfrak{m}}}$ is generated over \mathbb{Z}_p by the images of $\{T_x, S_x \mid x \in |U|\}$. Using also Lemma B.8, moreover, Λ/\mathfrak{m} is generated over \mathbb{F}_p by the images of $\{T_x \mid x \in |U|\}$.

To describe the multiplicity of $\bar{\rho}_m$ in $A_n[p]^{\mathrm{ss}}$, we denote, for each $\mathfrak{p} \in P_m$, by $d_{m,\mathfrak{p}}$ the degree of $(\mathcal{O}_n)_{\mathfrak{p}}$ over Λ/\mathfrak{m} and by $e_{m,\mathfrak{p}}$ the ramification index of $(\mathcal{O}_n)_{\mathfrak{p}}$ over \mathbb{Z}_p .

PROPOSITION B.12. *Denote by $A_n[p]^{\mathrm{ss}}$ the semisimplification of $A_n[p]$ as a module over $\mathbb{F}_p[G_F]$. Then the multiplicity of $\bar{\rho}_m$ in $A_n[p]^{\mathrm{ss}}$ is*

$$t_m := \sum_{\mathfrak{p} \in I_m} d_{m,\mathfrak{p}} e_{m,\mathfrak{p}}.$$

This is also the multiplicity of the semisimplification of $A_n[p]$ as a Λ -module of Λ/\mathfrak{m} .

Proof. We choose an isogeny $\xi : A_n \rightarrow A'$ where A' is an abelian variety with endomorphism ring equal to \mathcal{O}_n . The induced map $A_n[p^n] \rightarrow A'[p^n]$ has kernel and cokernel bounded independently of n . Thus it will suffice to show that $A'[p^n]^{\text{ss}[\tilde{\mathfrak{m}}]}(K^{\text{sep}}) \cong (\tilde{\Lambda}/\tilde{\mathfrak{m}})^{nt_m}$ for large n , where $[\tilde{\mathfrak{m}}]$ denotes the $\tilde{\mathfrak{m}}$ -torsion of the semisimplification. For this, it suffices to show the latter for $n = 1$. However,

$$A'[p]^{\text{ss}}(K^{\text{sep}}) \cong \bigoplus_{\mathfrak{p}} (\varepsilon_{\mathfrak{p}} A'_{\text{ét}}[p])^{\text{ss}}(K^{\text{sep}}).$$

Now each $(\varepsilon_{\mathfrak{p}} A'_{\text{ét}}[p])^{\text{ss}}(K^{\text{sep}})$ is isomorphic to $(\mathcal{O}_n/\mathfrak{p})^{e_{m,\mathfrak{p}}}$ for $\tilde{\mathfrak{m}} = \mathfrak{p} \cap \tilde{\Lambda}$. Moreover, for $\mathfrak{p} \in P_m$, each $\mathcal{O}_n/\mathfrak{p}$ is isomorphic to $(\Lambda/\mathfrak{m})^{d_{m,\mathfrak{p}}}$ as a vector space over Λ/\mathfrak{m} , and so the result is proved. \square

Finally, from the p -adic setting we deduce some consequences on the Frobenius action and on ramification for the mod p situation. For \mathfrak{m} as above, define $X_n^{\text{m-ord}} \subset X_n$ as the subset of those x such that for one $\mathfrak{p} \in P_m$ one has $v_{\mathfrak{p}}(T_x) = 0$ together with the generic point of X_n .¹²

THEOREM B.13. (a) *At $x \in |X_n^{\text{m-ord}}|$, the representation $\bar{\rho}_{\mathfrak{m}}$ is unramified and one has*

$$\bar{\rho}_{\mathfrak{m}}(\text{Frob}_x) = T_x \pmod{\mathfrak{m}} \quad \text{and} \quad \rho_{\Phi_{\mathfrak{p},\mathfrak{p}}}(\text{Frob}_x) = \bar{\rho}_{\mathfrak{m}}(\text{Frob}_x) \pmod{\mathfrak{p}} \tag{B6}$$

for those $\mathfrak{p} \in P_m$ such that $v_{\mathfrak{p}}(T_x) = 0$; the right equality uses the natural inclusion $\Lambda/\mathfrak{m} \hookrightarrow \mathcal{O}_n/\mathfrak{p}$. In particular, T_x is non-zero modulo \mathfrak{m} .

(b) *For $x \in X_n \setminus X_n^{\text{m-ord}}$, one has $T_x \equiv 0 \pmod{\mathfrak{m}}$.*

We cannot say anything about ramification or the Frobenius action at the places $x \in X_n \setminus X_n^{\text{m-ord}}$.

Proof. The first assertion in (a) follows from the definition of P_m and Corollary B.4. From the proof of Corollary B.9, one deduces the first formula in (B6). The second is immediate from the proof of Proposition B.12. Part (b) is clear from the definitions of $X_n^{\text{m-ord}}$ and of P_m . \square

We end this appendix with some remarks on more general levels than $\mathcal{K}_1(\mathfrak{n})$. Let $\mathcal{K} \subset \text{GL}_2(\tilde{A})$ be a compact open subgroup and choose a non-zero ideal \mathfrak{n}' of A such that $\mathcal{K} \supset \mathcal{K}(\mathfrak{n}')$. Let Φ' be a cuspidal automorphic form for $\text{GL}_2(\mathbb{A}_F)$ that is invariant under \mathcal{K} and that is an eigenform for all T_x , $x \nmid \mathfrak{n}'$. By multiplicity one for GL_2 , and Lemma B.1, there exists a unique minimal level \mathfrak{n} (i.e., $\mathfrak{n} \subset A$ is maximal under inclusion) and a cuspidal eigenform Φ of level $\mathcal{K}_1(\mathfrak{n})$ such that Φ and Φ' have the same eigenvalues for all $x \nmid \mathfrak{n}\mathfrak{n}'$; in fact, one has $\mathfrak{n}|\mathfrak{n}'$. Moreover, the automorphic representation generated by Φ and by Φ' are isomorphic and have the same field of definition E_{Φ} . From Corollary B.5, Theorem B.13 and the above paragraph, one deduces the following theorem.

THEOREM B.14. *Let Φ' be a cuspidal automorphic eigenform for $\text{GL}_2(\mathbb{A}_F)$ of level \mathfrak{n}' . Let \mathfrak{p} be a prime of $E_{\Phi'}$ above p . Denote by $X_{\Phi',\mathfrak{p}}^{\text{ord}}$ the places $x \in X_{\mathfrak{n}'}$ at which g_x has a root α_x in $(E_{\Phi'})_{\mathfrak{p}}$ of slope zero.*

(a) *The set $X_{\mathfrak{n}'} \setminus X_{\Phi',\mathfrak{p}}^{\text{ord}}$ is finite.*

(b) *There exists a unique homomorphism $\rho_{\Phi',\mathfrak{p}} : G_F \rightarrow \text{GL}_1((E_{\Phi'})_{\mathfrak{p}})$ that is unramified over $X_{\Phi',\mathfrak{p}}^{\text{ord}}$ and such that*

$$\forall x \in X_{\Phi',\mathfrak{p}}^{\text{ord}} : \rho_{\Phi',\mathfrak{p}}(\text{Frob}_x) = \alpha_{x,\mathfrak{p}}.$$

¹² The notation $X_n^{\text{m-ord}}$ is chosen to emphasize the similarities with X_n^{ord} ; it could also be motivated by the notion of an ordinary p -divisible group.

(c) One has $\rho_{\Phi',p} = \rho_{\Phi,p}$ for a unique newform Φ of some level \mathfrak{n} dividing \mathfrak{n}' . At all $x \in X_{\mathfrak{n}} \setminus X_{\Phi,p}^{\text{ord}}$ the representation $\rho_{\Phi',p}$ is infinitely ramified.

(d) Denote by $\mathbb{T}_{\mathbb{Z},\mathfrak{n}'}$ the \mathbb{Z} -Hecke-algebra for $\mathcal{K}(\mathfrak{n}')$, and let \mathfrak{P} denote the minimal prime defining Φ' , so that $E_{\Phi'} = \text{Frac}(\mathbb{T}_{\mathbb{Z},\mathfrak{n}'}/\mathfrak{P})$. Let \mathfrak{m} be any maximal ideal that contains \mathfrak{P} and p , and denote by \mathfrak{p} a maximal ideal of $E_{\Phi'}$ under \mathfrak{m} . Then there exists a unique homomorphism $\bar{\rho}_{\Phi',\mathfrak{m}} : G_F \rightarrow \text{GL}_1(\mathbb{T}_{\mathbb{Z},\mathfrak{n}'}/\mathfrak{m})$ that is unramified outside $X_{\Phi',p}^{\text{ord}}$ and such that

$$\forall x \in X_{\Phi',p}^{\text{ord}} : \bar{\rho}_{\Phi',\mathfrak{m}}(\text{Frob}_x) \equiv \alpha_{x,p} \equiv T_x \pmod{\mathfrak{m}}.$$

(e) One has $\rho_{\Phi',p}(\text{Frob}_x) \equiv \bar{\rho}_{\Phi',\mathfrak{m}}(\text{Frob}_x) \pmod{\mathfrak{p}}$ under $\mathbb{T}_{\mathbb{Z},\mathfrak{n}'}/\mathfrak{m} \hookrightarrow \mathcal{O}_{\mathfrak{n}}/\mathfrak{p}$.

(f) For \mathfrak{n} as in (c) and $x \in X_{\mathfrak{n}} \setminus X_{\mathfrak{n}}^{\text{m-ord}}$, one has $T_x \equiv 0 \pmod{\mathfrak{m}}$.

(g) If $\bar{\rho}_{\Phi',\mathfrak{m}}$ is ramified at $x \in X_{\mathfrak{n}'}$, then $T_x \equiv 0 \pmod{\mathfrak{m}}$.

Proof. Let Φ be a cuspidal eigenform as was identified in the paragraph preceding the theorem. Then $X_{\mathfrak{n}'} \setminus X_{\Phi',p}^{\text{ord}} \subset X_{\mathfrak{n}} \setminus X_{\Phi,p}^{\text{ord}}$ is clear and (a) follows. For (b), define $\rho_{\Phi',p} := \rho_{\Phi,p}$ and use the uniqueness from Corollary B.5. Corollary B.5 also yields (c). For (d), observe that we can define $\rho_{\Phi',p}(\text{Frob}_x)$ as $\bar{\rho}_{\Phi',\mathfrak{m}}(\text{Frob}_x) \pmod{\mathfrak{p}}$, but then restrict the domain to $\mathbb{T}_{\mathbb{Z},\mathfrak{n}'}/\mathfrak{m}$. Parts (e) and (f) are then obvious from Theorem B.13. Assertion (g) is implied by (d) and the definition of $X_{\Phi'}^{\text{ord}}$. \square

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