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CONGRUENCES FOR THE $(p-1)$TH APÉRY NUMBER

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Abstract

We prove two conjectural congruences on the $(p-1)$th Apéry number, which were recently proposed by Z.-H. Sun.

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1. Introduction

In 1979, Apéry [2] introduced the numbers

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \text{and} \quad A'_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

in his ingenious proof of the irrationality of $\zeta(2)$ and $\zeta(3)$. These numbers are now known as Apéry numbers. Since the appearance of the Apéry numbers, their interesting arithmetic properties have been gradually discovered. For instance, Beukers [3] showed that for primes $p \geq 5$ and positive integers m, r,

$$A_{mp^r-1} \equiv A_{mp^{r-1}-1} \pmod{p^{3r}},$$
$$A'_{mp^r-1} \equiv A'_{mp^{r-1}-1} \pmod{p^{3r}}.$$

In 2012, Sun [13] proved that, for any prime $p \geq 5$,

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5}.$$

Here the nth Bernoulli number B_n is defined as

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

In the past two decades, congruence properties for Apéry numbers and similar numbers have been widely studied (see, for example, [3–7, 11, 13–16]).

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Our interest concerns two conjectural congruences on the $(p-1)$th Apéry number, which were recently proposed by Sun [16, Conjectures 2.1 and 2.2].

CONJECTURE 1.1 (Z.-H. Sun). For any prime $p \geq 5$,

$$A_{p-1} \equiv 1 + \tfrac{2}{3}p^3 B_{p-3} \pmod{p^4}, \tag{1.1}$$

$$A'_{p-1} \equiv 1 + \tfrac{5}{3}p^3 B_{p-3} \pmod{p^4}. \tag{1.2}$$

The aim of this paper is to prove (1.1) and (1.2) by establishing the following generalisations.

THEOREM 1.2. *Let $p \geq 7$ be a prime. Then*

$$A_{p-1} \equiv 1 + p^3\big(\tfrac{4}{3}B_{p-3} - \tfrac{1}{2}B_{2p-4}\big) + \tfrac{1}{9}p^4 B_{p-3} \pmod{p^5}. \tag{1.3}$$

THEOREM 1.3. *Let $p \geq 7$ be a prime. Then*

$$A'_{p-1} \equiv 1 + p^3\big(\tfrac{10}{3}B_{p-3} - \tfrac{5}{4}B_{2p-4}\big) + \tfrac{5}{18}p^4 B_{p-3} \pmod{p^5}. \tag{1.4}$$

By taking $k = 1$ and $b = p-3$ in Kummer's congruence,

$$\frac{B_{k(p-1)+b}}{k(p-1)+b} \equiv \frac{B_b}{b} \pmod{p}$$

(see [12, page 193]), we obtain

$$B_{2p-4} \equiv \tfrac{4}{3}B_{p-3} \pmod{p}. \tag{1.5}$$

Substituting (1.5) into (1.3) and (1.4) gives (1.1) and (1.2) for primes $p \geq 7$. It is routine to check that (1.1) and (1.2) also hold for $p = 5$.

We prove Theorem 1.2 in Section 2 and Theorem 1.3 in Section 3.

2. Proof of Theorem 1.2

Since

$$\binom{p-1+k}{k} = \frac{p}{p+k}\binom{p+k}{k}, \tag{2.1}$$

we have

$$A_{p-1} = \sum_{k=0}^{p-1} \frac{p^2}{(p+k)^2}\binom{p-1}{k}^2\binom{p+k}{k}^2. \tag{2.2}$$

Note that

$$\binom{p-1}{k}\binom{p+k}{k} = \frac{(p^2-1^2)(p^2-2^2)\cdots(p^2-k^2)}{k!^2} \equiv (-1)^k(1-p^2H_k^{(2)}) \pmod{p^4}, \tag{2.3}$$

where $H_n^{(r)}$ denotes the nth generalised harmonic number of order r,

$$H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r},$$

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with the convention that $H_n = H_n^{(1)}$. It follows from (2.2) and (2.3) that

$$A_{p-1} = 1 + \sum_{k=1}^{p-1} \frac{p^2}{(p+k)^2} \binom{p-1}{k}^2 \binom{p+k}{k}^2$$
$$\equiv 1 + p^2 \sum_{k=1}^{p-1} \frac{1 - 2p^2 H_k^{(2)}}{(p+k)^2} \pmod{p^6}. \tag{2.4}$$

Furthermore,

$$\frac{1}{(p+k)^2} \equiv \frac{1}{k^2} - \frac{2p}{k^3} + \frac{3p^2}{k^4} \pmod{p^3}. \tag{2.5}$$

Substituting (2.5) into (2.4) gives

$$A_{p-1} \equiv 1 + p^2 H_{p-1}^{(2)} - 2p^3 H_{p-1}^{(3)} + 3p^4 H_{p-1}^{(4)} - 2p^4 \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k^2} \pmod{p^5}. \tag{2.6}$$

For $1 \le k \le p-1$,

$$H_k^{(2)} + H_{p-k}^{(2)} \equiv H_k^{(2)} + \sum_{i=1}^{p-k} \frac{1}{(p-i)^2} \pmod{p}$$
$$= H_{p-1}^{(2)} + \frac{1}{k^2}$$
$$\equiv \frac{1}{k^2} \pmod{p},$$

and so

$$\frac{H_k^{(2)}}{k^2} + \frac{H_{p-k}^{(2)}}{(p-k)^2} \equiv \frac{1}{k^4} \pmod{p}.$$

It follows that

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k^2} = \sum_{k=1}^{(p-1)/2} \left(\frac{H_k^{(2)}}{k^2} + \frac{H_{p-k}^{(2)}}{(p-k)^2} \right) \equiv \sum_{k=1}^{(p-1)/2} \frac{1}{k^4} \pmod{p}. \tag{2.7}$$

By [12, Theorem 5.2(a)],

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^4} \equiv 0 \pmod{p} \tag{2.8}$$

for any prime $p \ge 7$. From (2.7) and (2.8),

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k^2} \equiv 0 \pmod{p}. \tag{2.9}$$

The following two congruences are special cases of results of Lehmer [8, page 353]:

$$H_{p-1}^{(3)} \equiv 0 \pmod{p^2}, \tag{2.10}$$
$$H_{p-1}^{(4)} \equiv 0 \pmod{p}, \tag{2.11}$$

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for any prime $p \geq 7$. Combining (2.6) and (2.9)–(2.11) gives

$$A_{p-1} \equiv 1 + p^2 H_{p-1}^{(2)} \pmod{p^5}. \tag{2.12}$$

Taking $k = 2$ in [12, Theorem 5.1(a)] and simplifying,

$$H_{p-1}^{(2)} \equiv \left(\tfrac{4}{3}B_{p-3} - \tfrac{1}{2}B_{2p-4}\right)p + \left(\tfrac{4}{9}B_{p-3} - \tfrac{1}{4}B_{2p-4}\right)p^2 \pmod{p^3}. \tag{2.13}$$

Substituting (1.5) into (2.13) yields

$$H_{p-1}^{(2)} \equiv \left(\tfrac{4}{3}B_{p-3} - \tfrac{1}{2}B_{2p-4}\right)p + \tfrac{1}{9}p^2 B_{p-3} \pmod{p^3}. \tag{2.14}$$

Now (1.3) follows from (2.12) and (2.14).

3. Proof of Theorem 1.3

In order to prove Theorem 1.3, we need the following combinatorial identity.

Lemma 3.1. *For any nonnegative integer n,*

$$\sum_{k=1}^{n} \frac{(-1)^k}{k}\binom{n}{k}\binom{n+1+k}{k} = -2H_n + \frac{(-1)^n - 1}{n+1}. \tag{3.1}$$

Proof. Since

$$\binom{n+1+k}{k} = \frac{n+1+k}{n+1}\binom{n+k}{k},$$

we have

$$\sum_{k=1}^{n} \frac{(-1)^k}{k}\binom{n}{k}\binom{n+1+k}{k} = \sum_{k=1}^{n} \frac{(-1)^k}{k}\binom{n}{k}\binom{n+k}{k} + \frac{1}{n+1}\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}. \tag{3.2}$$

By the Chu–Vandermonde identity,

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n}\binom{n}{n-k}\binom{-n-1}{k} - 1 = (-1)^n - 1. \tag{3.3}$$

On the other hand, by [1, (2.2)],

$$\sum_{k=1}^{n} \frac{(-1)^k}{k}\binom{n}{k}\binom{n+k}{k} = -2H_n. \tag{3.4}$$

Now (3.1) follows from (3.2)–(3.4). □

Proof of Theorem 1.3. By (2.1) and (2.3),

$$\begin{aligned} A'_{p-1} &= 1 + \sum_{k=1}^{p-1} \frac{p}{p+k}\binom{p-1}{k}^2\binom{p+k}{k} \\ &\equiv 1 + p\sum_{k=1}^{p-1} \frac{(-1)^k}{p+k}\binom{p-1}{k}(1 - p^2 H_k^{(2)}) \pmod{p^5}. \end{aligned} \tag{3.5}$$

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Taking $n = p-1$ and $x = p$ in the partial fraction decomposition

$$\sum_{k=0}^{n} \frac{(-1)^k}{x+k}\binom{n}{k} = \frac{n!}{x(x+1)\cdots(x+n)},$$

we arrive at

$$\sum_{k=0}^{p-1} \frac{(-1)^k}{p+k}\binom{p-1}{k} = \frac{1}{p\binom{2p-1}{p-1}}.$$

It follows that

$$p\sum_{k=1}^{p-1} \frac{(-1)^k}{p+k}\binom{p-1}{k} = p\sum_{k=0}^{p-1} \frac{(-1)^k}{p+k}\binom{p-1}{k} - 1 = \frac{1}{\binom{2p-1}{p-1}} - 1. \tag{3.6}$$

We need the following congruence of McIntosh (see [9, (6)]):

$$\binom{2p-1}{p-1} \equiv 1 - p^2 H_{p-1}^{(2)} \pmod{p^5} \tag{3.7}$$

for any prime $p \geq 7$. Substituting (3.7) into (3.6) and using the fact that $H_{p-1}^{(2)} \equiv 0 \pmod{p}$, we arrive at

$$p\sum_{k=1}^{p-1} \frac{(-1)^k}{p+k}\binom{p-1}{k} \equiv p^2 H_{p-1}^{(2)} \pmod{p^5}. \tag{3.8}$$

On the other hand, using $\binom{p-1}{k} \equiv (-1)^k(1 - pH_k) \pmod{p^2}$,

$$p^3\sum_{k=1}^{p-1} \frac{(-1)^k}{p+k}\binom{p-1}{k} H_k^{(2)} \equiv p^3\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{p+k} - p^4\sum_{k=1}^{p-1} \frac{H_k H_k^{(2)}}{p+k} \pmod{p^5}. \tag{3.9}$$

By [10, (55)],

$$\sum_{k=1}^{p-1} \frac{H_k H_k^{(2)}}{p+k} \equiv \sum_{k=1}^{p-1} \frac{H_k H_k^{(2)}}{k} \equiv 0 \pmod{p}. \tag{3.10}$$

Since

$$\frac{1}{p+k} \equiv \frac{1}{k} - \frac{p}{k^2} \pmod{p^2},$$

by (2.9), we arrive at

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{p+k} \equiv \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} - p\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k^2} \equiv \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} \pmod{p^2}. \tag{3.11}$$

Combining (3.9)–(3.11) gives

$$p^3\sum_{k=1}^{p-1} \frac{(-1)^k}{p+k}\binom{p-1}{k} H_k^{(2)} \equiv p^3\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} \pmod{p^5}. \tag{3.12}$$

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Letting $n = p-1$ in (3.1),

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k}\binom{p+k}{k} = -2H_{p-1}.$$

It follows from (2.3) and the above that

$$\sum_{k=1}^{p-1} \frac{1-p^2H_k^{(2)}}{k} \equiv -2H_{p-1} \pmod{p^4}$$

and so

$$p^2\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} \equiv 3H_{p-1} \pmod{p^4},$$

which implies that

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} \equiv \frac{3}{p^2}H_{p-1} \pmod{p^2}. \tag{3.13}$$

Substituting (3.13) into (3.12) gives

$$p^3\sum_{k=1}^{p-1} \frac{(-1)^k}{p+k}\binom{p-1}{k}H_k^{(2)} \equiv 3pH_{p-1} \pmod{p^5}. \tag{3.14}$$

From [9, (6) and (7)],

$$pH_{p-1} \equiv -\frac{p^2}{2}H_{p-1}^{(2)} \pmod{p^5}. \tag{3.15}$$

Finally, combining (3.5), (3.8), (3.14) and (3.15) gives

$$A'_{p-1} \equiv 1 + \tfrac{5}{2}p^2H_{p-1}^{(2)} \pmod{p^5}. \tag{3.16}$$

Now (1.4) follows from (2.14) and (3.16). □

REMARK 3.2. On WeChat, Professor Z.-W. Sun independently conjectured two extensions of (1.1) and (1.2), namely,

$$A_{p-1} \equiv 1 - 2pH_{p-1} \pmod{p^5}, \tag{3.17}$$

$$A'_{p-1} \equiv 1 - 5pH_{p-1} \pmod{p^5} \tag{3.18}$$

for primes $p \geq 7$, which have simpler forms than (1.3) and (1.4). We remark that (3.17) and (3.18) can be deduced from (2.12), (3.15) and (3.16).

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