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Abstract of Australasian PhD thesis

A functional equation involving vector mean values

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Consider the functional equation

$$f(A(x, y)) = B(f(x), f(y)) ,$$

where A and B are vector mean values whose domains are open convex subsets of the euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^s$ respectively. The main part of this thesis is concerned with existence and uniqueness theorems for what we have called, by analogy with differential equations, boundary value problems and initial value problems, the latter being defined when $s = 1$ only. These problems consist of the functional equation together with either, the specification of the value of f at $m + 1$ distinct points (the boundary conditions), or the specification of the value and m partial derivatives of f at just one point (the initial conditions).

The first two chapters are both introductory and preparatory. Although their main purpose is to develop those properties of vector mean values needed for the subsequent discussion on boundary value problems and initial value problems, some of the results given therein have intrinsic value as part of the theory of vector mean values.

The whole of Chapter 3 is devoted to proving Theorem 3.1, an existence and uniqueness theorem for a wide class of boundary value problems. This states that given a boundary value problem whose boundary conditions are defined at $m + 1$ vectors whose closed A-hull (this concept is defined in

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Chapter 2 is non-empty, then there is a continuous solution of the boundary value problem which has an open domain and is unique and maximal in the sense that it extends any other continuous solution with open domain. This is more general than earlier results in a similar vein given by A.N. Kolmogorov, M. Nagumo, J. Aczél, and E. Hille, and in the introduction to this chapter we compare Theorem 3.1 against these earlier results so that it may be seen in perspective.

In Chapter 4 we turn our attention to initial value problems and concentrate on the real case $(m = 1)$. We explore the relationship between the existence and uniqueness of continuous solutions to a given initial value problem, and the "differentiability" of the mean values A and B, and we obtain two main results.

The first of these is Theorem 4.1 which states that given a family of initial value problems

$$f\big(A(x, y)\big) = B\big(f(x), f(y)\big), \quad f(b) = s, \quad f'(b) = \lambda,$$

where b and s are fixed, and λ varies through the positive real numbers, and given the existence of a continuous solution to one of these initial value problems, then each initial value problem in the family has a unique continuous solution if and only if the mappings $x \mapsto A(b, x)$ and $x \mapsto A\big(A(b, x), x\big)$ have derivatives at b with values in $(0, 1)$. A generalization of this theorem to the vector case $(m > 1)$ is given at the end of the chapter.

The other main result is Theorem 4.8 which is also for the real case and states that if the mappings $(x, y) \mapsto A(x, y)$ and $(u, v) \mapsto B(u, v)$ have partial derivatives everywhere, then every initial value problem has a unique continuous solution.